ON THE LOWER BOUND OF THE DISCREPANCY OF (t,s)-SEQUENCES: II

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Dedicated to the 100th anniversary of Professor N.M. Korobov

ABSTRACT. Let $(\mathbf{x}(n))_{n\geq 1}$ be an $s-$dimensional Niederreiter-Xing's sequence in base b. Let $D((\mathbf{x}(n))_{n=1}^N)$ be the discrepancy of the sequence $(\mathbf{x}(n))_{n=1}^N$. It is known that $ND((\mathbf{x}(n))_{n=1}^N) = O(\ln^s N)$ as $N \to \infty$. In this paper, we prove that this estimate is exact. Namely, there exists a constant $K > 0$, such that

$$\inf_{\mathbf{w}\in[0,1)^s} \sup_{1\leq N\leq b^m} ND((\mathbf{x}(n)\oplus\mathbf{w})_{n=1}^N) \geq Km^s \quad \text{for } m = 1,2,\dots .$$

We also get similar results for other explicit constructions of (t,s)-sequences.

Key words: low discrepancy sequences, (t,s)-sequences, (t,m,s)-nets
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1. Introduction.

1.1 Let $(\beta_n^{(s)})_{n\geq 1}$ be a sequence in unit cube $[0,1)^s$, $(\beta_{n,N}^{(s)})_{n=0}^{N-1}$ points set in $[0,1)^s$, $J_{\mathbf{y}} = [0,y_1)\times\cdots\times[0,y_s)$,

$$\Delta(J_{\mathbf{y}}, (\beta_{n,N}^{(s)})_{k=1}^N) = \#\{1\leq n\leq N \mid \beta_{n,N}^{(s)} \in J_{\mathbf{y}}\} - Ny_1\dots y_s. \tag{1.1}$$

We define the star discrepancy of a $(\beta_{n,N}^{(s)})_{n=0}^{N-1}$ as

$$D^*(N) = D^*((\beta_{n,N}^{(s)})_{n=0}^{N-1}) = \sup_{0<y_1,\dots,y_s\leq 1} \Big|\frac{1}{N}\Delta(J_{\mathbf{y}}, (\beta_{n,N}^{(s)})_{n=1}^N)\Big|. \tag{1.2}$$

Definition 1. *A sequence $(\beta_n^{(s)})_{n\geq 0}$ is of* low discrepancy *(abbreviated l.d.s.) if* $\mathrm{D}((\beta_n^{(s)})_{n=0}^{N-1}) = O(N^{-1}(\ln N)^s)$ *for* $N\to\infty$.

Definition 2. *A sequence of point sets $((\beta_{n,N}^{(s)})_{n=0}^{N-1})_{N=1}^\infty$ is of low discrepancy (abbreviated l.d.p.s.) if* $\mathrm{D}((\beta_{n,N}^{(s)})_{n=0}^{N-1}) = O(N^{-1}(\ln N)^{s-1})$, *for* $N\to\infty$.

For examples of such a sequence, see, e.g., [BC], [DiPi], and [Ni].
In 1954, Roth proved that there exists a constant $C_s > 0$, such that

$$ND^*((\beta_{n,N}^{(s)})_{n=0}^{N-1}) > C_s(\ln N)^{\frac{s-1}{2}}, \quad \text{and} \quad \overline{\lim} ND^*((\beta_n^{(s)})_{n=0}^{N-1})(\ln N)^{-s/2} > 0$$

for all N-point sets $(\beta_{n,N}^{(s)})_{n=0}^{N-1}$ and all sequences $(\beta_n^{(s)})_{n\geq 0}$.

According to the well-known conjecture (see, e.g., [BC, p.283], [DiPi, p.67], [Ni, p.32]), these estimates can be improved

$$ND^*((\beta_{n,N}^{(\ddot{s})})_{n=0}^{N-1})(\ln N)^{-\ddot{s}+1} > C'_{\ddot{s}} \quad \text{and} \quad \overline{\lim_{N\to\infty}} N(\ln N)^{-\dot{s}} D^*((\beta_n^{(\dot{s})})_{n=1}^{N}) > 0 \tag{1.3}$$

for all N-point sets $(\beta_{n,N}^{(\ddot{s})})_{n=0}^{N-1}$ and all sequences $(\beta_n^{(\dot{s})})_{n\geq 0}$ with some $C'_{\ddot{s}} > 0$.

In 1972, W. Schmidt proved (1.3) for $\dot{s} = 1$ and $\ddot{s} = 2$. In [FaCh], (1.3) is proved for a class of $(t, 2)-$sequences.

In 1989, Beck [Be1] proved that $ND^*(N) \geq \dot{c} \ln N (\ln\ln N)^{1/8-\epsilon}$ for $s = 3$ and some $\dot{c} > 0$. In 2008, Bilyk, Lacey and Vagharshakyan (see [Bi, p.147], [BiLa, p.2]), proved in all dimensions $s \geq 3$ that there exists some $\dot{c}(s), \eta > 0$ for which the following estimate holds for all N-point sets : $ND^*(N) > \dot{c}(s)(\ln N)^{\frac{s-1}{2}+\eta}$.

There exists another conjecture on the lower bound for the discrepancy function: there exists a constant $\dot{c}_3 > 0$, such that

$$ND^*((\beta_{k,N})_{k=0}^{N-1}) > \dot{c}_3(\ln N)^{s/2}$$

for all N-point sets $(\beta_{k,N})_{k=0}^{N-1}$ (see [Bi, p.147], [BiLa, p.3] and [ChTr, p.153]).

A subinterval E of $[0,1)^s$ of the form

$$E = \prod_{i=1}^{s} [a_i b^{-d_i}, (a_i+1)b^{-d_i}),$$

with $a_i, d_i \in Z,\ d_i \geq 0,\ 0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$ is called an *elementary interval in base $b \geq 2$.*

Definition 3. *Let $0 \leq t \leq m$ be an integer. A (t, m, s)*-net in base *b is a point set $\mathbf{x}_0, ..., \mathbf{x}_{b^m-1}$ in $[0,1)^s$ such that $\#\{n \in [0, b^m-1] | x_n \in E\} = b^t$ for every elementary interval E in base b with* $\mathrm{vol}(E) = b^{t-m}$.

Definition 4. *Let $t \geq 0$ be an integer. A sequence $\mathbf{x}_0, \mathbf{x}_1, ...$ of points in $[0,1)^s$ is a (t, s)*-sequence in base *b if, for all integers $k \geq 0$ and $m \geq t$, the point set consisting of $\mathbf{x}_n$ with $kb^m \leq n < (k+1)b^m$ is a (t, m, s)-net in base b.*

By [Ni, p. 56,60], (t, m, s)-nets and (t, s)-sequences are of low discrepancy.

See reviews on (t, m, s)-nets and (t, s)-sequences in [DiPi] and [Ni].

For $x = \sum_{j\geq 1} x_i b^{-i}$, and $y = \sum_{j\geq 1} y_i b^{-i}$ where $x_i, y_i \in Z_b := \{0, 1, ..., b-1\}$, we define the ($b$-adic) digital shifted point v by $v = x \oplus y := \sum_{j\geq 1} v_i b^{-i}$, where

$v_i \equiv x_i + y_i \mod(b)$ and $v_i \in Z_b$. For higher dimensions $s > 1$, let $\mathbf{y} = (y_1, ..., y_s) \in [0,1)^s$. For $\mathbf{x} = (x_1, ..., x_s) \in [0,1)^s$ we define the (b-adic) digital shifted point $\mathbf{v}$ by $\mathbf{v} = \mathbf{x} \oplus \mathbf{y} = (x_1 \oplus y_1, ..., x_s \oplus y_s)$. For $n_1, n_2 \in [0, b^m)$, we define $n_1 \oplus n_2 := (n_1/b^m \oplus n_2/b^m)b^m$.

For $x = \sum_{j\geq 1} x_i p_i^{-i}$, where $x_i \in Z_b$, $x_i = 0$ $(i = 1, ..., k)$ and $x_{k+1} \neq 0$, we define the absolute valuation $\|.\|_b$ of x by $\|x\|_b = b^{-k-1}$. Let $\|n\|_b = b^k$ for $n \in [b^k, b^{k+1})$.

Definition 5. *A point set* $(\mathbf{x}_n)_{0\leq n<b^m}$ *in* $[0,1)^s$ *is* $d-$admissible *in base* b *if*

$$\min_{0\leq k<n<b^m} \|\mathbf{x}_n \ominus \mathbf{x}_k\|_b > b^{-m-d} \quad \text{where} \quad \|\mathbf{x}\|_b := \prod_{i=1}^{s} \left\| x_j^{(i)} \right\|_b. \tag{1.4}$$

A sequence $(\mathbf{x}_n)_{n\geq 0}$ *in* $[0,1)^s$ *is* $d-$admissible *in base* b *if* $\inf_{n>k\geq 0} \|n \ominus k\|_b \|\mathbf{x}_n \ominus \mathbf{x}_k\|_b \geq b^{-d}$.

Let $(\mathbf{x}_n)_{n\geq 0}$ be a $d-$admissible (t,s)-sequence in base b. In [Le4], we proved for all $m \geq 9s^2(d+t)$ that

$$1 + \max_{1\leq N\leq b^m} ND^*((\mathbf{x}_n \oplus \mathbf{w})_{0\leq n<N}) \geq b^{-d} K_{d,t,s+1}^{-s} m^s \tag{1.5}$$

with some $\mathbf{w} \in [0,1)^s$ and $K_{d,t,s} = 4(d+t)(s-1)^2$.

In this paper we consider some known constructions of (t,s)-sequences (e.g., Niederreiter's sequences, Xing-Niederreiter's sequences, Halton type (t,s)-sequences) and we prove that they have $d-$admissible properties. Moreover, we prove that for these sequences the bound (1.5) is true for all $\mathbf{w} \in [0,1)^s$. This result supports conjecture (1.3) (see also [Be2], [LaPi], [Le2] and [Le3]).

We describe the structure of the paper. In Section 2, we fix some definitions. In Section 3, we state our results. In Section 4, we prove our outcomes.

2. Definitions and auxiliary results.

2.1 Notation and terminology for algebraic function fields. For the theory of algebraic function fields, we follow the notation and terminology in the books [St] and [Sa].

Let b be an arbitrary prime power, $\mathsf{k} = \mathbb{F}_b$ a finite field with b elements, $\mathsf{k}(x) = \mathbb{F}_b(x)$ the rational function field over $\mathbb{F}_b$, and $\mathsf{k}[x] = \mathbb{F}_b[x]$ the polynomial ring over $\mathbb{F}_b$. For $\alpha = f/g$, $f, g \in \mathsf{k}[x]$, let

$$\nu_\infty(\alpha) = \deg(g) - \deg(f) \tag{2.1}$$

be the degree valuation of $\mathsf{k}(x)$. We define the field of Laurent series as

$$\mathsf{k}((x)) := \Big\{ \sum_{i=m}^{\infty} a_i x^i \mid m \in \mathbb{Z},\ a_i \in \mathsf{k} \Big\}.$$

A finite extension field F of $\mathsf{k}(x)$ is called an algebraic function field over k. Let k is algebraically closed in F. We express this fact by simply saying that F/k is an algebraic function field. The genus of F/k is denoted by g.

A place $\mathcal{P}$ of F is, by definition, the maximal ideal of some valuation ring of F. We denote by $O_{\mathcal{P}}$ the valuation ring corresponding to $\mathcal{P}$ and we denote by $\mathbb{P}_F$ the set of places of F. For a place $\mathcal{P}$ of F, we write $\nu_{\mathcal{P}}$ for the normalized discrete valuation of F corresponding to $\mathcal{P}$, and any element $t \in F$ with $\nu_{\mathcal{P}}(t) = 1$ is called a local parameter (prime element) at $\mathcal{P}$.

The field $F_{\mathcal{P}} := O_{\mathcal{P}}/\mathcal{P}$ is called the residue field of F with respect to $\mathcal{P}$. The degree of a place $\mathcal{P}$ is defined as $\deg(\mathcal{P}) = [F_{\mathcal{P}} : \mathsf{k}]$. We denote by $\mathrm{Div}(F)$ the set of divisors of F/k.

Let $y \in F \setminus \{0\}$ and denote by $Z(y)$, respectively $N(y)$, the set of zeros, respectively poles, of y. Then we define the zero divisor of y by $(y)_0 = \sum_{\mathcal{P}\in Z(y)} \nu_{\mathcal{P}}(y)\mathcal{P}$ and the pole divisor of y by $(y)_\infty = \sum_{\mathcal{P}\in N(y)} \nu_{\mathcal{P}}(y)\mathcal{P}$. Furthermore, the principal divisor of y is given by $\mathrm{div}(y) = (y)_0 - (y)_\infty$.

Theorem A (Approximation Theorem). [St, Theorem 1.3.1] *Let F/k be a function field, $\mathcal{P}_1, ..., \mathcal{P}_n \in \mathbb{P}_F$ pairwise distinct places of F/k, $x_1, ..., x_n \in F$ and $r_1, ..., r_n \in \mathbb{Z}$. Then there is some $y \in F$ such that*

$$\nu_{\mathcal{P}_i}(y - x_i) = r_i \quad \text{for} \quad i = 1, ..., n.$$

The completion of F with respect to $\nu_{\mathcal{P}}$ will be denoted by $F^{(\mathcal{P})}$. Let t be a local parameter of $\mathcal{P}$. Then $F^{(\mathcal{P})}$ is isomorphic to $F_{\mathcal{P}}((t))$ (see [Sa, Theorem 2.5.20]), and an arbitrary element $\alpha \in F^{(P)}$ can be uniquely expanded as (see [Sa, p. 293])

$$\alpha = \sum_{i=\nu_{\mathcal{P}}(\alpha)}^{\infty} S_i t^i \quad \text{where} \quad S_i = S_i(t,\alpha) \in F_{\mathcal{P}} \subseteq F^{(P)}. \tag{2.2}$$

The derivative $\frac{d\alpha}{dt}$, or differentiation with respect to t, is defined by (see [Sa, Definition 9.3.1])

$$\frac{d\alpha}{dt} = \sum_{i=\nu_{\mathcal{P}}(\alpha)}^{\infty} i S_i t^{i-1}. \tag{2.3}$$

For an algebraic function field F/k, we define its set of differentials (or Hasse differentials, H-differentials) as

$$\Delta_F = \{y\, dz \mid y \in F,\ z \text{ is a separating element for } F/\mathsf{k}\}$$

(see [St, Definition 4.1.7]).

Proposition A. ([St, Proposition 4.1.8] or [Sa, Theorem 9.3.13]) *Let $z \in F$ be separating. Then every differential $\gamma \in \Delta_F$ can be written uniquely as $\gamma = y\, dz$ for some $y \in F$.*

We define the order of $\alpha\ \mathrm{d}\beta$ at $\mathcal{P}$ by

$$\nu_{\mathcal{P}}(\alpha\ \mathrm{d}\beta) := \nu_{\mathcal{P}}(\alpha\ \mathrm{d}\beta/\mathrm{d}t), \tag{2.4}$$

where t is any local parameter for $\mathcal{P}$ (see [Sa, Definition 9.3.8]).

Let Ω_F be the set of all Weil differentials of F/k. There exists a $F-$linear isomorphism of the differential module Δ_F onto Ω_F (see [St, Theorem 4.3.2] or [Sa, Theorem 9.3.15]).

For $0 \neq \omega \in \Omega_F$, there exists a uniquely determined divisor $\mathrm{div}(\omega) \in \mathrm{Div}(F)$. Such a divisor $\mathrm{div}(\omega)$ is called a canonical divisor of F/k. (see [St, Definition 1.5.11]). For a canonical divisor $\dot{W}$, we have (see [St, Corollary 1.5.16])

$$\deg(\dot{W}) = 2g-2 \quad \text{and} \quad \ell(\dot{W}) = g. \tag{2.5}$$

Let $\alpha\ \mathrm{d}\beta$ be a nonzero H-differential in F and let ω the corresponding Weil differential. Then (see [Sa, Theorem 9.3.17], [St, ref. 4.35])

$$\nu_{\mathcal{P}}(\mathrm{div}(\omega)) = \nu_{\mathcal{P}}(\alpha\ \mathrm{d}\beta), \quad \text{for all} \quad \mathcal{P} \in \mathbb{P}_F. \tag{2.6}$$

Let $\alpha\ \mathrm{d}\beta$ be a H-differential, t a local parameter of $\mathcal{P}$, and

$$\alpha\ \mathrm{d}\beta = \sum_{i=\nu_{\mathcal{P}}(\alpha)}^{\infty} S_i t^i \mathrm{d}t \in F^{(\mathcal{P})}.$$

Then the residue of $\alpha\ \mathrm{d}\beta$ (see [Sa, Definition 9.3.10) is defined by

$$\mathrm{Res}_{\mathcal{P}}(\alpha\ \mathrm{d}\beta) := \mathrm{Tr}_{F_{\mathcal{P}}/\mathsf{k}}(S_{-1}) \in \mathsf{k}. \tag{2.7}$$

Let

$$\mathrm{Res}_{\mathcal{P},t}(\alpha) := \mathrm{Res}_{\mathcal{P}}(\alpha \mathrm{d}t). \tag{2.8}$$

Theorem B (Residue Theorem). ([St, Corollary 4.3.3], [Sa Theorem 9.3.14]) *Let $\alpha\ \mathrm{d}\beta$ be any H-differential. Then $\mathrm{Res}_{\mathcal{P}}(\alpha\ \mathrm{d}\beta) = 0$ for almost all places $\mathcal{P}$. Furthermore,*

$$\sum_{\mathcal{P}\in\mathbb{P}_F} \mathrm{Res}_{\mathcal{P}}(\alpha\ \mathrm{d}\beta) = 0.$$

For a divisor $\mathcal{D}$ of F/k, let $\mathcal{L}(\mathcal{D})$ denote the Riemann-Roch space

$$\mathcal{L}(\mathcal{D}) = \mathcal{L}_F(\mathcal{D}) = \mathcal{L}_{F/\mathsf{k}}(\mathcal{D}) = \{y \in F \setminus 0 \mid \mathrm{div}(y) + \mathcal{D} \geq 0\} \cup \{0\}.$$

Then $\mathcal{L}(\mathcal{D})$ is a finite-dimensional vector space over F, and we denote its dimension by $\ell(\mathcal{D})$. By [St, Corollary 1.4.12], $\ell(\mathcal{D}) = \{0\}$ for $\deg(\mathcal{D}) < 0$.

Theorem C (Riemann-Roch Theorem). [St, Theorem 1.5.15, and St, Theorem 1.5.17] *Let W be a canonical divisor of F/k. Then for each divisor $A \in \mathrm{div}(F)$, $\ell(A) = \deg(A) + 1 - g + \ell(W - A)$, and*

$$\ell(A) = \deg(A) + 1 - g \quad \text{for} \quad \deg(A) \geq 2g-1.$$

Let $P \in \mathbb{P}_F$, $e_P = \deg(P)$, and let $F' = FF_P$ be the compositum field (see [Sa, Theorem 5.4.4]). By [St, Proposition 3.6.1] F_P is the full constant field of F'.

For a place $P \in \mathbb{P}_F$, we define its conorm (with respect to F'/F) as

$$\mathrm{Con}_{F'/F}(P) := \sum_{P'|P} e(P'|P)P', \tag{2.9}$$

where the sum runs over all places $P' \in \mathbb{P}_{F'}$ lying over P (see [St, Definition 3.1.8.]) and $e(P'|P)$ is the ramification index of P' over P.

Theorem D. ([St, Theorem 3.6.3]) *In an algebraic constant field extension $F' = FF_P$ of F/k, the following hold:*

(a) F'/F is unramified (i.e., $e(P'|P) = 1$ for all $P \in \mathbb{P}_F$ and all $P' \in \mathbb{P}_{F'}$ with $P'|P$).
(b) F'/F_P has the same genus as F/k.
(c) For each divisor $A \in \mathrm{Div}(F)$, we have $\deg(\mathrm{Con}_{F'/F}(A)) = \deg(A)$.
(d) For each divisor $A \in \mathrm{Div}(F)$, $\ell(\mathrm{Con}_{F'/F}(A)) = \ell(A)$. More precisely: Every basis of $\mathcal{L}_{F/\mathsf{k}}(A)$ is also a basis of $\mathcal{L}_{F'/F_P}(\mathrm{Con}_{F'/F}(A))$.

Theorem E. ([St, Proposition 3.1.9]) *For $0 \neq x \in F$ let $(x)_0^F$, $(x)_\infty^F$, $\mathrm{div}(x)^F$, resp. $(x)_0^{F'}$, $(x)_\infty^{F'}$, $\mathrm{div}(x)^{F'}$ denote the zero, pole, principal divisor of x in $\mathrm{Div}(F)$ resp. in $\mathrm{Div}(F')$. Then*

$$\mathrm{Con}_{F'/F}((x)_0^F) = (x)_0^{F'},\ \mathrm{Con}_{F'/F}((x)_\infty^F) = (x)_\infty^{F'} \text{ and } \mathrm{Con}_{F'/F}(\mathrm{div}(x)^F) = \mathrm{div}(x)^{F'}.$$

Let $\mathfrak{B}_1, ..., \mathfrak{B}_\mu$ be all the places of F'/F_P lying over P. By [St, Proposition 3.1.4.], [St, Definition 3.1.5.] and Theorem D(a), we have

$$\nu_{\mathfrak{B}_i}(\alpha) = \nu_P(\alpha) \quad \text{for} \quad \alpha \in F, \quad 1 \leq i \leq \mu. \tag{2.10}$$

We will denote by $F^{(P)}$ resp. $F'^{(\mathfrak{B}_i)}$ $(1 \leq i \leq \mu)$ the completion of F resp. F' with respect to the valuation ν_P resp. $\nu_{\mathfrak{B}_i}$. Applying [Sa, p.132, 133], we obtain

$$F \subseteq F^{(P)} \subseteq F'^{(\mathfrak{B}_i)} \quad \text{and} \quad F \subseteq F' \subseteq F'^{(\mathfrak{B}_i)}, \quad 1 \leq i \leq \mu.$$

Let t be a local parameter of $\mathcal{P}$, and let $\alpha \in F^{(P)}$. By (2.10), we have $\nu_{\mathfrak{B}_i}(t) = 1$. Consider the local expansion (2.2). Using (2.10), we get $\nu_{\mathfrak{B}_i}(\alpha) = \nu_P(\alpha)$. Hence

$$\nu_{\mathfrak{B}_i}(\alpha) = \nu_P(\alpha) \quad \text{for} \quad \alpha \in F' \cap F^{(P)} \quad 1 \leq i \leq \mu. \tag{2.11}$$

Theorem F. ([LiNi, Theorem 2.24]) *Let M be a finite extension of the finite field L, both considered as vector spaces over L. Then the linear transformations from M into L are exactly the mappings K_β, $\beta \in F$ where $K_\beta = \mathrm{Tr}_{M/L}(\beta\alpha)$ for all $\alpha \in F$.*

Furthermore, we have $K_\beta \neq K_\gamma$ whenever β and γ are distinct elements of L.

Theorem G. ([St, Proposition 3.3.3] or [LiNi, Definition 2.30, and p.58]) *Let L be a finite field and M a finite extension of L. Consider a basis $\{\alpha_1, ..., \alpha_m\}$ of M/L. Then there are uniquely determined elements $\beta_1, ..., \beta_m$ of M, such that*

$$\mathrm{Tr}_{M/L}(\alpha_i \beta_j) = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.12}$$

The set $\beta_1, ..., \beta_m$ is a basis of M/L as well; it is called the dual basis of $\{\alpha_1, ..., \alpha_m\}$ (with respect to the trace).

2.2 Digital sequences and $(\mathbf{T}, s)$ sequences ([DiPi, Section 4]).

Definition 6. ([DiPi, Definition 4.30]) *For a given dimension $s \geq 1$, an integer base $b \geq 2$, and a function $\mathbf{T} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\mathbf{T}(m) \leq m$ for all $m \in \mathbb{N}_0$, a sequence $(\mathbf{x}_0, \mathbf{x}_1, ...)$ of points in $[0,1)^s$ is called a $(\mathbf{T}, s)$-sequence in base b if for all integers $m \geq 0$ and $k \geq 0$, the point set consisting of the points $x_{kb^m}, ..., x_{kb^m + b^m - 1}$ forms a $(\mathbf{T}(m), m, s)$-net in base b.*

Lemma A. ([DiPi, Lemma 4.38]) *Let $(\mathbf{x}_0, \mathbf{x}_1, ...)$ be a $(\mathbf{T}, s)$-sequence in base b. Then, for every m, the point set $\{\mathbf{y}_0, \mathbf{y}_1, ..., \mathbf{y}_{b^m - 1}\}$ with $\mathbf{y}_k := (\mathbf{x}_k, k/b^m)$, $0 \leq k < b^m$, is an (r(m),m, s+1)-net in base b with $r(m) := \max\{\mathbf{T}(0), ..., \mathbf{T}(m)\}$.*

Repeating the proof of this lemma, we obtain

Lemma 1. *Let $(\mathbf{x}_n)_{n \geq 0}$ be a sequence in $[0,1)^s$, $m_n \in \mathbb{N}$, $m_i > m_j$ for $i > j$, and let $(\mathbf{x}_n, n/b^{m_k})_{0 \leq n < b^{m_k}}$ be a $(t, m_k, s+1)$-net in base b for all $k \geq 1$. Then $(\mathbf{x}_n)_{n \geq 0}$ is a (t, s)-sequence in base b.*

Lemma B. ([Ni, Lemma 3.7]) *Let $(\mathbf{x}_n)_{n \geq 0}$ be a sequence in $[0,1)^s$. For $N \geq 1$, let H be the point set consisting of $(\mathbf{x}_n, n/N) \in [0,1)^{s+1}$ for $n = 0, ..., N-1$. Then*

$$1 + \max_{1 \leq M \leq N} M\mathrm{D}^*((\mathbf{x}_n)_{n=0}^{M-1}) \geq N\mathrm{D}^*((\mathbf{x}_n, n/N)_{n=0}^{N-1}).$$

Definition 7. ([DiNi, Definition 1]) *Let $m, s \geq 1$ be integers. Let $C^{(1,m)}, ...,$ $C^{(s,m)}$ be $m \times m$ matrices over $\mathbb{F}_b$. Now we construct b^m points in $[0,1)^s$. For $n = 0, 1, ..., b^m - 1$, let $n = \sum_{j=0}^{m-1} a_j(n) b^j$ be the b-adic expansion of n. Choose a bijection $\phi :\ Z_b := \{0, 1, ..., b-1\} \mapsto \mathbb{F}_b$ with $\phi(0) = \bar{0}$, the neutral element of addition in $\mathbb{F}_b$. Let $|\phi(a)| := |a|$ for $a \in Z_b$. We identify n with the row vector*

$$\mathbf{n} = (\bar{a}_0(n), ..., \bar{a}_{m-1}(n)) \in \mathbb{F}_b^m \quad \text{with} \quad \bar{a}_i(n) = \phi(a_i(n)),\ 0 \leq i \leq m-1. \tag{2.13}$$

We map the vectors

$$y_n^{(i)} = (y_{n,1}^{(i)}, ..., y_{n,m}^{(i)}) := \mathbf{n} C^{(i,m)\top} \in \mathbb{F}_b^m \tag{2.14}$$

to the real numbers

$$x_n^{(i)} = \sum_{j=1}^{m} \phi^{-1}(y_{n,j}^{(i)})/b^j \tag{2.15}$$

to obtain the point

$$\mathbf{x}_n := (x_n^{(1)}, ..., x_n^{(s)}) \in [0,1)^s. \tag{2.16}$$

The point set $\{\mathbf{x}_0, ..., \mathbf{x}_{b^m-1}\}$ *is called a* digital net *(over* $\mathbb{F}_b$*) (with* generating matrices $(C^{(1,m)}, ..., C^{(s,m)})$*).*

For $m = \infty$*, we obtain a sequence* $\mathbf{x}_0, \mathbf{x}_1, ...$ *of points in* $[0,1)^s$ *which is called a* digital sequence $($*over* $\mathbb{F}_b)$ $($*with* generating matrices $(C^{(1,\infty)}, ..., C^{(s,\infty)}))$.

We abbreviate $C^{(i,m)}$ as $C^{(i)}$ for $m \in \mathbb{N}$ and for $m = \infty$.

Definition 8. *Let* $0 \leq D(1) \leq D(2) \leq D(3) \leq ...$ *be a sequence of integers. A sequence* $(\mathbf{x}_n)_{n \geq 0}$ *in* $[0,1)^s$ *is* $\mathbf{D}-$admissible *in base b if*

$$\min_{0 \leq k < n < b^m} \|\mathbf{x}_n \ominus \mathbf{x}_k\|_b > b^{-m-D(m)} \quad \text{where} \quad \|\mathbf{x}\|_b := \prod_{i=1}^{s} \left\| x_j^{(i)} \right\|_b, \tag{2.17}$$

$\|x\|_b = b^{-k-1}$, $x = \sum_{j \geq 1} x_i p_i^{-i}$ with $x_i \in Z_b$, $x_i = 0$ $(i = 1, ..., k)$ and $x_{k+1} \neq 0$.

Note that for $D(m) = d, \ m = 1, 2, ...$ this definition is equal to Definition 5. It is easy to see that condition (2.17) coincides for the case of digital sequences with the following inequality

$$\min_{0 < n < b^m} \|\mathbf{x}_n\|_b > b^{-m-D(m)}, \quad m = 1, 2, ... \, . \tag{2.18}$$

2.3 Duality theory (see [DiPi, Section 7], [DiNi], [NiPi], [Skr]).

Let $\mathcal{N}$ be an arbitrary $\mathbb{F}_b$-linear subspace of $\mathbb{F}_b^{sm}$. Let H be a matrix over $\mathbb{F}_b$ consisting of sm columns such that the row-space of H is equal to $\mathcal{N}$. Then we define the dual space $\mathcal{N}^\perp \subseteq \mathbb{F}_b^{sm}$ of $\mathcal{N}$ to be the null space of H (see [DiPi, p. 244]). In other words, $\mathcal{N}^\perp$ is the orthogonal complement of $\mathcal{N}$ relative to the standard inner product in F_b^{sm},

$$\mathcal{N}^\perp = \{A \in \mathbb{F}_b^{sm} \mid B \cdot A = 0 \quad \text{for all} \ \ B \in \mathcal{N}\}. \tag{2.19}$$

For any vector $\mathbf{a} = (a_1, ..., a_m) \in \mathbb{F}_b^m$, let

$$v_m(\mathbf{a}) = 0 \text{ if } \mathbf{a} = \mathbf{0} \quad \text{and} \quad v_m(\mathbf{a}) = \max\{j \,:\, a_j \neq 0\} \text{ if } \mathbf{a} \neq \mathbf{0}. \tag{2.20}$$

Then we extend this definition to $\mathbb{F}_b^{ms}$ by writing a vector $\mathbf{A} \in \mathbb{F}_b^{ms}$ as the concatenation of s vectors of length m, i.e. $\mathbf{A} = (\mathbf{a}_1, ..., \mathbf{a}_s) \in \mathbb{F}_b^{ms}$ with $\mathbf{a}_i \in \mathbb{F}_b^m$ for $1 \leq i \leq s$ and putting

$$V_m(\mathbf{A}) = \sum_{1 \leq i \leq s} v_m(\mathbf{a}_i). \tag{2.21}$$

Definition 9. For any nonzero $\mathbb{F}_b^m$-linear subspace $\mathcal{N}$ of $\mathbb{F}_b^{ms}$, the minimum distance of $\mathcal{N}$ is defined by

$$\delta_m(\mathcal{N}) = \min\{V_m(\mathbf{A}) \mid \mathbf{A} \in \mathcal{N} \setminus \{\mathbf{0}\}\}.$$

We define a weight function on $\mathbb{F}_b^{ms}$ dual to the weight function V_m (2.21). For any vector $\mathbf{a} = (a_1, ..., a_m) \in \mathbb{F}_b^m$, let

$$v_m^{\perp}(\mathbf{a}) = m+1 \text{ if } \mathbf{a} = \mathbf{0} \quad \text{and} \quad v_m^{\perp}(\mathbf{a}) = \min\{j \,:\, a_j \neq 0\} \text{ if } \mathbf{a} \neq \mathbf{0}. \tag{2.22}$$

Then we extend this definition to $\mathbb{F}_b^{ms}$ by writing a vector $\mathbf{A} \in \mathbb{F}_b^{ms}$ as the concatenation of s vectors of length m, i.e. $\mathbf{A} = (\mathbf{a}_1, ..., \mathbf{a}_s) \in \mathbb{F}_b^{ms}$ with $\mathbf{a}_i \in \mathbb{F}_b^m$ for $1 \leq i \leq s$ and putting

$$V_m^{\perp}(\mathbf{A}) = \sum_{1 \leq i \leq s} v_m^{\perp}(\mathbf{a}_i). \tag{2.23}$$

Definition 10. For any nonzero $\mathbb{F}_b^m$-linear subspace $\mathcal{N}$ of $\mathbb{F}_b^{ms}$, the maximum distance of $\mathcal{N}$ is defined by

$$\delta_m^{\perp}(\mathcal{N}) = \max\{V_m^{\perp}(\mathbf{A}) \mid \mathbf{A} \in \mathcal{N} \setminus \{\mathbf{0}\}\}. \tag{2.24}$$

Definition 11. ([DiPi], Definition 7.4) Let k, m, s be positive integers. The system $\{\dot{\mathbf{c}}_j^{(i)} \in \mathbb{F}_b^m \mid 1 \leq j \leq m,\ 1 \leq i \leq s\}$ is called a $(k, m, s) - \textit{system}$ over $\mathbb{F}_b$ if for any $k_1, ..., k_s \in \mathbb{N}_0$ with $0 \leq k_i \leq m$ for $1 \leq i \leq s$ and $k_1 + ... + k_s = k$ the system

$$\{\dot{\mathbf{c}}_j^{(i)} \in \mathbb{F}_b^m \mid 1 \leq j \leq k_i,\ 1 \leq i \leq s\}$$

is linearly independent over $\mathbb{F}_b$.

For a given $(k, m, s)-$ system $\{\dot{\mathbf{c}}_j^{(i)} \in \mathbb{F}_b^m \mid 1 \leq j \leq m,\ 1 \leq i \leq s\}$ let $\dot{C}^{(i)}, 1 \leq i \leq s$ be the $m \times m$ matrix with the row vectors $\dot{\mathbf{c}}_1^{(i)}, ..., \dot{\mathbf{c}}_m^{(i)}$. With these $m \times m$ matrices over is linearly independent over $\mathbb{F}_b$, we build up the matrix

$$\dot{C} = (\dot{C}^{(1)\top} | \dot{C}^{(2)\top} | ... | \dot{C}^{(s)\top}) \in \mathbb{F}_b^{m \times sm}.$$

Let $\dot{\mathcal{C}}$ denote the row space of the matrix $\dot{C}$. The dual space is then given by

$$\dot{\mathcal{C}}^{\perp} = \{A \in \mathbb{F}_b^{sm} \mid B \cdot A = \mathbf{0} \quad \text{for all} \ \ B \in \dot{\mathcal{C}}\}.$$

Lemma C. ([DiPi, Theorem 7.5]) *The system* $\{\dot{\mathbf{c}}_j^{(i)} \in \mathbb{F}_b^m \mid 1 \le j \le m, \ 1 \le i \le s\}$ *is a* $(k, m, s)-$*system over* $\mathbb{F}_b$ *if and only if the dual space* $\dot{\mathcal{C}}^{\perp}$ *of the row space* $\dot{\mathcal{C}}$ *satisfies* $\delta_m(\dot{\mathcal{C}}^{\perp}) \ge k+1$.

Let $C^{(1)}, ..., C^{(s)} \in \mathbb{F}_b^{\infty\times\infty}$ be generating matrices of a digital sequence $\mathbf{x}_n(C)_{n\ge 0}$ over $\mathbb{F}_b$. For any $m \in \mathbb{N}$, we denote the $m \times m$ left-upper sub-matrix of $C^{(i)}$ by $[C^{(i)}]_m$. The matrices $[C^{(1)}]_m, ..., [C^{(s)}]_m$ are then the generating matrices of a digital net. We define the overall generating matrix of this digital net by

$$[C]_m = ([C^{(1)}]_m^{\top} | [C^{(2)}]_m^{\top} | ... | [C^{(s)}]_m^{\top}) \in F_b^{m\times sm}, \qquad m = 1, 2, ... \tag{2.25}$$

Let $\mathcal{C}_m$ denote the row space of the matrix $[C]_m$ i.e.,

$$\mathcal{C}_m = \Big\{ \Big(\sum_{r=0}^{m-1} c_{j,r}^{(i)} \bar{a}_r(n) \Big)_{0\le j\le m-1, 1\le i\le s} \mid 0 \le n < b^m \Big\}. \tag{2.26}$$

The dual space is then given by

$$\mathcal{C}_m^{\perp} = \{A \in \mathbb{F}_b^{sm} \mid B \cdot A = \mathbf{0} \quad \text{for all} \ \ B \in \mathcal{C}_m\}. \tag{2.27}$$

Consider a matrix

$$\tilde{C}_m = (\tilde{C}_m^{(1)\top} | \tilde{C}_m^{(2)\top} | ... | \tilde{C}_m^{(s)\top}) \in \mathbb{F}_b^{m\times sm}$$

with row space $\tilde{\mathcal{C}}_m = \mathcal{C}_m^{\perp}$. Let $\tilde{\mathbf{c}}_j^{(i)} = (\tilde{c}_{j,1}^{(i)}, ..., \tilde{c}_{j,m}^{(i)})$ with $j \in [1, m]$ are row vectors of the matrix $\tilde{C}_m^{(i)}$, $i = 1, ..., s$. Hence

$$\tilde{\mathcal{C}}_m = \mathcal{C}_m^{\perp} = \Big\{ \Big(\sum_{r=0}^{m-1} \tilde{c}_{j,r}^{(i)} \bar{a}_r(n) \Big)_{0\le j\le m-1, 1\le i\le s} \mid 0 \le n < b^m \Big\}. \tag{2.28}$$

Let $\tilde{\mathbf{c}}_j^{(*,i)} = (\tilde{c}_{j,m-1}^{(i)}, ..., \tilde{c}_{j,1}^{(i)}, \tilde{c}_{j,0}^{(i)})$, $j = 0, ..., m-1$, $i = 1, ..., s$. Consider the matrix $\tilde{C}_m^{(*,i)}$, with row vectors $\tilde{\mathbf{c}}_j^{(*,i)}$, $j = 0, ..., m-1$, $i = 1, ..., s$.

Let $\tilde{C}_m^{(*)} = (\tilde{C}_m^{(*,1)\top} | ... | \tilde{C}_m^{(*,s)^{\top}})$. The row space of $\tilde{C}_m^{(*)}$ is then given by

$$\tilde{\mathcal{C}}_m^{(*)} = \Big\{ \Big(\sum_{r=0}^{m-1} \tilde{c}_{m-j-1,r}^{(i)} \bar{a}_r(n) \Big)_{0\le j\le m-1, 1\le i\le s} \mid 0 \le n < b^m \Big\}. \tag{2.29}$$

Using (2.14) and (2.26), we get

$$\mathcal{C}_m = \{(y_{n,1}^{(1)}, ..., y_{n,m}^{(1)}, ..., y_{n,1}^{(s)}, ..., y_{n,m}^{(s)}) \mid 0 \le n < b^m\}. \tag{2.30}$$

Let

$$\mathcal{Y}_m = \{(y_n^{(*,1)}, ..., y_n^{(*,s)}) = (y_{n,m}^{(1)}, ..., y_{n,1}^{(1)}, ..., y_{n,m}^{(s)}, ..., y_{n,1}^{(s)}) \mid 0 \le n < b^m\}, \tag{2.31}$$

where $y_n^{(*,i)} := (y_{n,m}^{(i)}, ..., y_{n,2}^{(1)}, y_{n,1}^{(i)})$, $1 \leq i \leq s$.

Bearing in mind (2.27), (2.30) and (2.28), we get

$$\sum_{i=1}^{s} \sum_{r=0}^{m-1} \sum_{j=0}^{m-1} \tilde{c}_{m-j-1,r}^{(i)} \bar{a}_r(n_1) y_{n_2,m-j}^{(i)} = \sum_{i=1}^{s} \sum_{r=0}^{m-1} \sum_{j=0}^{m-1} \tilde{c}_{j,r}^{(i)} \bar{a}_r(n_1) y_{n_2,j+1}^{(i)} = 0, \ \ 0 \leq n_1, n_2 < b^m.$$

Now, from (2.27), (2.31) and (2.29), we derive that $\tilde{\mathcal{C}}_m^{(*)}$ is the dual space of $\mathcal{Y}_m$:

$$\tilde{\mathcal{C}}_m^{(*)\perp} = \mathcal{Y}_m.$$

Proposition B. *Let $C^{(1)}, ..., C^{(s)} \in \mathbb{F}_b^{\infty \times \infty}$ be generating matrices of a digital sequence $\mathbf{x}_n(C)_{n \geq 0}$ over $\mathbb{F}_b$. Then $\mathbf{x}_n(C)_{n \geq 0}$ is $\mathbf{D}-$*admissible *in base b if and only if for all $m \in \mathbb{N}$ the system $\{\tilde{\mathbf{c}}_j^{(*,i)} \in \mathbb{F}_b^m \mid 1 \leq j \leq m, \ 1 \leq i \leq s\}$ is a $(m(s-1) - D(m) + s, m, s)-$system over $\mathbb{F}_b$.*

Proof. Applying Lemma C, we get that the system $\{\tilde{\mathbf{c}}_j^{(*,i)} \in \mathbb{F}_b^m \mid 0 \leq j \leq m-1, \ 1 \leq i \leq s\}$ is a $(m(s-1) - D(m) + s, m, s)-$system over $\mathbb{F}_b$ if and only if the dual space $\tilde{\mathcal{C}}_m^{(*)\perp} = \mathcal{Y}_m$ of the row space $\tilde{\mathcal{C}}_m^{(*)}$ satisfies $\delta_m(\mathcal{Y}_m) \geq m(s-1) - D(m) + s + 1 =: \alpha_m$.

By Definition 9, we have

$$\delta_m(\mathcal{Y}_m) \geq \alpha_m \Leftrightarrow \sum_{i=1}^{s} v_m(\mathbf{b}_i) \geq \alpha_m \quad \text{for all} \quad (\mathbf{b}_1, ..., \mathbf{b}_s) \in \mathcal{Y}_m \setminus \{\mathbf{0}\}.$$

Using (2.31), we obtain

$$\delta_m(\mathcal{Y}_m) \geq \alpha_m \Leftrightarrow \sum_{i=1}^{s} v_m(y_n^{(*,i)}) \geq \alpha_m \quad \text{for all} \quad n \in \{1, ..., b^m - 1\}.$$

From (2.15), (2.20), (2.22), (2.31) and Definition 5, we derive

$$\log_b(\|x_n^{(i)}\|_b) = -v_m^{\perp}(y_n^{(i)}) = v_m(y_n^{(*,i)}) - m - 1, \quad 1 \leq i \leq s.$$

Therefore

$$\delta_m(\mathcal{Y}_m) \geq \alpha_m \Leftrightarrow \min_{1 \leq n < b^m} \sum_{i=1}^{s} (m + 1 - v_m^{\perp}(y_n^{(i)})) \geq \alpha_m \Leftrightarrow \min_{1 \leq n < b^m} \sum_{i=1}^{s} -v_m^{\perp}(y_n^{(i)})$$

$$= \min_{1 \leq n < b^m} \sum_{i=1}^{s} \log_b(\|\mathbf{x}_n\|_b) \geq \alpha_m - (m+1)s = -m - D(m) + 1.$$

Hence $\delta_m(\mathcal{Y}_m) \geq \alpha_m$ if and only if $\min_{1 \leq n < b^m} \|\mathbf{x}_n\|_b > b^{-m-D(m)}$.

By Definition 8, Proposition B is proved. □

We will also need the following assertion.

Proposition C. ([DiPi, Proposition 7.22] *For $s \in \mathbb{N}$, $s \geq 2$, the matrices $C^{(1)}, ..., C^{(s)}$ generate a digital $(\mathbf{T}, s)$-sequence if and only if for all $m \in \mathbb{N}$ we have*

$$\mathbf{T}(m) \geq m - \delta_m(C_m^{\perp}) + 1, \quad \text{for all} \quad m \in \mathbb{N}.$$

2.4 Admissible latices.

Let $\mathsf{k}(x) = \mathbb{F}_b(x)$ be the rational function field over $\mathbb{F}_b$, $\mathsf{k}[x] = \mathbb{F}_b[x]$ the polynomial ring over $\mathbb{F}_b$, and let $\mathsf{k}((x))$ be the perfect completion of k with respect to valuation (2.1).

A *lattice* Γ in $\mathsf{k}((x))^s$ is the image of $(\mathsf{k}[x])^s$ under an invertible $\mathsf{k}((x))$-linear mapping of the vector space $\mathsf{k}((x))^s$ into itself. The points of Γ will be called lattice points. We will consider only unimodular lattices.

Define the norm of a vector $\boldsymbol{\gamma} = (\gamma_1, ..., \gamma_s) \in \mathsf{k}((x))^s$ as $|\boldsymbol{\gamma}| := \max_{1 \leq i \leq s} |\gamma_i|$, where $|\gamma_i| = b^{-\nu_\infty(\gamma_i)}$ and ν_∞ is the discrete exponential valuation (2.1).

Now let $< y, z >$ be a standard inner product ($< y, z >= y_1 z_1 + ... + y_s z_s$ for $y = (y_1, ..., y_s)$ and $z = (z_1, ..., z_s)$).

The dual (or polar) lattice $\Gamma^\perp$ of a lattice Γ is defined by $\Gamma^\perp = \{\mathbf{x} \in \mathsf{k}((x))^s \mid < \mathbf{x}, \mathbf{y} >$ is a polynomial for all $\mathbf{y} \in \Gamma\}$.

First, we describe Mahler's variant of Minkowski's theorem on a convex body in a field of series for the following special case:

The first successive minimum λ_1 is defined as the norm of a nonzero shortest vector $\mathbf{b}_1$ of a lattice Γ in $\mathsf{k}((x))^s$. For $2 \leq i \leq s$, a ith successive minimum λ_i of Γ is recursively defined as the norm of a smallest vector $\mathbf{b}_i$ in Γ that is linearly independent of $\mathbf{b}_1, ..., \mathbf{b}_{i-1}$ over $\mathsf{k}((x))$.

As an immediate consequence, we get

$$0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_s.$$

We have a famous theorem due to Mahler (see [Ma], [Te2, p. 33]).

Theorem H. *Let $\lambda_1, ..., \lambda_s$ be the successive minima of a lattice Γ and let $\lambda_1^\perp, ..., \lambda_s^\perp$ be the successive minima of the dual lattice $\Gamma^\perp$. We then have*

$$\lambda_1 \lambda_2 ... \lambda_s = \lambda_1^\perp \lambda_2^\perp ... \lambda_s^\perp = 1, \qquad \lambda_j \lambda_{s-j+1}^\perp = 1 \quad \text{for} \quad 1 \leq j \leq s.$$

Hence $\lambda_1^{s-1} \lambda_s \leq 1$ and

$$\lambda_1 \leq \lambda_s^{-1/(s-1)}. \tag{2.32}$$

Definition 12. *A lattice $\Gamma \subset \mathsf{k}((x))^s$ is $d-$admissible if*

$$\mathrm{Nm}(\Gamma) = \inf_{\gamma \in \Gamma \setminus \{0\}} \mathrm{Nm}(\gamma) / \det(\Gamma) \geq b^{-d}, \quad \text{where} \quad \mathrm{Nm}(\gamma) = \prod_{1 \leq i \leq s} |\gamma_i|.$$

A lattice $\Gamma \subset \mathsf{k}((x))^s$ is said to be admissible if Γ is $d-$admissible with some real d.

Proposition D. *Let a lattice $\Gamma \subset \mathsf{k}((x))^s$ be $d-$admissible,* $\det(\Gamma) = 1$. *Then the dual lattice $\Gamma^\perp$ is $(d+1)(s-1)+2-$admissible.*

Proof. Suppose that there exists $\gamma^{\perp} = (\gamma_1^{\perp}, ..., \gamma_s^{\perp}) \in \Gamma^{\perp} \setminus \{0\}$ with $\mathrm{Nm}(\gamma^{\perp}) =$

b^{-a}, $\infty > a > c := (d+1)(s-1)+2$, $a = a_1 s + a_2$, $a_1 = [a/s]$ and $a_2 \in \{0, ..., s-1\}$. We have that $a_1 > (c-s-1)/s$. Consider the following unimodular diagonal matrix $U = \mathrm{diag}(u_1, ..., u_s)$, where $u_i = \gamma_i^{\perp} x^{a_1}$ for $1 \leq i < s$ and $u_s = \gamma_s^{\perp} x^{a_1+a_2}$.

Let $\dot{\gamma} := \gamma^{\perp} U^{-1} = (x^{-a_1}, ..., x^{-a_1}, x^{-a_1-a_2})$. Therefore $|\dot{\gamma}| \leq b^{-a_1} < b^{-(c-s-1)/s}$. It is easy that $\dot{\gamma} \in \Gamma^{\perp} U^{-1}$ and

$$\lambda_1^{\perp}(\Gamma^{\perp} U^{-1}) \leq |\dot{\gamma}| < b^{-(c-s-1)/s}. \tag{2.33}$$

Note that $(U\Gamma)^{\perp} = \Gamma^{\perp} U^{-1}$, $\mathrm{Nm}(\mathbf{y}) \leq |\mathbf{y}|^s$ for $\mathbf{y} \in \mathsf{k}((x))^s$, and

$$b^{-d} \leq \mathrm{Nm}(\Gamma) = \mathrm{Nm}(U\Gamma) \leq \inf_{\gamma \in U\Gamma \setminus \mathbf{0}} |\gamma|^s = (\lambda_1(U\Gamma))^s. \tag{2.34}$$

Using (2.32) and (2.33), we get

$$b^{-d/s} \leq \lambda_1(U\Gamma) \leq (\lambda_s(U\Gamma))^{-1/(s-1)} = (\lambda_1^{\perp}(\Gamma^{\perp} U^{-1}))^{1/(s-1)} < b^{-\frac{c-s-1}{(s-1)s}}. \tag{2.35}$$

Thus $-d/s < -(c-s-1)/(s^2-s)$ and

$$d > (c-s-1)/(s-1) = ((d+1)(s-1)+2-s-1)/(s-1) = d.$$

We have a contradiction.

Now suppose that there exists $\gamma^{\perp} \in \Gamma^{\perp} \setminus \{0\}$ with $\mathrm{Nm}(\gamma^{\perp}) = 0$. Let $\gamma_i^{\perp} \neq 0$ for $i \in J \subset \{1, ..., s\}$, $\gamma_i^{\perp} = 0$ for $i \in \bar{J} = \{1, ..., s\} \setminus J$, $a = \mathrm{card}(J) \in [1, s-1]$, $s \in \bar{J}$, and let $b^f := \prod_{i \in J} |\gamma_i^{\perp}|$.

Let $\dot{\gamma} := (\dot{\gamma}_1, ..., \dot{\gamma}_s)$ with $\dot{\gamma}_i = x^{-c}$ for $i \in J$ and $\dot{\gamma}_i = 0$ for $i \in \bar{J}$, where $c = 2d(s-a)$. Therefore $|\dot{\gamma}| = b^{-c}$.

Consider the following diagonal matrix $U = \mathrm{diag}(u_1, ..., u_s)$, where $u_i = \gamma_i^{\perp} x^c$ for $i \in J$, $u_i = x^{-c_1}$ for $i \in \bar{J} \setminus \{s\}$, and $u_s = x^{-c_1 - f}$, with $c_1 = 2ad$.

Note that $\log_b |\det(U)| = f + ac - (s-a)c_1 - f = 2ad(s-a) - 2(s-a)ad = 0$. Hence U is a unimodular matrix.

It is easy to see that $\dot{\gamma} = \gamma^{\perp} U^{-1} \in \Gamma^{\perp} U^{-1}$, and $\lambda_1^{\perp}(\Gamma^{\perp} U^{-1}) \leq |\dot{\gamma}| = b^{-c} < b^{-d}$.

By (2.34) and (2.35), we get

$$b^{-d/s} \leq \lambda_1(U\Gamma) \leq (\lambda_s(U\Gamma))^{-1/(s-1)} = (\lambda_1^{\perp}(\Gamma^{\perp} U^{-1}))^{1/(s-1)} \leq b^{-c/(s-1)} < b^{-d/s}.$$

We have a contradiction. Therefore Proposition D is proved. □

Remark 1. In [Le1, Theorem 3.2], we proved the following analog of the main theorem of the duality theory (see, [DiPi, Section 7], [NiPi] and [Skr]): if a unimodular lattice $\Gamma \mathsf{k}((x))^{s+1}$ is $d-$admissible, then from the dual lattice $\Gamma^{\perp}$

we can get a (t,s)-sequence $(\mathbf{x}_n)_{n\geq 0}$ with $t=d-s$. Using Definition 5, Definition 12, and Proposition D, we get that $(\mathbf{x}_n)_{n\geq 0}$ is $(d+1)s+2-$admissible. In [Le5] and in this paper we consider a more general object. We consider nets in $[0,1)^s$ having simultaneously both (t,m,s) properties and d-admissible properties. The d-admissible properties have a direct connection to the notion of the weight in the duality theory (see Definition 5, Definition 8 - Definition 11, Lemma C and Proposition B). Thus we can consider this paper as a part of the duality theory.

2.5 Auxiliary results.

Lemma D. ([Le4, Lemma 1]) *Let* $\dot{s}\geq 2$, $d\geq 1$, $(\mathbf{x}_n)_{0\leq n<b^{\tilde{m}}}$ *be a* $d-$*admissible* $(t,\tilde{m},\dot{s})$*-net in base* b, $d_0=d+t$, $\hat{e}\in\mathbb{N}$, $0<\epsilon\leq(2d_0\hat{e}(\dot{s}-1))^{-1}$, $\dot{m}=[\tilde{m}\epsilon]$, $\ddot{m}_i=0$, $\dot{m}_i=d_0\hat{e}\dot{m}$ $(1\leq i\leq\dot{s}-1)$, $\ddot{m}_{\dot{s}}=\tilde{m}-(\dot{s}-1)\dot{m}_1-t\geq 1$, $\dot{m}_{\dot{s}}=\ddot{m}_{\dot{s}}+\dot{m}_1$, $B_i\subset\{0,...,\dot{m}-1\}$ $(1\leq i\leq\dot{s})$, $\mathbf{w}\in E^{\dot{s}}_{\tilde{m}}$ *and let* $\gamma^{(i)}=\gamma^{(i)}_1/b+...+\gamma^{(i)}_{\dot{m}_i}/b^{\dot{m}_i}$,

$$\gamma^{(i)}_{\ddot{m}_i+d_0(\hat{j}_i\hat{e}+\check{j}_i)+\check{j}_i}=0 \text{ for } 1\leq\check{j}_i<d_0, \qquad \gamma^{(i)}_{\ddot{m}_i+d_0(\hat{j}_i\hat{e}+\check{j}_i)+\check{j}_i}=1 \text{ for } \check{j}_i=d_0 \tag{2.36}$$

and $\hat{j}_i\in\{0,...,\dot{m}-1\}\setminus B_i$, $0\leq\check{j}_i<\hat{e}$, $1\leq i\leq\dot{s}$, $\boldsymbol{\gamma}=(\gamma^{(1)},...,\gamma^{(\dot{s})})$, $B=\#B_1+...+\#B_{\dot{s}}$ *and* $\tilde{m}\geq 4\epsilon^{-1}(\dot{s}-1)(1+\dot{s}B)+2t$. *Let there exists* $n_0\in[0,b^{\tilde{m}})$ *such that* $[(\mathbf{x}_{n_0}\oplus\mathbf{w})^{(i)}]_{\dot{m}_i}=\gamma^{(i)}$, $1\leq i\leq\dot{s}$. *Then*

$$\Delta((\mathbf{x}_n\oplus\mathbf{w})_{0\leq n<b^{\tilde{m}}},J_{\boldsymbol{\gamma}})\leq -b^{-d}\big(\hat{e}\epsilon(2(\dot{s}-1))^{-1}\big)^{\dot{s}-1}\tilde{m}^{\dot{s}-1}+b^{t+s}d_0\hat{e}B\tilde{m}^{\dot{s}-2}. \tag{2.37}$$

Corollary 1. *With notations as above. Let* $\dot{s}\geq 3$, $\tilde{r}\geq 0$, $\tilde{m}=m-\tilde{r}$, $(\mathbf{x}_n)_{0\leq n<b^{\tilde{m}}}$ *be a* $d-$*admissible* $(t,\tilde{m},\dot{s})$*-net in base* b, $d_0=d+t$, $\hat{e}\in\mathbb{N}$, $\epsilon=\eta(2d_0\hat{e}(\dot{s}-1))^{-1}$, $0<\eta\leq 1$, $\dot{m}=[\tilde{m}\epsilon]$, $\ddot{m}_i=0$, $\dot{m}_i=d_0\hat{e}\dot{m}$, $\ddot{m}_{\dot{s}}=\tilde{m}-(\dot{s}-1)\dot{m}_1-t\geq 1$, $\dot{m}_{\dot{s}}=\ddot{m}_{\dot{s}}+\dot{m}_1$, $B_i\subset\{0,...,\dot{m}-1\}$, $\bar{B}_i=\{0,...,\dot{m}-1\}\setminus B_i$, $1\leq i\leq\dot{s}$, $B=\#B_1+...+\#B_{\dot{s}}$. *Suppose that*

$$\{(x^{(i)}_{n,\ddot{m}_i+d_0\hat{e}\hat{j}_i+\check{j}_i}\mid\hat{j}_i\in\bar{B}_i,\ \check{j}_i\in[1,d_0\hat{e}],\ i\in[1,\dot{s}])\mid n\in[0,b^m)\}=Z^{\mu}_b, \tag{2.38}$$

with $m\geq 2t+8(d+t)\hat{e}(\dot{s}-1)^2\eta^{-1}+2^{2\dot{s}}b^{d+\dot{s}+t}(d+t)^{\dot{s}}\hat{e}(\dot{s}-1)^{2(\dot{s}-1)}\eta^{-\dot{s}+1}B+4(\dot{s}-1)\tilde{r}$ *and* $\mu=d_0\hat{e}(\dot{s}\dot{m}-B)$. *Then there exists* $n_0\in[0,b^{\tilde{m}})$ *such that* $[(\mathbf{x}_{n_0}\oplus\mathbf{w})^{(i)}]_{\dot{m}_i}=\gamma^{(i)}$, $1\leq i\leq\dot{s}$, *and for each* $\mathbf{w}\in E^{\dot{s}}_{\tilde{m}}$, *we have*

$$b^{\tilde{m}}D^*((\mathbf{x}_n\oplus\mathbf{w})_{0\leq n<b^{\tilde{m}}})\geq\Big|\Delta((\mathbf{x}_n\oplus\mathbf{w})_{0\leq n<b^{\tilde{m}}},J_{\boldsymbol{\gamma}})\Big|\geq 2^{-2}b^{-d}K^{-\dot{s}+1}_{d,t,\dot{s}}\eta^{\dot{s}-1}m^{\dot{s}-1}$$

with $K_{d,t,\dot{s}}=4(d+t)(\dot{s}-1)^2$.

Proof. Let $\boldsymbol{\gamma}(n,\mathbf{w}) = \boldsymbol{\gamma} = (\gamma^{(1)},...,\gamma^{(\dot{s})})$ with $\gamma^{(i)} := [(\mathbf{x}_n \oplus \mathbf{w})^{(i)}]_{\dot{m}_i}$, $i \in [1,\dot{s}]$. Using (2.38), we get that there exists $n_0 \in [0, b^{\tilde{m}})$ such that $\boldsymbol{\gamma}(n_0,\mathbf{w})$ satisfy (2.36). Hence (2.37) is true. Taking into account (1.2) and that $\mathbf{w} \in E^{\dot{s}}_{\tilde{m}}$ is arbitrary, we get the assertion in Corollary 1. □

Let $\phi: \ Z_b \mapsto \mathbb{F}_b$ be a bijection with $\phi(0) = \bar{0}$, and let $x^{(i)}_{n,j} = \phi^{-1}(y^{(i)}_{n,j})$ for $1 \leq i \leq s$, $j \geq 1$ and $n \geq 0$. We obtain from Corollary 1 :

Corollary 2. *Let $\dot{s} \geq 3$, $\tilde{r} \geq 0$, $\tilde{m} = m - \tilde{r}$, $(\mathbf{x}_n)_{0 \leq n < b^{\tilde{m}}}$ be a $d-$admissible $(t,\tilde{m},\dot{s})$-net in base b, $d_0 = d+t$, $\hat{e} \in \mathbb{N}$, $\epsilon = \eta(2d_0\hat{e}(\dot{s}-1))^{-1}$, $0 < \eta \leq 1$, $\dot{m} = [\tilde{m}\epsilon]$, $\ddot{m}_i = 0$, $\dot{m}_i = d_0\hat{e}\dot{m}$, $\ddot{m}_{\dot{s}} = \tilde{m} - (\dot{s}-1)\dot{m}_1 - t \geq 1$, $\dot{m}_{\dot{s}} = \ddot{m}_{\dot{s}} + \dot{m}_1$, $B_i \subset \{0,...,\dot{m}-1\}$, $\bar{B}_i = \{0,...,\dot{m}-1\} \setminus B_i$, $1 \leq i \leq \dot{s}$, $B = \#B_1 + ... + \#B_{\dot{s}}$. Suppose that*

$$\{(y^{(i)}_{n,\ddot{m}_i + d_0\hat{e}\hat{j}_i + \check{j}_i} \mid \hat{j}_i \in \bar{B}_i,\ \check{j}_i \in [1,d_0\hat{e}],\ i \in [1,\dot{s}]) \mid n \in [0,b^m)\} = \mathbb{F}^{\mu}_b,$$

with $m \geq 2t + 8(d+t)\hat{e}(\dot{s}-1)^2\eta^{-1} + 2^{2\dot{s}}b^{d+\dot{s}+t}(d+t)^{\dot{s}}\hat{e}(\dot{s}-1)^{2(\dot{s}-1)}\eta^{-\dot{s}+1}B + 4(\dot{s}-1)\tilde{r}$ and $\mu = d_0\hat{e}(\dot{s}\dot{m} - B)$. Then there exists $n_0 \in [0,b^{\tilde{m}})$ such that $[(\mathbf{x}_{n_0} \oplus \mathbf{w})^{(i)}]_{\dot{m}_i} = \gamma^{(i)}$, $1 \leq i \leq \dot{s}$, and for each $\mathbf{w} \in E^{\dot{s}}_{\tilde{m}}$, we have

$$b^{\tilde{m}} D^*((\mathbf{x}_n \oplus \mathbf{w})_{0 \leq n < b^{\tilde{m}}}) \geq \Big|\Delta((\mathbf{x}_n \oplus \mathbf{w})_{0 \leq n < b^{\tilde{m}}}, J_{\boldsymbol{\gamma}})\Big| \geq 2^{-2}b^{-d}K^{-\dot{s}+1}_{d,t,\dot{s}}\eta^{\dot{s}-1}m^{\dot{s}-1}.$$

With notations as above, we consider the case of (t,s)-sequence in base b:

Corollary 3. *Let $s \geq 2$, $d \geq 1$, $(\mathbf{x}_n)_{n \geq 0}$ be a $d-$admissible (t,s) sequence in base b, $d_0 = d+t$, $\hat{e} \in \mathbb{N}$, $\epsilon = \eta(2d_0\hat{e}s)^{-1}$, $0 < \eta \leq 1$, $\dot{m} = [m\epsilon]$, $\ddot{m}_i = 0$, $1 \leq i \leq s$, $\ddot{m}_{s+1} = t - 1 + (s-1)d_0\hat{e}\dot{m}$, $B^{'}_i \subset \{0,...,\dot{m}-1\}$, $\bar{B}^{'}_i = \{0,...,\dot{m}-1\} \setminus B^{'}_i$, $1 \leq i \leq s+1$, $B = \#B^{'}_1 + ... + \#B^{'}_{s+1}$. Suppose that*

$$\{(y^{(i)}_{n,\ddot{m}_i + d_0\hat{e}\hat{j}_i + \check{j}_i} \mid \hat{j}_i \in \bar{B}^{'}_i,\ \check{j}_i \in [1,d_0\hat{e}],\ i \in [1,s],$$
$$\bar{a}_{\ddot{m}_{s+1} + d_0\hat{e}\tilde{j}_{s+1} + \check{j}_{s+1}}(n),\ \tilde{j}_{s+1} \in \bar{B}^{'}_{s+1},\ \check{j}_{s+1} \in [1,d_0\hat{e}],) \mid n \in [0,b^m)\} = \mathbb{F}^{\mu}_b.$$

with $\mu = d_0\hat{e}((s+1)\dot{m} - B)$, and $m \geq 2t + 8(d+t)\hat{e}s^2\eta^{-1} + 2^{2s+2}b^{d+s+t+1}(d+t)^{s+1}\hat{e}s^{2s}\eta^{-s}B$. Then

$$1 + \min_{0 \leq Q < b^m} \min_{\mathbf{w} \in E^s_m} \max_{1 \leq N \leq b^m} N D^*((\mathbf{x}_{n \oplus Q} \oplus \mathbf{w})_{0 \leq n < N}) \geq 2^{-2}b^{-d}K^{-s}_{d,t,s+1}\eta^s m^s.$$

Proof. Using Lemma B, we have

$$1+\sup_{1\leq N\leq b^m} ND^*((\mathbf{x}_{n\oplus Q}\oplus\mathbf{w})_{0\leq n<N})\geq b^m D^*((\mathbf{x}_{n\oplus Q}\oplus\mathbf{w},n/b^m)_{0\leq n<b^m})$$
$$=b^m D^*((\mathbf{x}_n\oplus\mathbf{w},(n\ominus Q)/b^m)_{0\leq n<b^m}).$$

By (1.4) and [DiPi, Lemma 4.38], we have that $((\mathbf{x}_n,n/b^m)_{0\leq n<b^m})$ is a $d-$admissible $(t,m,s+1)-$net in base b. We apply Corollary 2 with $\dot{s}=s+1$, $\tilde{r}=0$, $B_i^{'}=B_i$, $1\leq i<\dot{s}$, $B_{\dot{s}}^{'}=\{\dot{m}-j-1|j\in B_{\dot{s}}\}$, $\hat{j}_{s+1}=\dot{m}-\tilde{j}_{s+1}-1$, $\check{j}_{s+1}=d_0\hat{e}-\check{j}_{s+1}+1$, and $x_n^{(s+1)}=n/b^m$. Taking into account that $y_{n,m-j}^{(s+1)}=\bar{a}_j(n)$ $(0\leq j<m)$, we get $y^{(s+1)}_{n,m-\ddot{m}_{s+1}-d_0\hat{e}\dot{m}-1+d_0\hat{e}\hat{j}_{s+1}+\check{j}_{s+1}}=\bar{a}_{\ddot{m}_{s+1}+d_0\hat{e}\tilde{j}_{s+1}+\check{j}_{s+1}}(n)$, and Corollary 3 follows. □

Lemma 2. *Let $\dot{s}\geq 2$, $d_0\geq 1$, $\hat{e}\geq 1$, $\dot{m}\geq 1$, $\dot{m}_1=d_0\hat{e}\dot{m}$, $\ddot{m}_i\in[0,m-\dot{m}_1]$ $(1\leq i\leq\dot{s})$, $m\geq\dot{s}\dot{m}_1$, $\dot{m}\geq r$, and let*

$$\Phi:=\{(y^{(1)}_{n,\ddot{m}_1+1},...,y^{(1)}_{n,\ddot{m}_1+\dot{m}_1},...,y^{(\dot{s})}_{n,\ddot{m}_{\dot{s}}+1},...,y^{(\dot{s})}_{n,\ddot{m}_{\dot{s}}+\dot{m}_1})|n\in[0,b^m)\}\subseteq\mathbb{F}_b^{\dot{s}\dot{m}_1}. \tag{2.39}$$

Suppose that Φ is a $\mathbb{F}_b$ linear subspace of $\mathbb{F}_b^{\dot{s}\dot{m}_1}$ and $\dim_{\mathbb{F}_b}(\Phi)=\dot{s}\dot{m}_1-r$. Then there exists $B_i\in\{0,...,\dot{m}-1\}$, $1\leq i\leq\dot{s}$, with $B=\#B_1+...+\#B_{\dot{s}}\leq r$ and

$$\Psi=\mathbb{F}_b^{d_0\hat{e}(\dot{s}\dot{m}-B)}, \tag{2.40}$$

where

$$\Psi=\{(y^{(i)}_{n,\ddot{m}_i+d_0\hat{e}(\dot{j}_i-1)+\ddot{j}_i}\mid\dot{j}_i\in\bar{B}_i,\ \ddot{j}_i\in[1,d_0\hat{e}],\ i\in[1,\dot{s}])\mid n\in[0,b^m)\} \tag{2.41}$$

with $\bar{B}_i=\{0,...,\dot{m}-1\}\setminus B_i$.

Proof. Let $\hat{r}=\dot{s}\dot{m}_1-r$, and let $\mathfrak{f}_1,...,\mathfrak{f}_{\hat{r}}$ be a basis of Φ with

$$\mathfrak{f}_\mu=(f^{(1)}_{\mu,\ddot{m}_1+1},...,f^{(1)}_{\mu,\ddot{m}_1+\dot{m}_1},...,f^{(s)}_{\mu,\ddot{m}_s+1},...,f^{(s)}_{\mu,\ddot{m}_s+\dot{m}_1}),\ 1\leq\mu\leq\hat{r}.$$

Let

$$v(\mathfrak{f}_\mu)=\max\{\ddot{m}_i+(i-1)\dot{m}_1+j\mid f^{(i)}_{\mu,\ddot{m}_i+j}\neq 0,\ j\in[1,\dot{m}_1],i\in[1,\dot{s}]\}\ \text{for}\ \ \mu\in[1,\hat{r}].$$

Without loss of generality, assume now that $v(\mathfrak{f}_i)\leq v(\mathfrak{f}_j)$ for $1\leq i<j\leq\hat{r}$. Let $v(\mathfrak{f}_j)=\ddot{m}_{l_1}+(l_1-1)\dot{m}_1+l_2$, and let $\dot{\mathfrak{f}}_k=\mathfrak{f}_k-\mathfrak{f}_j f^{(l_1)}_{k,\ddot{m}_{l_1}+l_2}/f^{(l_1)}_{j,\ddot{m}_{l_1}+l_2}$ for $1\leq k\leq j-1$.
We have $v(\dot{\mathfrak{f}}_k)<v(\mathfrak{f}_j)$ for all $1\leq k\leq j-1$.

By repeating this procedure for $j=\hat{r},\hat{r}-1,...,2$, we obtain a basis $\hat{\mathfrak{f}}_1,...,\hat{\mathfrak{f}}_{\hat{r}}$ of Φ with $v(\hat{\mathfrak{f}}_i)<v(\hat{\mathfrak{f}}_j)$ for $1\leq i<j\leq\hat{r}$. Let

$$A_i=\{\ddot{m}_i+j\mid v(\hat{\mathfrak{f}}_\mu)=(i-1)\dot{m}_1+\ddot{m}_i+j,\ 1\leq j\leq\dot{m}_1,1\leq\mu\leq\hat{r}\},i\in[1,\dot{s}].$$

Taking into account that $\hat{\mathfrak{f}}_1,...,\hat{\mathfrak{f}}_{\hat{r}}$ is a basis of Φ, we get from (2.39)

$$\{(y_{n,j}^{(i)} \mid j \in A_i,\ i \in [1,\dot{s}]) \mid n \in [0,b^m)\} = \mathbb{F}_b^{\dot{s}\dot{m}_1 - r}. \tag{2.42}$$

Now let

$$\bar{B}_i := \{\dot{j}_i \in [0,\dot{m}_1) \mid \exists \ddot{j}_i \in [1,d_0\hat{e}], \text{ with } \ddot{m}_i + \dot{j}_i d_0 \hat{e} + \ddot{j}_i \in A_i)\},\ i \in [1,\dot{s}].$$

It is easy to see that $B = \#B_1 + ... + \#B_{\dot{s}} \leq r$, where $\bar{B}_i = \{0,...,\dot{m}-1\} \setminus B_i$.

Bearing in mind (2.41), we obtain (2.40) from (2.42). Hence Lemma 2 is proved. □

3. Statements of results.

If $s = 2$ for the case of nets, or $s = 1$ for the case of sequences, then (1.5) follows from the W. Schmidt estimate (1.3) (see [Ni, p.24]). In this paper we take $s \geq 2$ for the case of sequences, and $s \geq 3$ for the case of nets.

3.1 Generalized Niederreiter sequence. In this subsection, we introduce a generalization of the Niederreiter sequence due to Tezuka (see [Te2, Section 6.1.2], [DiPi, Section 8.1.2]). By [Te2, p.165], the Sobol's sequence [DiPi, Section 8.1.2], the Faure's sequence [DiPi, Section 8.1.2]) and the original Niederreiter sequence [DiPi, Section 8.1.2]) are particular cases of a generalized Niederreiter sequence.

Let b be a prime power and let $p_1,...,p_s \in F_b[x]$ be pairwise coprime polynomials over $\mathbb{F}_b$. Let $e_i = \deg(p_i) \geq 1$ for $1 \leq i \leq s$. For each $j \geq 1$ and $1 \leq i \leq s$, the set of polynomials $\{y_{i,j,k}(x)\ :\ 0 \leq k < e_i\}$ needs to be linearly independent $(\bmod\ p_i(x))$ over $\mathbb{F}_b$. For integers $1 \leq i \leq s$, $j \geq 1$ and $0 \leq k < e_i$, consider the expansions

$$\frac{y_{i,j,k}(x)}{p_i(x)^j} = \sum_{r \geq 0} a^{(i)}(j,k,r) x^{-r-1} \tag{3.1}$$

over the field of formal Laurent series $F_b((x^{-1}))$. Then we define the matrix $C^{(i)} = (c_{j,r}^{(i)})_{j \geq 1, r \geq 0}$ by

$$c_{j,r}^{(i)} = a^{(i)}(Q+1,k,r) \in \mathbb{F}_b \qquad \text{for} \qquad 1 \leq i \leq s,\ j \geq 1,\ r \geq 0,$$

where $j-1 = Qe_i + k$ with integers $Q = Q(i,j)$ and $k = k(i,j)$ satisfying $0 \leq k < e_i$.

A digital sequence $(\mathbf{x}_n)_{n \geq 0}$ over $\mathbb{F}_b$ generated by the matrices $C^{(1)},...,C^{(s)}$ is called a generalized Niederreiter sequence (see [DiPi, p.266]).

Theorem I. (see [DiPi, p.266]) *The generalized Niederreiter sequence with generating matrices, defined as above, is a digital (t, s)-sequence over $\mathbb{F}_b$ with $t = e_0 - s$ and*

$e_0 = e_1 + ... + e_s$.

Theorem 1. *With the notations as above, $(\mathbf{x}_n)_{n\geq 0}$ is $d-$admissible with $d = e_0$.*
(a) For $s \geq 2$, $e = e_1 e_2 \cdots e_s$, $\eta_1 = s/(s+1)$ $m \geq 9(d+t)es(s+1)$ and $K_{d,t,s} = 4(d+t)(s-1)^2$, we have

$$1 + \min_{0\leq Q<b^m} \min_{\mathbf{w}\in E_m^s} \max_{1\leq N\leq b^m} ND^*((\mathbf{x}_{n\oplus Q} \oplus \mathbf{w})_{0\leq n<N}) \geq 2^{-2} b^{-d} K_{d,t,s+1}^{-s} \eta_1^s m^s.$$

(b) Let $s \geq 3$, $\eta_2 \in (0,1)$ and $m \geq 8(d+t)e(s-1)^2\eta_2^{-1} + 2(1+t)\eta_2^{-1}(1-\eta_2)^{-1}$. Suppose that $\min_{m/2-t\leq je_{i_0}\leq m, 0\leq k<e_{i_0}} (1 - \deg(y_{i_0,j,k}(x))j^{-1}e_{i_0}^{-1}) \geq \eta_2$ for some $i_0 \in [1,s]$. Then

$$\min_{\mathbf{w}\in E_m^s} b^m D^*((\mathbf{x}_n \oplus \mathbf{w})_{0\leq n<b^m}) \geq 2^{-2} b^{-d} K_{d,t,s}^{-s+1} \eta_2^{s-1} m^{s-1}.$$

3.2 Xing-Niederreiter sequence (see [DiPi, Section 8.4]). Let $F/\mathbb{F}_b$ be an algebraic function field with full constant field $\mathbb{F}_b$ and genus $g = g(F/\mathbb{F}_b)$. Assume that $F/\mathbb{F}_b$ has at least one rational place P_∞, and let G be a positive divisor of $F/\mathbb{F}_b$ with $\deg(G) = 2g$ and $P_\infty \notin \mathrm{supp}(G)$. Let $P_1, ..., P_s$ be s distinct places of $F/\mathbb{F}_b$ with $P_i \neq P_\infty$ for $1 \leq i \leq s$. Put $e_i = \deg(P_i)$ for $1 \leq i \leq s$.

By [DiPi, p.279], we have that there exists a basis $w_0, w_1, ..., w_g$ of $\mathcal{L}(G)$ over $\mathbb{F}_b$ such that

$$\nu_{P_\infty}(w_u) = n_u \quad \text{for} \quad 0 \leq u \leq g,$$

where $0 = n_0 < n_1 < .... < n_g \leq 2g$. For each $1 \leq i \leq s$, we consider the chain

$$\mathcal{L}(G) \subset \mathcal{L}(G + P_i) \subset \mathcal{L}(G + 2P_i) \subset ...$$

of vector spaces over $\mathbb{F}_b$. By starting from the basis $w_0, w_1, ..., w_g$ of $\mathcal{L}(G)$ and successively adding basis vectors at each step of the chain, we obtain for each $n \in \mathbb{N}$ a basis

$$\{w_0, w_1, ..., w_g, k_{i,1}, k_{i,2}, ..., k_{i,ne_i}\} \tag{3.2}$$

of $\mathcal{L}(G + nP_i)$. We note that we then have

$$k_{i,j} \in \mathcal{L}(G + ([(j-1)/e_i + 1)]P_i) \quad \text{for} \quad 1 \leq i \leq s \quad \text{and} \quad j \geq 1. \tag{3.3}$$

By the Riemann-Roch theorem, there exists a local parameter z at P_∞, e.g., with

$$\deg((z)_\infty) \leq 2g + e_1 \qquad \text{for} \qquad z \in \mathcal{L}(G + P_1 - P_\infty) \setminus \mathcal{L}(G + P_1 - 2P_\infty). \tag{3.4}$$

For $r \in \mathbb{N} \cup \{0\}$, we put

$$z_r = \begin{cases} z^r & \text{if } r \notin \{n_0, n_1, ..., n_g\}, \\ w_u & \text{if } r = n_u \text{ for some } u \in \{0, 1, ..., g\}. \end{cases} \tag{3.5}$$

Note that in this case $\nu_{P_\infty}(z_r) = r$ for all $r \in \mathbb{N} \cup \{0\}$. For $1 \le i \le s$ and $j \in \mathbb{N}$, we have $k_{i,j} \in \mathcal{L}(G + nP_i)$ for some $n \in \mathbb{N}$ and also $P_\infty \notin \operatorname{supp}(G + nP_i)$, hence $\nu_{P_\infty}(k_j^{(i)}) \ge 0$. Thus we have the local expansions

$$k_{i,j} = \sum_{r=0}^{\infty} a_{j,r}^{(i)} z_r \quad \text{for } 1 \le i \le s \quad \text{and } j \in \mathbb{N}, \tag{3.6}$$

where all coefficients $a_{j,r}^{(i)} \in \mathbb{F}_b$. For $1 \le i \le s$ and $j \in \mathbb{N}$, we now define the sequences

$$\begin{aligned} \mathbf{c}_j^{(i)} &= (c_{j,0}^{(i)}, c_{j,1}^{(i)}, ...) := (a_{j,n}^{(i)})_{n \in \mathbb{N}_0 \setminus \{n_0,...,n_g\}} \\ &= (\widehat{a_{j,n_0}^{(i)}}, a_{j,n_0+1}^{(i)}, ..., \widehat{a_{j,n_1}^{(i)}}, a_{j,n_1+1}^{(i)}, ..., \widehat{a_{j,n_g}^{(i)}}, a_{j,n_g+1}^{(i)}, ...) \in \mathbb{F}_b^{\mathbb{N}}, \end{aligned} \tag{3.7}$$

where the hat indicates that the corresponding term is deleted. We define the matrices $C^{(1)}, ..., C^{(s)} \in \mathbb{F}_b^{\mathbb{N}\times\mathbb{N}}$ by

$$C^{(i)} = (\mathbf{c}_1^{(i)}, \mathbf{c}_2^{(i)}, \mathbf{c}_3^{(i)}, ...)^\top \quad \text{for} \quad 1 \le i \le s, \tag{3.8}$$

i.e., the vector $\mathbf{c}_j^{(i)}$ is the jth row vector of $C^{(i)}$ for $1 \le i \le s$.

Theorem J (see [DiPi, Theorem 8.11]). *With the above notations, we have that the matrices $C^{(1)}, ..., C^{(s)}$ given by (3.8) are generating matrices of the Xing-Niederreiter (t,s)-sequence $(\mathbf{x}_n)_{n\ge0}$ with $t = g + e_0 - s$ and $e_0 = e_1 + ... + e_s$.*

Theorem 2. *With the above notations, $(\mathbf{x}_n)_{n\ge0}$ is $d-$admissible, where $d = g + e_0$.*
(a) For $s \ge 2$, $e = e_1...e_s$, $m \ge 9(d+t)es^2\eta_1^{-1}$ and $K_{d,t,s} = 4(d+t)(s-1)^2$, we have

$$1 + \min_{0\le Q<b^m} \min_{\mathbf{w}\in E_m^s} \max_{1\le N\le b^m} N\mathrm{D}^*((\mathbf{x}_{n\oplus Q} \oplus \mathbf{w})_{0\le n<N}) \ge 2^{-2}b^{-d}K_{d,t,s+1}^{-s}\eta_1^s m^s$$

with $\eta_1 = (1 + \deg((z)_\infty))^{-1}$ (see (3.4)).
(b) Let $s \ge 3$, $\eta_2 \in (0,1)$ and $m \ge 8(d+t)e(s-1)^2\eta_2^{-1} + 2(1 + 2g + \eta_2 t)\eta_2^{-1}(1 - \eta_2)^{-1}$. Suppose that $\min_{m/2-t\le j\le m} \nu_{P_\infty}(k_{i_0,j})/j \ge \eta_2$, for some $i_0 \in [1,s]$. Then

$$\min_{\mathbf{w}\in E_m^s} b^m D^*((\mathbf{x}_n \oplus \mathbf{w})_{0\le n<b^m}) \ge 2^{-2}b^{-d}K_{d,t,s}^{-s+1}\eta_2^{s-1}m^{s-1}. \tag{3.9}$$

3.3 Niederreiter-Özbudak nets (see [DiPi, Section 8.2]). Let $F/\mathbb{F}_b$ be an algebraic function field with full constant field $\mathbb{F}_b$ and genus $g = g(F/\mathbb{F}_b)$. Let $s \ge 2$, and let $P_1, ..., P_s$ be s distinct places of F with degrees $e_1, ..., e_s$. For $1 \le i \le s$, let ν_{P_i} be the normalized discrete valuation of F corresponding to P_i, let t_i be a local parameter at P_i. Further, for each $1 \le i \le s$, let F_{P_i} be the residue class field of P_i, i.e., $F_{P_i} = O_{P_i}/P_i$, and let $\vartheta_i = (\vartheta_{i,1}, ..., \vartheta_{i,e_i}) : F_{P_i} \to \mathbb{F}_b^{e_i}$ be an $\mathbb{F}_b$-linear vector space isomorphism. Let $m > g + \sum_{i=1}^s (e_i - 1)$. Choose an arbitrary divisor G of $F/\mathbb{F}_b$ with $\deg(G) = ms - m + g - 1$ and define $a_i := \nu_{P_i}(G)$ for $1 \le i \le s$. For each $1 \le i \le s$, we define an F_b-linear map $\theta_i \ : \ \mathcal{L}(G) \to \mathbb{F}_b^m$ on the Riemann-Roch space $\mathcal{L}(G) = \{y \in F \setminus 0 : \operatorname{div}(y) + G \ge 0\} \cup \{0\}$. We fix i and repeat the following definitions related to θ_i for each $1 \le i \le s$.

Note that for each $f \in \mathcal{L}(G)$ we have $\nu_{P_i}(f) \ge -a_i$, and so the local expansion of f at P_i has the form

$$f = \sum_{j=-a_i}^{\infty} S_j(t_i, f) t_i^j, \quad \text{with} \quad S_j(t_i, f) \in F_{P_i}, \ j \ge -a_i. \tag{3.10}$$

We denote $S_j(t_i, f)$ by $f_{i,j}$. Let $m_i = [m/e_i]$ and $r_i = m - e_i m_i$. Note that $0 \le r_i < e_i$. For $f \in \mathcal{L}(G)$, the image of f under $\theta_i^{(G)}$, for $1 \le i \le s$, is defined as

$$\theta_i^{(G)}(f) = (\theta_{i,1}(f), ..., \theta_{i,m}(f)) := (\mathbf{0}_{r_i}, \vartheta_i(f_{i,-a_i+m_i-1}), ..., \vartheta_i(f_{i,-a_i})) \in \mathbb{F}_b^m, \tag{3.11}$$

where we add the r_i-dimensional zero vector $\mathbf{0}_{r_i} = (0, ..., 0) \in \mathbb{F}_b^{r_i}$ in the beginning. Now we set

$$\theta^{(G)}(f) := (\theta_1^{(G)}(f), ..., \theta_s^{(G)}(f)) \in \mathbb{F}_b^{ms}, \tag{3.12}$$

and define the $\mathbb{F}_b$-linear map

$$\theta^{(G)} \ : \ \mathcal{L}(G) \to \mathbb{F}_b^{ms}, \qquad f \ \mapsto \ \theta^{(G)}(f).$$

The image of $\theta^{(G)}$ is denoted by

$$\mathcal{N}_m = \mathcal{N}_m(P_1, ..., P_s; G) := \{\theta^{(G)}(f) \in \mathbb{F}_b^{ms} \mid f \in \mathcal{L}(G)\}. \tag{3.13}$$

According to [DiPi, p.274],

$$\dim(\mathcal{N}_m) = \dim(\mathcal{L}(G)) \ge \deg(G) + 1 - g = ms - m \quad \text{for} \quad m > g - s + e_1 + ... + e_s.$$

Using the Riemann-Roch theorem, we get

$$\dim(\mathcal{N}_m) = ms - m \quad \text{for} \quad m > g - s + e_1 + ... + e_s, \ s \ge 3. \tag{3.14}$$

Let $\mathcal{N}_m^\perp = \mathcal{N}_m^\perp(P_1, ..., P_s; G)$ be the dual space of $\mathcal{N}_m(P_1, ..., P_s; G)$ (see (2.27)). The space $\mathcal{N}_m^\perp$ can be viewed as the row space of a suitable $m \times ms$ matrix C over $\mathbb{F}_b$. Finally, we consider the digital net $\mathcal{P}_1(\mathcal{N}_m^\perp) = \{\mathbf{x}_n(C) | n \in [0, b^m)\}$ with overall generating matrix C (see (2.25)).

Let $\tilde{x}_i(h_i) = \sum_{j=1}^m \phi^{-1}(h_{i,j}) b^{-j}$, where $h_i = (h_{i,1}, ..., h_{i,m}) \in F_b^m$ $(i = 1, ..., s)$ and let $\tilde{\mathbf{x}}(\mathbf{h}) = (\tilde{x}_1(h_1), ..., \tilde{x}_s(h_s))$ where $\mathbf{h} = (h_1, ..., h_s)$. From (2.15), (2.16) and (2.26), we derive

$$\mathcal{P}_1 := \mathcal{P}_1(\mathcal{N}_m^\perp) = \{\tilde{\mathbf{x}}(\mathbf{h}) \mid \mathbf{h} \in \mathcal{N}_m^\perp(P_1, ..., P_s; G)\}. \tag{3.15}$$

Theorem K (see [DiPi, Corollary 8.6]).*With the above notations, we have that $\mathcal{P}_1$ is a (t,m,s)-net over $\mathbb{F}_b$ with $t = g + e_0 - s$ and $e_0 = e_1 + ... + e_s$.*

To obtain a $d-$admissible net, we will consider also the following net:

$$\mathcal{P}_2 := \{(\{b^{r_1}z_1\},...,\{b^{r_s}z_s\}) \mid \mathbf{z} = (z_1,...,z_s) \in \mathcal{P}_1\}. \tag{3.16}$$

Without loss of generality, let

$$e_s = \min_{1\leq i\leq s} e_i. \tag{3.17}$$

Theorem 3. *Let $s \geq 3$, $m_0 = 2^{2s+3}b^{d+t+s}(d+t)^s(s-1)^{2s-1}(g+e_0)e\eta^{-s+1}$ and $\eta = (1+\deg((t_s)_\infty))^{-1}$. Then*

$$\min_{\mathbf{w}\in E_m^s}\max_{1\leq N\leq b^m} N\mathrm{D}^*(\mathcal{P}_1 \oplus \mathbf{w}) \geq 2^{-2}b^{-d}K_{d,t,s}^{-s+1}\eta^{-s+1}m^{s-1}, \quad \text{for} \quad m \geq m_0,$$

$\mathcal{P}_2$ is a $d-$admissible $(t, m-r_0, s)$-net in base b with $d = g + e_0$, $t = g + e_0 - s$, and

$$\min_{\mathbf{w}\in E_{m-r_0}^s} b^m\mathrm{D}^*((\mathcal{P}_2 \oplus \mathbf{w})) \geq 2^{-2}b^{-d}K_{d,t,s}^{-s+1}\eta^{s-1}m^{-s+1}, \quad \text{for} \quad m \geq m_0,$$

where $\mathcal{P}_i \oplus \mathbf{w} := \{\mathbf{z}\oplus\mathbf{w} \mid \mathbf{z}\in\mathcal{P}_i\}$.

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3.4 Halton-type sequence (see [NiYe]). Let $F/\mathbb{F}_b$ be an algebraic function field with full constant field $\mathbb{F}_b$ and genus $g = g(F/\mathbb{F}_b)$. We assume that $F/\mathbb{F}_b$ has at least one rational place, that is, a place of degree 1. Given a dimension $s \geq 1$, we choose $s+1$ distinct places P_1,...,P_{s+1} of F with $\deg(P_{s+1}) = 1$. The degrees of the places P_1,...,P_s are arbitrary and we put $e_i = \deg(P_i)$ for $1 \leq i \leq s$. Denote by O_F the holomorphy ring given by

$$O_F = \bigcap_{P\neq P_{s+1}} O_P,$$

where the intersection is extended over all places $P \neq P_{s+1}$ of F, and O_P is the valuation ring of P. We arrange the elements of O_F into a sequence by using the fact that

$$O_F = \bigcup_{m=0}^{\infty} \mathcal{L}(mP_{s+1}).$$

The terms of this sequence are denoted by $f_0, f_1, ...$ and they are obtained as follows. Consider the chain

$$\mathcal{L}(0) \subseteq L(P_{s+1}) \subseteq L(2P_{s+1}) \subseteq \cdots$$

of vector spaces over $\mathbb{F}_b$. At each step of this chain, the dimension either remains the same or increases by 1. From a certain point on, the dimension always increases by 1 according to the Riemann-Roch theorem. Thus we can construct a sequence $v_0, v_1, \ldots$ of elements of O_F such that

$$\{v_0, v_1, \ldots, v_{\ell(mP_{s+1})-1}\} \tag{3.18}$$

is a $\mathbb{F}_b$-basis of $\mathcal{L}(mP_{s+1})$. For $n \in \mathbb{N}$, let

$$n = \sum_{r=0}^{\infty} a_r(n) b^r \quad \text{with all } \ a_r(n) \in Z_b$$

be the digit expansion of n in base b. Note that $a_r(n) = 0$ for all sufficiently large r. We fix a bijection $\phi : Z_b \to \mathbb{F}_b$ with $\phi(0) = \bar{0}$. Then we define

$$f_n = \sum_{r=0}^{\infty} \bar{a}_r(n) v_r \in O_F \quad \text{with} \quad \bar{a}_r(n) = \phi(a_r(n)) \quad \text{for } \ n = 0, 1, \ldots . \tag{3.19}$$

Note that the sum above is finite since for each $n \in \mathbb{N}$ we have $a_r(n) = 0$ for all sufficiently large r. By the Riemann-Roch theorem, we have

$$\{\tilde{f} \mid \tilde{f} \in \mathcal{L}((m+g-1)P_{s+1})\} = \{f_n \mid n \in [0, b^m)\} \quad \text{for} \quad m \geq g. \tag{3.20}$$

For each $i = 1, \ldots, s$, let $\wp_i$ be the maximal ideal of O_F corresponding to P_i. Then the residue class field $F_{P_i} := O_F/\wp_i$ has order b^{e_i} (see [St, Proposition 3.2.9]). We fix a bijection

$$\sigma_{P_i} : F_{P_i} \to Z_{b^{e_i}}. \tag{3.21}$$

For each $i = 1, \ldots, s$, we can obtain a local parameter $t_i \in O_F$ at $\wp_i$, by applying the Riemann-Roch theorem and choosing

$$t_i \in \mathcal{L}(kP_{s+1} - P_i) \setminus \mathcal{L}(kP_{s+1} - 2P_i) \tag{3.22}$$

for a suitably large integer k. We have a local expansion of f_n at $\wp_i$ of the form

$$f_n = \sum_{j \geq 0} f_{n,j}^{(i)} t_i^j \quad \text{with all } \ f_{n,j}^{(i)} \in F_{P_i}, \ n = 0, 1, \ldots . \tag{3.23}$$

We define the map $\xi : O_F \to [0,1]^s$ by

$$\xi(f_n) = \Big(\sum_{j=0}^{\infty} \sigma_{P_1}(f_{n,j}^{(1)}) b^{-e_1(j+1)}, \ldots, \sum_{j=0}^{\infty} \sigma_{P_s}(f_{n,j}^{(s)}) (b^{-e_s(j+1)} \Big). \tag{3.24}$$

Now we define the sequence $\mathbf{x}_0, \mathbf{x}_1, \ldots$ of points in $[0,1]^s$ by

$$\mathbf{x}_n = \xi(f_n) \quad \text{for} \quad n = 0, 1, \ldots . \tag{3.25}$$

From [NiYe, Theorem 1], we get the following theorem :

Theorem L. *With the notation as above, we have that* $(\mathbf{x}_n)_{n \geq 0}$ *is a* (t,s)*-sequence in base* b *with* $t = g + e_0 - s$ *and* $e_0 = e_1 + \ldots + e_s$.

By Lemma 17, $(\mathbf{x}_n)_{n\geq 0}$ is $d-$admissible with $d = g + e_0$. Using [Le4, Theorem 2], we get

$$1 + \max_{1\leq N\leq b^m,} ND^*((\mathbf{x}_{n\oplus Q} \oplus \mathbf{w})_{0\leq n<N}) \geq 2^{-2}b^{-d}K^{-s}_{d,t,s+1}m^s \tag{3.26}$$

for some $Q \in [0, b^m)$ and $\mathbf{w} \in E^s_m$.

In order to obtain (3.26) for every Q and $\mathbf{w}$, we choose a specific sequence $v_0, v_1, ...$ as follows. Let

$$t_{s+1} \in \mathcal{L}(([(2g+1)/e_1] + 1)P_1 - P_{s+1}) \setminus \mathcal{L}(([(2g+1)/e_1] + 1)P_1 - 2P_{s+1}).$$

It is easy to see that

$$\nu_{P_{s+1}}(t_{s+1}) = 1, \;\; \nu_{P_i}(t_{s+1}) \geq 0, \; i \in [2, s] \;\text{ and }\; \deg((t_{s+1})_\infty) \leq 2g + e_1 + 1. \tag{3.27}$$

By (3.18) and the Riemann-Roch theorem, we have $\nu_{P_{s+1}}(v_i) = -i - g$ for $i \geq g$. Hence

$$v_i = \sum_{j\leq i+g} v_{i,j}t^{-j}_{s+1} \quad \text{with} \quad \text{all} \quad v_{i,j} \in \mathbb{F}_b, \quad v_{i,i+g} \neq 0, \quad i \geq g. \tag{3.28}$$

Using the orthogonalization procedure, we can construct a sequence $v_0, v_1, ...$ such that $\{v_0, v_1, ..., v_{\ell(mP_{s+1})-1}\}$ is a $\mathbb{F}_b$-basis of $\mathcal{L}(mP_{s+1})$,

$$v_{i,i+g} = 1, \quad \text{and} \quad v_{i,j+g} = 0 \quad \text{for} \quad j \in [g, i), \quad i \geq g. \tag{3.29}$$

Subsequently, we will use just this sequence.

Theorem 4. *With the above notations, $(\mathbf{x}_n)_{n\geq 0}$ is $d-$admissible, where $d = g + e_0$.*
(a) For $s \geq 2$, $m \geq 2^{2s+3}b^{d+t+s+1}(d+t)^{s+1}s^{2s}e(g+1)(e_0+s)\eta_1^{-s}$ and $\eta_1 = (1 + \deg((t_{s+1})_\infty))^{-1}$, we have

$$1 + \min_{0\leq Q<b^m} \min_{\mathbf{w}\in E^s_m} \max_{1\leq N\leq b^m} ND^*((\mathbf{x}_{n\oplus Q} \oplus \mathbf{w})_{0\leq n<N}) \geq 2^{-2}b^{-d}K^{-s}_{d,t,s+1}\eta_1^s m^s. \tag{3.30}$$

(b) Let $s \geq 3$, $m \geq 2^{2s+3}b^{d+t+s}(d+t)^s(s-1)^{2s-1}(g+e_0)e\eta_2^{-s+1}$, $e_s = \min_{1\leq i\leq s} e_i$ and $\eta_2 = (1 + \deg((t_s)_\infty))^{-1}$. Then

$$\min_{\mathbf{w}\in E^s_m} b^m D^*((\mathbf{x}_n \oplus \mathbf{w})_{0\leq n<b^m}) \geq 2^{-2}b^{-d}K^{-s+1}_{d,t,s}\eta_2^{s-1}m^{s-1}. \tag{3.31}$$

3.5. Niederreiter-Xing sequence.

Let $F/\mathbb{F}_b$ be an algebraic function field with full constant field $\mathbb{F}_b$ and genus $g = g(F/\mathbb{F}_b)$. Assume that $F/\mathbb{F}_b$ has at least $s+1$ rational places. Let $P_1, ..., P_{s+1}$ be $s+1$ distinct rational places of F. Let $G_m = m(P_1 + ... + P_s) - (m - g + 1)P_{s+1}$, and let t_i be a local parameter at P_i, $1 \leq i \leq s+1$. For any $f \in \mathcal{L}(G_m)$ we have $\nu_{P_i}(f) \geq m$, and so the local expansion of f at P_i has the form

$$f = \sum_{j=-m}^{\infty} f_{i,j}t_i^j, \quad \text{with} \quad f_{i,j} \in \mathbb{F}_b, \; j \geq -m, \; 1 \leq i \leq s.$$

For $1 \le i \le s$, we define the $\mathbb{F}_b$-linear map $\psi_{m,i}(f) \ : \ \mathcal{L}(G_m) \to \mathbb{F}_b^m$ by

$$\psi_{m,i}(f) = (f_{i,-1},...,f_{i,-m}) \in \mathbb{F}_b^m, \quad \text{for} \quad f \in \mathcal{L}(G_m).$$

Let

$$\mathcal{M}_m = \mathcal{M}_m(P_1,...,P_s;G_m) := \{(\psi_{m,1}(f),...,\psi_{m,s}(f)) \in \mathbb{F}_b^{ms} \mid f \in \mathcal{L}(G_m)\}. \tag{3.32}$$

Let $C^{(1)},...,C^{(s)} \in \mathbb{F}_b^{\infty\times\infty}$ be the generating matrices of a digital sequence $\mathbf{x}_n(C)_{n\ge 0}$, and let $(\mathcal{C}_m)_{m\ge 1}$ be the associated sequence of row spaces of overall generating matrices $[C]_m$, $m = 1,2,...$ (see (2.25)).

Theorem M. (see [DiPi, Theorem 7.26 and Theorem 8.9]) *There exist matrices $C^{(1)},...,C^{(s)}$ such that $\mathbf{x}_n(C)_{n\ge 0}$ is a digital (t,s)-sequence with $t = g$ and $\mathcal{C}_m = \mathcal{M}_m^{\perp}(P_1,...,P_s;G_m)$ for $m \ge g+1$, $s \ge 2$.*

According to [DiNi, p.411] and [DiPi, p.275], the construction of digital sequences of Niederreiter and Xing [NiXi] can be achieved by using the above approach. We propose the following way to get $\mathbf{x}_n(C)_{n\ge 0}$.

We consider the H-differential dt_{s+1}. Let ω be the corresponding Weil differential, $\mathrm{div}(\omega)$ the divisor of ω, and $W := \mathrm{div}(dt_{s+1}) = \mathrm{div}(\omega)$. By (2.5), we have $\deg(W) = 2g-2$. Similarly to (3.18)-(3.29), we can construct a sequence $\dot{v}_0, \dot{v}_1,...$ of elements of F such that $\{\dot{v}_0, \dot{v}_1, ..., \dot{v}_{\ell((m-g+1)P_{s+1}+W)-1}\}$ is a $\mathbb{F}_b$-basis of
$L_m := \mathcal{L}((m-g+1)P_{s+1}+W)$ and

$$\dot{v}_r \in L_{r+1} \setminus L_r, \quad \nu_{P_{s+1}}(\dot{v}_r) = -r+g-2, \ r \ge g, \ \text{ and } \ \dot{v}_{r,r+2-g} = 1, \ \dot{v}_{r,j} = 0 \tag{3.33}$$

for $2 \le j < r+2-g$, where

$$\dot{v}_r := \sum_{j \le r-g+2} \dot{v}_{r,j} t_{s+1}^{-j} \quad \text{for} \quad \dot{v}_{r,j} \in \mathbb{F}_b \ \text{ and } \ r \ge g.$$

According to Proposition A, we have that there exists $\tau_i \in F$ $(1 \le i \le s)$, such that $\mathrm{d}t_{s+1} = \tau_i \mathrm{d}t_i$ for $1 \le i \le s$.

Bearing in mind (2.4), (2.6) and (3.33), we get

$$\nu_{P_i}(\dot{v}_r\tau_i) = \nu_{P_i}(\dot{v}_r\tau_i \mathrm{d}t_i) = \nu_{P_i}(\dot{v}_r \mathrm{d}t_{s+1}) \ge \nu_{P_i}(\mathrm{div}(\mathrm{d}t_{s+1}) - W) = 0, \quad 1 \le i \le s, \ r \ge 0.$$

We consider the following local expansions

$$\dot{v}_r\tau_i \ := \ \sum_{j=0}^{\infty} \dot{c}_{j,r}^{(i)} t_i^j, \quad \text{where all} \quad \dot{c}_{j,r}^{(i)} \in \mathbb{F}_b, \ 1 \le i \le s, \ j \ge 0. \tag{3.34}$$

Now let $\dot{C}^{(i)} = (\dot{c}_{j,r}^{(i)})_{j,r\ge 0}$, $1 \le i \le s$, and let $\dot{\mathcal{C}}_m$ be the row space of overall generating matrix $[\dot{C}]_m$ (see (2.25)).

Theorem 5. *With the above notations,* $\mathbf{x}_n(\dot{C})_{n\geq 0}$ *is a digital* $d-$*admissible* (t,s)-*sequence, satisfying the bounds (3.30) and (3.31), with* $d = g+s$, $t = g$, *and* $\dot{C}_m = \mathcal{M}_m^{\perp}(P_1,...,P_s;G_m)$ *for all* $m \geq g+1$.

3.6 General $d-$**admissible digital** (t,s)**-sequences**. In [KrLaPi], discrepancy bounds for index-transformed uniformly distributed sequences was studied. In this subsection, we consider a lower bound of such a sequences.

Let $s \geq 2$, $d \geq 1$, $t \geq 0$, $d_0 = d+t$ and $m_k = s^2 d_0(2^{2k+2}-1)$ for $k = 1,2,\ldots$. Let $C^{(s+1)} = (c_{i,j}^{(s+1)})_{i,j\geq 1}$ be a $\mathbb{N}\times\mathbb{N}$ matrix over $\mathbb{F}_b$, and let $[C^{(s+1)}]_{m_k}$ be a non-singular matrix, $k = 1,2,\ldots$. For $n \in [0,b^{m_k})$, let $\mathbf{h}_k(n) = (h_{k,1}(n),...,h_{k,m_k}(n)) = \mathbf{n}[C^{(s+1)}]_{m_k}^{\top}$ and $h_k(n) = \sum_{j=1}^{m} \phi^{-1}(h_{k,j}(n))b^{j-1}$ $(k \geq 1)$. We have $h_k(l) \neq h_k(n)$ for $l \neq n$, $l,n \in [0,b^{m_k})$. Let $h_k^{-1}(h_k(n)) = n$ for $n \in [0,b^{m_k})$. It is easy to see that h_k^{-1} is a bijection from $[0,b^{m_k})$ to $[0,b^{m_k})$ $(k = 1,2,...)$.

Theorem 6. *Let* $(\mathbf{x}_n)_{n\geq 0}$ *be a digital* $d-$*admissible* (t,s)*-sequence in base b. Then there exists a matrix* $C^{(s+1)}$ *and a sequence* $(h^{-1}(n))_{n\geq 0}$ *such that* $[C^{(s+1)}]_{m_k}$ *is non-singular,* $h^{-1}(n) = h_l^{-1}(n) = h_k^{-1}(n)$ *for* $n \in [0,b^{m_k})$ $(l > k,\ k = 1,2,...)$, $(\mathbf{x}_{h^{-1}(n)})_{n\geq 0}$ *a* $d-$*admissible* (t,s)*-sequence in base b, and*

$$1 + \min_{0\leq Q<b^{m_k}, \mathbf{w}\in E_{m_k}^s} \max_{1\leq N\leq b^{m_k}} ND^*((\mathbf{x}_{h^{-1}(n)\oplus Q} \oplus \mathbf{w})_{0\leq n<N}) \geq 2^{-2}b^{-d}K_{d,t,s+1}^{-s} m_k^s, \quad k \geq 1 .$$

Remark 2. Halton-type sequences were introduced in [Te1] for the case of rational function fields over finite fields. Generalizations to the general case of algebraic function field were obtained in [Le1] and [NiYe]. The constructions in [Le1] and [NiYe] are similar. The difference is that the construction in [NiYe] is more simple, but the construction in [Le1] a somewhat more general.

Remark 3. We note that all explicit constructions of this article are expressed in terms of the residue of a differential and are similar to the Halton construction (see, e.g., (4.6), (4.28), (4.62) and (4.113)-(4.121)). The earlier constructions of $(t,s)-$sequences using differentials, see e.g. [MaNi].

4. Proof of theorems.

4.1. **Generalized Niederreiter sequence. Proof of Theorem 1.** Using [Le4, Lemma 2] and [Te3, Theorem 1], we obtain that $(\mathbf{x}_n)_{n\geq 0}$ is $d-$admissible with $d = e_0$.

We apply Corollary 3 with $B_i^{'} = \emptyset$, $1 \leq i \leq s+1$, $B = 0$, $\hat{e} = e = e_1 e_2 \cdots e_s$, $d_0 = d+t$, $\epsilon = \eta_1(2sd_0e)^{-1}$ and $\eta_1 = s/(s+1)$. In order to prove the first assertion in Theorem 1, it is sufficient to verify that

$$\Lambda_1 = \mathbb{F}_b^{(s+1)d_0e[m\epsilon]}, \qquad \text{for} \qquad m \geq 9(d+t)es(s+1), \tag{4.1}$$

where

$$\Lambda_1 = \{(y^{(1)}_{n,1}, ..., y^{(1)}_{n,d_1}, ..., y^{(s)}_{n,1}, ..., y^{(s)}_{n,d_s}, \bar{a}_{d_{s+1,1}}(n), ..., \bar{a}_{d_{s+1,2}}(n)) \mid n \in [0, b^m)\}$$

with

$$d_i = \dot{m}_i = d_0e[m\epsilon] \ \ (1 \leq i \leq s), \quad d_{s+1,1} = \ddot{m}_{s+1} + 1 := t + (s-1)d_0e[m\epsilon], \tag{4.2}$$

$d_{s+1,2} = \dot{m}_{s+1} := t - 1 + sd_0e[m\epsilon]$, and $n = \sum_{0 \leq j \leq m-1} a_j(n)b^j$.

Suppose that (4.1) is not true. Then there exists $b_{i,j} \in \mathbb{F}_b$ $(i, j \geq 1)$ such that

$$\sum_{i=1}^{s}\sum_{j=1}^{d_i} |b_{i,j}| + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} |b_{s+1,j}| > 0 \tag{4.3}$$

and

$$\sum_{i=1}^{s}\sum_{j=1}^{d_i} b_{i,j}y^{(i)}_{n,j} + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} b_{s+1,j}\bar{a}_j(n) = 0 \quad \text{for all} \quad n \in [0, b^m). \tag{4.4}$$

From (2.14) and (3.1), we have

$$y^{(i)}_{n,j} = \sum_{r=0}^{m-1} c^{(i)}_{j,r}\bar{a}_r(n),$$

with

$$c^{(i)}_{j,r} = a^{(i)}(Q+1, k, r) \in \mathbb{F}_b, \qquad j - 1 = Qe_i + k, \quad 0 \leq k < e_i, \tag{4.5}$$

$Q = Q(i, j)$, $k = k(i, j)$, where $a^{(i)}(j, k, r)$ are defined from the expansions

$$\frac{y_{i,j,k}(x)}{p_i(x)^j} = \sum_{r \geq 0} a^{(i)}(j, k, r)x^{-r-1}.$$

We consider the field $F = \mathbb{F}_b(x)$, the valuation ν_∞ (see (2.1)) and the place $P_\infty = \mathrm{div}(x^{-1})$. By (2.8), we get

$$a^{(i)}(j, k, r) = \operatorname*{Res}_{P_\infty, x^{-1}} \big(y_{i,j,k}(x)p_i(x)^{-j}x^{r+2}\big).$$

Hence

$$y^{(i)}_{n,j} = \operatorname*{Res}_{P_\infty, x^{-1}} \Big(\frac{y_{i,Q(i,j)+1,k(i,j)}(x)}{p_i(x)^{Q(i,j)+1}} \sum_{r=0}^{m-1} \bar{a}_r(n)x^{r+2}\Big) = \operatorname*{Res}_{P_\infty, x^{-1}} \Big(\frac{y_{i,Q(i,j)+1,k(i,j)}(x)}{p_i(x)^{Q(i,j)+1}} n(x)\Big) \tag{4.6}$$

with $n(x) = \sum_{j=0}^{m-1} \bar{a}_j(n) x^{j+2}$ for all $j \in [1, d_i]$, $i \in [1, s]$.
We have $\bar{a}_j(n) = \operatorname{Res}_{P_\infty, x^{-1}} (n(x) x^{-j-1})$. From (4.4), we derive

$$\operatorname{Res}_{P_\infty, x^{-1}} (n(x)\alpha) = 0 \text{ with } \alpha = \sum_{i=1}^{s} \sum_{j=1}^{d_i} b_{i,j} \frac{y_{i,Q(i,j)+1,k(i,j)}(x)}{p_i(x)^{Q(i,j)+1}} + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} b_{s+1,j} x^{-j-1} \tag{4.7}$$

for all $n \in [0, b^m)$. Consider the local expansion

$$\alpha = \sum_{r=0}^{\infty} \varphi_r x^{-r-1} \quad \text{with} \quad \varphi_r \in \mathbb{F}_b, \quad r \geq 0.$$

Applying (2.12) and (4.7), we derive

$$\operatorname{Res}_{P_\infty, x^{-1}} (n(x)\alpha) = \operatorname{Res}_{P_\infty, x^{-1}} \Big(\sum_{\mu=0}^{m-1} \bar{a}_\mu(n) x^{\mu+2} \sum_{r=0}^{\infty} \varphi_r x^{-r-1} \Big) = \sum_{\mu=0}^{m-1} \sum_{r=0}^{\infty} \bar{a}_\mu(n) \varphi_r$$

$$\times \operatorname{Res}_{P_\infty, x^{-1}} (x^{\mu+2-r-1}) = \sum_{\mu=0}^{m-1} \sum_{r=0}^{\infty} \bar{a}_\mu(n) \varphi_r \delta_{\mu,r} = \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) \varphi_\mu = 0$$

for all $n \in [0, b^m)$. Hence

$$\varphi_r = 0 \quad \text{for} \quad r \in [0, m-1] \quad \text{and} \quad \nu_\infty(\alpha) \geq m. \tag{4.8}$$

According to (4.5), we obtain

$$Q(i,j) + 1 \leq Q(i, d_i) + 1 \leq [(d_i - 1)/e_i] + 1 = d_i/e_i \text{ for } j \in [1, d_i], i \in [1, s].$$

By (4.7), we get

$$\alpha \in \mathcal{L}(G_1) \quad \text{with} \quad G_1 = \sum_{i=1}^{s} d_i/e_i \operatorname{div}(p_i(x)) + (d_{s+1,2} + 1)\operatorname{div}(x) - mP_\infty. \tag{4.9}$$

From (4.1) and (4.2), we have for $m \geq 2t + 8(d+t)es(s+1)$

$$\deg(G_1) = \sum_{i=1}^{s} d_i + d_{s+1,2} + 1 - m = sd_0e[m\epsilon] + t - 1 + sd_0e[m\epsilon] + 1 - m$$

$$\leq t - m(1 - 2sd_0e\epsilon) = t - m(1 - \eta_1) = t - m/(s+1) < 0.$$

Hence $\alpha = 0$.
Let g.c.d.$(x, p_j(x)) = 1$ for all $j \neq i$ with some $i \in [1, s]$. For example, let $i = 1$, and let $p_1(x) = x^{e_{1,1}} \dot{p}_1(x)$ with $e_{1,2} = \deg(\dot{p}_1(x))$, $e_1 = e_{1,1} + e_{1,2}$, $e_{1,1} \geq 0$, g.c.d.$(x, \dot{p}_1(x)) = 1$. According to (4.7), we get $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, where

$$\alpha_1 = \sum_{i=2}^{s} \sum_{j=1}^{d_i} b_{i,j} \frac{y_{i,Q(i,j)+1,k(1,j)}(x)}{p_i(x)^{Q(i,j)+1}}, \qquad \alpha_2 = \sum_{j=1}^{d_1} b_{1,j} \frac{\ddot{y}_{i,Q(1,j)+1,k(1,j)}(x)}{\dot{p}_1(x)^{Q(1,j)+1}}$$

$$\text{and} \quad \alpha_3 = \sum_{j=1}^{d_1} b_{1,j} \frac{\dot{y}_{1,Q(1,j)+1,k(1,j)}(x)}{x^{e_{1,1}(Q(1,j)+1)}} + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} \frac{b_{s+1,j}}{x^{j+1}}$$

with some polynomials $\dot{y}_{1,j,k}(x)$ and $\ddot{y}_{1,j,k}(x)$.

Using (4.2), we obtain for $s \geq 2$ and $j \in [1, d_1]$ that

$$d_{s+1,1} + 1 = t + 1 + (s-1)d_0 e[m\epsilon] > d_0 e[m\epsilon] = d_1 \geq e_{1,1}d_1/e_1 \geq e_{1,1}\deg(Q(1,d_1)+1).$$

We have that the polynomials $p_2, ..., p_s, \dot{p}_1$ and x are pairwise coprime over $\mathbb{F}_b$. By the uniqueness of the partial fraction decomposition of a rational function, we have that $\alpha_3 = 0$ and $b_{s+1,j} = 0$ for all $j \in [d_{s+1,1}, d_{s+1,2}]$.

Bearing in mind that $p_1, ..., p_s$ are pairwise coprime polynomials over $\mathbb{F}_b$, we obtain from [Te3, p.242] or [Te2, p. 166,167] that $b_{i,j} = 0$ for all $j \in [1, d_i]$ and $i \in [1, s]$.

By (4.3), we have the contradiction. Hence assertion (4.1) is true. Thus the first assertion in Theorem 1 is proved.

Now consider the second assertion in Theorem 1:

Let, for example, $i_0 = s$, i.e.

$$(4.10) \qquad \min_{m/2-t \leq je_s \leq m, 0 \leq k < e_s} (1 - \deg(y_{s,j,k}(x))j^{-1}e_s^{-1}) \geq \eta_2.$$

We apply Corollary 2 with $\dot{s} = s \geq 3$, $B_i = \emptyset$, $1 \leq i \leq s$, $B = 0$, $\tilde{r} = 0$, $m = \tilde{m}$, $d_0 = d + t$, $\hat{e} = e = e_1 e_2 \cdots e_s$, $\epsilon = \eta_2(2(s-1)d_0 e)^{-1}$. In order to prove the second assertion in Theorem 1, it is sufficient to verify that

$$(4.11) \qquad \Lambda_2 = \mathbb{F}_b^{sd_0 e[m\epsilon]} \qquad \text{for} \qquad m \geq 8(d+t)e(s-1)^2\eta_2^{-1} + 2(1+t)\eta_2^{-1}(1-\eta_2)^{-1},$$

where

$$\Lambda_2 = \{(y_{n,1}^{(1)}, ..., y_{n,d_1}^{(1)}, ..., y_{n,1}^{(s-1)}, ..., y_{n,d_{s-1}}^{(s-1)}, y_{n,d_{s,1}}^{(s)}, ..., y_{n,d_{s,2}}^{(s)}) \mid n \in [0, b^m)\},$$

with

$$(4.12) \qquad d_i = \dot{m}_i = d_0 e[m\epsilon],\ i \in [1, s),\ \ d_{s,1} = \ddot{m}_s + 1 := m - t + 1 - (s-1)d_0 e[m\epsilon]$$

and $d_{s,2} = \dot{m}_s := m - t - (s-2)d_0 e[m\epsilon]$.

Suppose that (4.11) is not true. Then there exists $b_{i,j} \in \mathbb{F}_b$ $(i, j \geq 1)$ such that

$$(4.13) \qquad \sum_{i=1}^{s-1} \sum_{j=1}^{d_i} |b_{i,j}| + \sum_{j=d_{s,1}}^{d_{s,2}} |b_{s,j}| > 0$$

and

$$(4.14) \qquad \sum_{i=1}^{s-1} \sum_{j=1}^{d_i} b_{i,j} y_{n,j}^{(i)} + \sum_{j=d_{s,1}}^{d_{s,2}} b_{s,j} y_{n,j}^{(s)} = 0 \quad \text{for all} \quad n \in [0, b^m).$$

Similarly to (4.7), we have

$$\operatorname*{Res}_{P_\infty, x^{-1}}(n(x)\alpha) = 0 \quad \text{for all} \quad n \in [0, b^m), \quad \text{with} \quad \alpha = \alpha_1 + \alpha_2,$$

where

$$\alpha_1 = \sum_{i=1}^{s-1}\sum_{j=1}^{d_i} b_{i,j}\frac{y_{i,Q(i,j)+1,k(i,j)}(x)}{p_i(x)^{Q(i,j)+1}} \quad \text{and} \quad \alpha_2 = \sum_{j=d_{s,1}}^{d_{s,2}} b_{s,j}\frac{y_{s,Q(s,j)+1,k(s,j)}(x)}{p_s(x)^{Q(s,j)+1}}. \tag{4.15}$$

Consider the local expansions

$$\alpha_1 = \sum_{r=0}^{\infty}\varphi_{1,r}x^{-r-1} \quad \text{and} \quad \alpha_2 = \sum_{r=0}^{\infty}\varphi_{2,r}x^{-r-1} \quad \text{with} \quad \varphi_{i,r}\in\mathbb{F}_b \quad i=1,2,\ r\geq 0.$$

Analogously to (4.8), we obtain from (4.14)

$$\varphi_{1,r}+\varphi_{2,r}=0 \quad \text{for all} \quad r\in[0,m-1]. \tag{4.16}$$

Taking into account that $j \leq (Q(s,j)+1)e_s$ and $d_{s,1}\geq m/2-t$, we get from (2.1) and (4.10) that

$$\nu_\infty\Big(\frac{y_{s,Q(s,j)+1,k(s,j)}(x)}{p_s(x)^{Q(s,j)+1}}\Big) = (Q(s,j)+1)e_s - \deg(y_{s,Q(s,j)+1,k(s,j)}(x)) =$$

$$(Q(s,j)+1)\Big(1-\frac{\deg(y_{s,Q(s,j)+1,k(s,j)}(x))}{(Q(s,j)+1)e_s}\Big)e_s \geq (Q(s,j)+1)e_s\eta_2 \geq \eta_2 j, \quad j\geq d_{s,1}.$$

Applying (4.15)-(4.16), we have $\varphi_{2,r}=0$ for $r<[\eta_2 d_{s,1}]$. Therefore $\varphi_{1,r}=0$ for $r<[\eta_2 d_{s,1}]$. Hence

$$\nu_\infty(\alpha_1)\geq[\eta_2 d_{s,1}].$$

Similarly to (4.9), we obtain

$$\alpha_1\in\mathcal{L}(G_2) \quad \text{with} \quad G_2=\sum_{i=1}^{s-1} d_i/e_i\mathrm{div}(p_i(x)) - [\eta_2 d_{s,1}]P_\infty.$$

From (4.11) and (4.12), we have that $m>2(1+t)\eta_2^{-1}(1-\eta_2)^{-1}$ and

$$\begin{aligned}\deg(G_2) &= \sum_{i=1}^{s-1} d_i - [d_{s,1}\eta_2] = (s-1)d_0e[m\epsilon] - [(m-t+1-(s-1)d_0e[m\epsilon])\eta_2] \\ &\leq (s-1)d_0e[m\epsilon] - (m-t-(s-1)d_0e[m\epsilon])\eta_2 + 1 = (1+\eta_2)(s-1)d_0e[m\epsilon] \\ &\quad -m\eta_2+1+t \leq m((1+\eta_2)((s-1)d_0e\epsilon-\eta_2)+1+t \\ &= m\eta_2((1+\eta_2)/2-1)+1+t = 1+t-m\eta_2(1-\eta_2)/2<0.\end{aligned}$$

Hence $\alpha_1=0$ and $\varphi_{1,r}=0$ for $r\geq 0$.
Using [Te3, p.242] or [Te2, p. 166,167], we get $b_{i,j}=0$ for all $j\in[1,d_i]$ and $i\in[1,s-1]$.
According to (4.16), we have $\varphi_{2,r}=0$ for $r\in[0,m-1]$. Thus $\nu_\infty(\alpha_2)\geq m$.
From (4.15), we obtain

$$\alpha_2\in\mathcal{L}(G_3) \quad \text{with} \quad G_3=[d_{s,2}/e_s+1]\mathrm{div}(p_s(x))-mP_\infty.$$

Applying (4.1) and (4.2), we derive for $m>2/\epsilon$ and $s\geq 3$

$$\deg(G_3)\leq m-t-(s-2)d_0e[m\epsilon]+e_s-m<0.$$

Hence $\alpha_2 = 0$.

By the uniqueness of the partial fraction decomposition of a rational function, we have from (4.15) that $b_{s+1,j} = 0$ for all $j \in [d_{s,1}, d_{s,2}]$.

By (4.13), we have a contradiction. Thus assertion (4.11) is true. Therefore Theorem 1 is proved. □

4.2. **Xing-Niederreiter sequence. Proof of Theorem 2. Lemma 3.** *Let $P \in \mathbb{P}_F$, t be a local parameter of P over F, $k_j \in F$, $\nu_P(k_j) = j$ $(j = 0, 1, ...)$. Then there exists $k_j^{\perp} \in F$ with $\nu_P(k_j^{\perp}) = -j$ $(j = 1, 2, ...)$, such that*

$$S_{-1}(t, k_{j_1} k_{j_2+1}^{\perp}) = \delta_{j_1, j_2} \qquad \text{for} \qquad j_1, j_2 \geq 0. \tag{4.17}$$

Proof. Let $k_1^{\perp} = (tk_0)^{-1}$. We see $\nu_P(k_j k_1^{\perp}) \geq 0$ for $j \geq 1$. Using (2.2) and (2.12), we get that (4.17) is true for $j_2 = 0$. Suppose that the assertion of the lemma is true for $0 \leq j_2 \leq j_0 - 1$, $j_0 \geq 1$. We take

$$k_{j_0+1}^{\perp} = \sum_{\mu=1}^{j_0} \rho_{\mu, j_0} k_{\mu}^{\perp} + (tk_{j_0})^{-1}, \quad \text{where} \quad \rho_{\mu, j_0} = S_{-1}(t, k_{\mu-1}(tk_{j_0})^{-1}). \tag{4.18}$$

We see that $\nu_P(k_{j_0+1}^{\perp}) = -j_0 - 1$. By the condition of the lemma and the assumption of the induction, we have $\nu_P(k_{j_1} k_{j_0+1}^{\perp}) \geq 0$ for $j_1 > j_0$ and

$$S_{-1}(t, k_{j_1} k_{j_0+1}^{\perp}) = \delta_{j_1, j_0} \quad \text{for} \quad j_1 \geq j_0. \tag{4.19}$$

Now consider the case $j_1 \in [0, j_0)$. Applying (4.18), we derive

$$S_{-1}(t, k_{j_1} k_{j_0+1}^{\perp}) = \sum_{\mu=1}^{j_0} \rho_{\mu, j_0} S_{-1}(t, k_{j_1} k_{\mu}^{\perp}) + S_{-1}(t, k_{j_1}(tk_{j_0})^{-1}).$$

Using (2.12), (4.18) and the assumption of the induction, we get

$$S_{-1}(t, k_{j_1} k_{j_0+1}^{\perp}) = \sum_{\mu=1}^{j_0} \rho_{\mu, j_0} \delta_{j_1, \mu-1} + S_{-1}(t, k_{j_1}(tk_{j_0})^{-1}) = \rho_{j_1+1, j_0} - \rho_{j_1+1, j_0} = 0.$$

Hence (4.19) is true for all $j_1 \geq 0$. By induction, Lemma 3 is proved. □

Lemma 4. $(\mathbf{x}_n)_{n \geq 0}$ *is $d-$admissible with $d = g + e_0$, where $e_0 = e_1 + ... + e_s$.*

Proof. Consider Definition 5. Taking into account that $(\mathbf{x}_n)_{n \geq 0}$ is a digital sequence in base b, we can take $k = 0$. Suppose that the assertion of the lemma is not true. By (1.4), there exists $\tilde{n} > 0$ such that $\|\tilde{n}\|_b \, \|\mathbf{x}_{\tilde{n}}\|_b < b^{-d} = b^{-g-e_0}$. Let $d_i = \dot{d}_i e_i + \ddot{d}_i$ with $0 \leq \ddot{d}_i < e_i$, $1 \leq i \leq s$, $\|\tilde{n}\|_b = b^{m-1}$ and let $\left\| \mathbf{x}_{\tilde{n}}^{(i)} \right\|_b =$

b^{-d_i-1}, $1 \le i \le s$. Hence $\tilde{n} \in [b^{m-1}, b^m)$, $x^{(i)}_{\tilde{n},d_i+1} \neq 0$,

$$x^{(i)}_{\tilde{n},j} = 0 \text{ for all } j \in [1,d_i],\ i \in [1,s] \quad \text{and} \quad \sum_{i=1}^{s}(d_i+1) - m \ge d = g + e_0.$$

By (2.14), we have

$$y^{(i)}_{\tilde{n},j} = 0 \quad \text{for all} \quad j \in [1, \dot{d}_i e_i],\ \ i \in [1,s] \quad \text{with} \quad \sum_{i=1}^{s} \dot{d}_i e_i \ge m + g. \tag{4.20}$$

Let

$$\{\dot{n}_0, ..., \dot{n}_{g-1}\} = \{0, 1, ..., 2g\} \setminus \{n_0, n_1, ..., n_g\} \quad \text{and} \quad \dot{n}_i = g + i + 1 \text{ for } i \ge g. \tag{4.21}$$

Let $n = \sum_{i=0}^{m-1} a_i(n) b^i$ with $a_i(n) \in Z_b$ $(i = 0, 1...)$, and let $\bar{a}_i(n) = \phi(a_i(n))$ $(i = 0, 1, ...)$ (see (2.13)). From (2.14), (3.6) and (3.7), we get

$$y^{(i)}_{n,j} = \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) c^{(i)}_{j,\mu} = \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) a^{(i)}_{j,\dot{n}_\mu} \qquad \text{for } j \in [1,m],\ i \in [1,s]. \tag{4.22}$$

By (3.5), we have

$$\nu_{P_\infty}(z_r) = r, \quad \text{for} \quad r \ge 0, \quad \text{and} \quad z_{n_u} = w_u \quad \text{with} \quad u = 0, 1, ..., g. \tag{4.23}$$

Using Lemma 3, (2.2) and (2.8), we obtain that there exists a sequence $(z_j^{\perp})_{j \ge 1}$ such that $\nu_{P_\infty}(z_j^{\perp}) = -j$ and

$$\operatorname*{Res}_{P_\infty,z}(z_i z_{j+1}^{\perp}) = S_{-1}(z, z_i z_{j+1}^{\perp}) = \delta_{i,j} \quad \text{for all} \quad i, j \ge 0. \tag{4.24}$$

We put

$$f_n = \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) z^{\perp}_{\dot{n}_\mu+1}. \tag{4.25}$$

Hence

$$\bar{a}_\mu(n) = \operatorname*{Res}_{P_\infty,z}(f_n z_{\dot{n}_\mu}) \quad \text{for} \quad 0 \le \mu \le m-1,\ n \in [0, b^m). \tag{4.26}$$

By (2.12) and (4.21), we have $\delta_{\dot{n}_\mu, n_u} = 0$ for all $0 \le u \le g,\ \mu \ge 0$.
Applying (4.23) and (4.24), we derive

$$\operatorname*{Res}_{P_\infty,z}(f_n w_u) = \operatorname*{Res}_{P_\infty,z}\Big(\sum_{\mu=0}^{m-1} \bar{a}_\mu(n) z^{\perp}_{\dot{n}_\mu+1}\, z_{n_u}\Big) \tag{4.27}$$

$$= \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) \operatorname*{Res}_{P_\infty,z}\big(z^{\perp}_{\dot{n}_\mu+1}\, z_{n_u}\big) = \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) \delta_{\dot{n}_\mu, n_u} = 0 \quad \text{for} \quad u = 0, 1, ..., g,\ n \ge 0.$$

According to (3.6) and (4.25), we have

$$\operatorname*{Res}_{P_\infty,z}(f_n k_{i,j}) = \operatorname*{Res}_{P_\infty,z}\Big(\sum_{\mu=0}^{m-1} \bar{a}_\mu(n) z^{\perp}_{\dot{n}_\mu+1} \sum_{r=0}^{\infty} a^{(i)}_{j,r} z_r\Big)$$

$$= \sum_{\mu=0}^{m-1}\sum_{r=0}^{\infty} \bar{a}_\mu(n) a^{(i)}_{j,r} \operatorname*{Res}_{P_\infty,z}(z^{\perp}_{\dot{n}_\mu+1}\, z_r) = \sum_{\mu=0}^{m-1}\sum_{r=0}^{\infty} \bar{a}_\mu(n) a^{(i)}_{j,r}\delta_{\dot{n}_\mu,r} = \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) a^{(i)}_{j,\dot{n}_\mu}.$$

From (4.22), we get

$$\operatorname*{Res}_{P_\infty,z}(f_n k_{i,j}) = y^{(i)}_{n,j} \quad \text{for all} \quad j \in [1,m],\ i \in [1,s],\ n \in [0, b^m). \tag{4.28}$$

Using (4.20) and (4.27), we derive

$$\operatorname*{Res}_{P_\infty,z}\Big(f_{\tilde{n}}\Big(\sum_{r=0}^{g} b_r w_r + \sum_{i=1}^{s}\sum_{j=1}^{\dot{d}_i e_i} b_{i,j} k_{i,j}\Big)\Big) = 0 \quad \text{for all} \quad b_i, b_{i,j} \in \mathbb{F}_b.$$

Taking into account that $(w_0, ..., w_g, k_{1,1}, ...k_{1,\dot{d}_1 e_1}, ..., k_{s,1}, ..., k_{s,\dot{d}_s e_s})$ is the basis of $\mathcal{L}(G + \sum_{i=1}^{s} \dot{d}_i P_i)$ (see (3.2)), we obtain

$$\operatorname*{Res}_{P_\infty,z}(f_{\tilde{n}}\gamma) = 0 \quad \text{for all} \quad \gamma \in \mathcal{L}(\dot{G}) \quad \text{with} \quad \dot{G} = G + \sum_{i=1}^{s} \dot{d}_i P_i. \tag{4.29}$$

By (4.20), we have

$$\deg(\dot{G} - (m+g+1)P_\infty) = 2g + \sum_{i=1}^{s} \dot{d}_i e_i - (m+g+1) \geq 2g + m + g - (m+g+1) = 2g-1.$$

Using the Riemann-Roch theorem, we get

$$\ddot{G} = (\dot{G} - (m+g)P_\infty) \setminus (\dot{G} - (m+g+1)P_\infty) \neq \emptyset.$$

We take $v \in \ddot{G}$. Hence $\nu_{P_\infty}(v) = m+g$.

From (3.5), we derive $v = \sum_{r\geq m+g} \hat{b}_r z_r$ with some $\hat{b}_r \in \mathbb{F}_b$ $(r \geq m+g)$ and $\hat{b}_{m+g} \neq 0$. According to (4.21), we have $\dot{n}_{m-1} = m+g$. Therefore $v = \sum_{r\geq \dot{n}_{m-1}} \hat{b}_r z_r$.

Taking into account that $\tilde{n} \in [b^{m-1}, b^m)$, we get $a_{m-1}(\tilde{n}) \neq 0$.
By (4.24), (4.25) and(4.29), we obtain

$$0 = \operatorname*{Res}_{P_\infty,z}(f_{\tilde{n}} v) = \sum_{\mu=0}^{m-1}\sum_{r\geq \dot{n}_{m-1}} a_\mu(\tilde{n})\hat{b}_r \operatorname*{Res}_{P_\infty,z}(z^{\perp}_{\dot{n}_\mu+1}\, z_r) = \sum_{\mu=0}^{m-1}\sum_{r\geq \dot{n}_{m-1}} a_\mu(\tilde{n})\hat{b}_r \delta_{\dot{n}_\mu,r}.$$

Bearing in mind that $\delta_{\dot{n}_\mu,r} = 1$ for $\mu \in [0, m-1]$, $r \geq \dot{n}_{m-1}$ if and only if $\mu = m-1$ and $r = \dot{n}_{m-1}$ (see (4.21)), we get $\operatorname{Res}_{P_\infty,z}(f_{\tilde{n}} v) = a_{m-1}(\tilde{n})\hat{b}_{\dot{n}_{m-1}} \neq 0$. We have a contradiction. Hence Lemma 4 is proved. $\square$

Lemma 5. *Let $s \geq 2$, $d_i = d_0 e[m\epsilon]$, $1 \leq i \leq s$, $d_{s+1,1} = t + (s-1)d_0 e[m\epsilon]$, $d_{s+1,2} = t - 1 + s d_0 e[m\epsilon]$, $d_0 = d + t$, $t = g + e_0 - s$, $e = e_1 ... e_s$ and $m \geq 2/\epsilon$. Then the system $\{w_0, w_1, ..., w_g\} \cup \{z^{j+g+1}\}_{d_{s+1,1} \leq j \leq d_{s+1,2}} \cup \{k_{i,j}\}_{1 \leq i \leq s, 1 \leq j \leq d_i}$ of elements of F is linearly independent over $\mathbb{F}_b$.*

Proof. Suppose that

$$\alpha := \sum_{j=0}^{g} b_{0,j} w_j + \sum_{i=1}^{s} \sum_{j=1}^{d_i} b_{i,j} k_{i,j} + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} b_{s+1,j} z^{j+g+1} = 0$$

for some $b_{i,j} \in \mathbb{F}_b$ and $\sum_{j=0}^{g} |b_{0,j}| + \sum_{i=1}^{s} \sum_{j=1}^{d_i} |b_{i,j}| + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} |b_{s+1,j}| > 0$. Let

$$\beta_1 = \sum_{j=0}^{g} b_{0,j} w_j, \; \beta_{2,i} = \sum_{j=1}^{d_i} b_{i,j} k_{i,j}, \;\; \beta_2 = \sum_{i=1}^{s} \beta_{2,i}, \; \beta_3 = \sum_{j=d_{s+1,1}}^{d_{s+1,2}} b_{s+1,j} z^{j+g+1}. \tag{4.30}$$

We have

$$\alpha = \beta_1 + \beta_2 + \beta_3 = 0. \tag{4.31}$$

Suppose that$\sum_{i=1}^{s} \sum_{j=1}^{d_i} |b_{i,j}| = 0$ and $\alpha = 0$. By (4.30) and (4.31), we have $\beta_1 + \beta_3 = 0$ and $\nu_{P_\infty}(\beta_1) \geq d_{s+1,1}$. Taking into account that $\beta_1 \in \mathcal{L}(G)$ with $\deg(G) = 2g$, we obtain from the Riemann-Roch theorem that $\beta_1 = 0$. Therefore $\sum_{j=0}^{g} |b_{0,j}| = 0$ and $\sum_{j=d_{s+1,1}}^{d_{s+1,2}} |b_{s+1,j}| = 0$. We have a contradiction.

According to [DiPi, Lemma 8.10], we get that if $\sum_{j=d_{s+1,1}}^{d_{s+1,2}} |b_{s+1,j}| = 0$ and $\alpha = 0$, then $\sum_{j=0}^{g} |b_{0,j}| = 0$ and $\sum_{i=1}^{s} \sum_{j=1}^{d_i} |b_{i,j}| = 0$. So, we will consider only the case then $\sum_{i=1}^{s} \sum_{j=1}^{d_i} |b_{i,j}| > 0$ and $\sum_{j=d_{s+1,1}}^{d_{s+1,2}} |b_{s+1,j}| > 0$.

Let $\sum_{j=1}^{d_h} |b_{h,j}| > 0$ for some $h \in [1, s]$, and let $\nu_{P_h}(z) \geq 0$.
By the construction of $k_{h,j}$, we have $\beta_{2,h} \notin \mathcal{L}(G)$ and $\beta_{2,h} \neq 0$. Applying (3.3) and (4.30), we obtain $\nu_P(\beta_{2,h}) \geq -\nu_P(G)$ for any place $P \neq P_h$ and hence we obtain that $\nu_{P_h}(\beta_{2,h}) \leq -\nu_{P_h}(G) - 1$ with $\nu_{P_h}(G) \geq 0$.

On the other hand, using (3.3) (4.30) and (4.31), we get

$$\nu_{P_h}(\beta_{2,h}) = \nu_{P_h}\Big(-\beta_1 - \sum_{i=1, i\neq h}^{s} \beta_{2,i} - \beta_3 \Big)$$

$$\geq \min\Big(\nu_{P_h}(\beta_1), \nu_{P_h}(\beta_3), \min_{1 \leq i \leq s, i \neq h} \nu_{P_h}(\beta_{2,i}) \Big) \geq -\nu_{P_h}(G).$$

We have a contradiction.

Now let $\nu_{P_h}(z) \leq -1$. Bearing in mind that $\sum_{j=d_{s+1,1}}^{d_{s+1,2}} |b_{s+1,j}| > 0$, we obtain that $\beta_3 \neq 0$, and $\nu_{P_h}(\beta_3) \leq -d_{s+1,1} - g - 1$. On the other hand, using (3.3) and

(4.31), we have

$$\nu_{P_h}(\beta_3) = \nu_{P_h}(\beta_1+\beta_2) \geq -\nu_{P_h}(G) - [(d_h-1)/e_h+1]e_h \geq -2g-d_h.$$

Taking into account that

$$d_{s+1,1}+g+1-(2g+d_h) = t+g+1+(s-2)d_0e[m\epsilon]-2g \geq t-g+1 \geq 1,$$

we have a contradiction. Thus Lemma 5 is proved. □

Lemma 6. *Let* $s \geq 2$, $d_0 = d+t$, $t = g+e_0-s$, $\epsilon = \eta_1(2sd_0e)^{-1}$, $\eta_1 = (1+\deg((z)_\infty))^{-1}$,

$$\Lambda_1 := \{(y_{n,1}^{(1)},...,y_{n,d_1}^{(1)},...,y_{n,1}^{(s)},...,y_{n,d_s}^{(s)},\bar{a}_{d_{s+1,1}}(n),...,\bar{a}_{d_{s+1,2}}(n)) \mid n \in [0,b^m)\},$$

where

$$d_i = \ddot{m}_i := d_0e[m\epsilon] \ \ (1 \leq i \leq s), \quad d_{s+1,1} = \ddot{m}_{s+1}+1 := t+(s-1)d_0e[m\epsilon], \tag{4.32}$$

$d_{s+1,2} = \dot{m}_{s+1} := t-1+sd_0e[m\epsilon]$, $e = e_1e_2\cdots e_s$, *and* $n = \sum_{0\leq j\leq m-1} a_j(n)b^j$. *Then*

$$\Lambda_1 = \mathbb{F}_b^{(s+1)d_0e[m\epsilon]}, \qquad \text{with} \quad m \geq 9(d+t)es^2\eta_1^{-1}. \tag{4.33}$$

Proof. Suppose that (4.33) is not true. Then there exists $b_{i,j} \in \mathbb{F}_b$ $(i,j \geq 1)$ such that

$$\sum_{i=1}^{s}\sum_{j=1}^{d_i}|b_{i,j}| + \sum_{j=d_{s+1,1}}^{d_{s+1,2}}|b_{s+1,j}| > 0 \tag{4.34}$$

and

$$\sum_{i=1}^{s}\sum_{j=1}^{d_i} b_{i,j}y_{n,j}^{(i)} + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} b_{s+1,j}\bar{a}_j(n) = 0 \quad \text{for all} \quad n \in [0,b^m). \tag{4.35}$$

From (4.26) and (4.28), we obtain for $n \in [0,b^m)$

$$\bar{a}_{j-1}(n) = \operatorname*{Res}_{P_\infty,z}(f_n z_{\dot{n}_{j-1}}) \quad \text{and} \quad y_{n,j}^{(i)} = \operatorname*{Res}_{P_\infty,z}(f_n k_{i,j}) \quad \text{with} \quad j \in [1,m],\ i \in [1,s].$$

Applying (3.5) and (4.21), we get $\dot{n}_{j-1} = g+j$ and $z_{\dot{n}_{j-1}} = z^{g+j}$ for $j \geq d_{s+1,1}$. Hence

$$\sum_{i=1}^{s}\sum_{j=1}^{d_i} b_{i,j}\operatorname*{Res}_{P_\infty,z}(f_n k_{i,j}) + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} b_{s+1,j}\operatorname*{Res}_{P_\infty,z}(f_n z^{g+j+1}) = \operatorname*{Res}_{P_\infty,z}(f_n\alpha_1) = 0 \tag{4.36}$$

with

$$\alpha_1 = \sum_{i=1}^{s}\sum_{j=1}^{d_i} b_{i,j}k_{i,j} + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} b_{s+1,j}z^{g+j+1} \quad \text{for} \quad n \in [0,b^m). \tag{4.37}$$

Let

$$b_{0,u} = -\sum_{i=1}^{s}\sum_{j=1}^{d_i} b_{i,j}a^{(i)}_{j,n_u}, \quad \beta_1 = \sum_{u=0}^{g} b_{0,u}w_u, \quad \beta_2 = \sum_{i=1}^{s}\sum_{j=1}^{d_i} b_{i,j}k_{i,j},$$

(4.38) $$\beta_3 = \sum_{j=d_{s+1,1}}^{d_{s+1,2}} b_{s+1,j}z^{g+j+1} \quad \text{and} \quad \alpha_2 = \beta_1+\beta_2+\beta_3 = \beta_1+\alpha_1.$$

By (4.34) and Lemma 5, we get

(4.39) $$\alpha_2 \neq 0.$$

Consider the local expansion

(4.40) $$\alpha_2 = \sum_{r=0}^{\infty} \varphi_r z_r \quad \text{with} \quad \varphi_r \in \mathbb{F}_b, \quad r \geq 0.$$

Using (3.5), (3.6) and (4.38), we have

(4.41) $$\varphi_{n_u} = 0 \quad \text{for} \quad 0 \leq u \leq g.$$

From (4.27), we derive $\operatorname{Res}_{P_\infty,z}(f_n w_u) = 0$ $(0 \leq u \leq g)$. By (4.36) and (4.38), we get

$$\operatorname{Res}_{P_\infty,z}(f_n\beta_1) = 0 \quad \text{and} \quad \operatorname{Res}_{P_\infty,z}(f_n\alpha_2) = 0 \quad \text{for all} \quad n \in [0, b^m).$$

Applying (4.24), (4.25) and (4.40), we obtain

$$\operatorname{Res}_{P_\infty,z}(f_n\alpha_2) = \operatorname{Res}_{P_\infty,z}\Big(\sum_{\mu=0}^{m-1} \bar{a}_\mu(n) z^{\perp}_{\dot{n}_\mu+1} \sum_{r=0}^{\infty}\varphi_r z_r\Big)$$

$$= \sum_{\mu=0}^{m-1}\sum_{r=0}^{\infty} \bar{a}_\mu(n)\varphi_r \operatorname{Res}_{P_\infty,z}(z^{\perp}_{\dot{n}_\mu+1}\, z_r) = \sum_{\mu=0}^{m-1}\sum_{r=0}^{\infty} \bar{a}_\mu(n)\varphi_r\delta_{\dot{n}_\mu,r} = \sum_{\mu=0}^{m-1} \bar{a}_\mu(n)\varphi_{\dot{n}_\mu} = 0$$

for all $n \in [0, b^m)$.
Hence $\varphi_{\dot{n}_\mu} = 0$ for $\mu \in [0, m-1]$. According to (4.21) and (4.41), we have

(4.42) $$\varphi_r = 0 \quad \text{for} \quad r \in [0, m+g].$$

Therefore

(4.43) $$\nu_{P_\infty}(\alpha_2) > m+g.$$

From (3.3) and (4.38), we derive

$$\beta_1+\beta_2 \in \mathcal{L}\big(G + \sum_{i=1}^{s}[(d_i-1)/e_i+1]P_i\big) \quad \text{and} \quad \beta_3 \in \mathcal{L}\big((d_{s+1,2}+g+1)(z)_\infty\big).$$

By (4.43), we obtain

$$\alpha_2 \in \mathcal{L}(G_1) \ \text{ with } \ G_1 = G + \sum_{i=1}^{s}[(d_i-1)/e_i+1]P_i + (d_{s+1,2}+g+1)(z)_\infty - (m+g+1)P_\infty.$$

Using (4.32), we have

$$\deg(G_1) = 2g + \sum_{i=1}^{s} d_i + (d_{s+1,2} + g + 1)\deg((z)_\infty) - (m + g + 1)$$
$$= 2g + sd_0e[m\epsilon] + (t + g + sd_0e[m\epsilon])(\eta_1^{-1} - 1) - (m + g + 1)$$
$$\leq 2g + (t + g)(\eta_1^{-1} - 1) + sd_0em\epsilon\eta_1^{-1} - (m + g + 1)$$
$$= g - 1 + (t + g)(\eta_1^{-1} - 1) - m(1 - sd_0e\epsilon\eta_1^{-1}) = g - 1 + (t + g)(\eta_1^{-1} - 1) - m/2 < 0$$

for $m \geq 9(d + t)es^2\eta_1^{-1} > 2(g - 1) + 2(t + g)(\eta_1^{-1} - 1)$ and $d = g + e_0$. Hence $\alpha_2 = 0$. By (4.39), we have a contradiction. Therefore assertion (4.35) is not true. Thus Lemma 6 is proved. □

End of the proof of Theorem 2. Using Lemma 4 and Theorem J, we get that $(\mathbf{x}(n))_{n\geq 0}$ is a $d-$admissible digital (t, s) sequence with $d = g + e_0$ and $t = g + e_0 - s$. Applying Lemma 6 and Corollary 3 with $B_i^{'} = \emptyset$, $1 \leq i \leq s + 1$, $B = 0$ and $\hat{e} = e = e_1e_2 \cdots e_s$, we get the first assertion in Theorem 2.

Consider the second assertion in Theorem 2 :
Let, for example, $i_0 = s$, i.e.

$$\nu_{P_\infty}(k_{s,j}) \geq \eta_2 j \quad \text{for} \quad j \geq m/2 - t, \quad \text{and} \quad \eta_2 \in (0, 1). \tag{4.44}$$

From (1.4), Lemma 4 and Theorem J, we get that $(\mathbf{x}(n))_{0\leq n<b^m}$ is a $d-$admissible digital (t, m, s)-net with $d = g + e_0$ and $t = g + e_0 - s$.

We apply Corollary 2 with $\dot{s} = s \geq 3$, $B_i = \emptyset$, $1 \leq i \leq s$, $B = 0$, $\tilde{r} = 0$, $m = \tilde{m}$, $\hat{e} = e = e_1e_2 \cdots e_s$, $d_0 = d + t$, $t = g + e_0 - s$ and $e_0 = e_1 + ... + e_s$. In order to prove the second assertion in Theorem 2, it is sufficient to verify that

$$\Lambda_2 = \mathbb{F}_b^{sd_0e[m\epsilon]} \quad \text{for} \quad m \geq 8(d + t)e(s - 1)^2\eta_2^{-1} + 2(1 + 2g + \eta_2 t)\eta_2^{-1}(1 - \eta_2)^{-1}, \tag{4.45}$$

where

$$\Lambda_2 = \{(y_{n,1}^{(1)}, ..., y_{n,d_1}^{(1)}, ..., y_{n,1}^{(s-1)}, ..., y_{n,d_{s-1}}^{(s-1)}, y_{n,d_{s,1}}^{(s)}, ..., y_{n,d_{s,2}}^{(s)}) \mid n \in [0, b^m)\}$$

with

$$d_i = \dot{m}_i := d_0e[m\epsilon],\ i \in [1, s),\ \ d_{s,1} = \ddot{m}_s + 1 := m - t + 1 - (s - 1)d_0e[m\epsilon], \tag{4.46}$$

$d_{s,2} = \dot{m}_s := m - t - (s - 2)d_0e[m\epsilon]$, and $\epsilon = \eta_2(2(s - 1)d_0e)^{-1}$.

Suppose that (4.45) is not true. Then there exists $b_{i,j} \in \mathbb{F}_b$ $(i, j \geq 1)$ such that

$$\sum_{i=1}^{s-1}\sum_{j=1}^{d_i} |b_{i,j}| + \sum_{j=d_{s,1}}^{d_{s,2}} |b_{s,j}| > 0 \tag{4.47}$$

and

$$\sum_{i=1}^{s-1}\sum_{j=1}^{d_i} b_{i,j} y_{n,j}^{(i)} + \sum_{j=d_{s,1}}^{d_{s,2}} b_{s,j} y_{n,j}^{(s)} = 0 \quad \text{for all} \quad n \in [0, b^m).$$

Similarly to (4.36), we get

$$\operatorname*{Res}_{P_\infty,z}(f_n \alpha_1) = 0 \quad \text{for all} \quad n \in [0, b^m), \text{with} \quad \alpha_1 = \alpha_2 - \beta_1$$

where $\alpha_2 = \beta_1 + \beta_2 + \beta_3$, with

(4.48) $$\beta_1 = \sum_{u=0}^{g} b_{0,u} w_u, \quad \beta_2 = \sum_{i=1}^{s-1}\sum_{j=1}^{d_i} b_{i,j} k_{i,j} \quad \text{and} \quad \beta_3 = \sum_{j=d_{s,1}}^{d_{s,2}} b_{s,j} k_{s,j}$$

and $b_{0,u} = -\sum_{i=1}^{s-1}\sum_{j=1}^{d_i} b_{i,j} a_{j,n_u}^{(i)} - \sum_{j=d_{s_1}}^{d_{s_2}} b_{s,j} a_{j,n_u}^{(s)}$. Consider the local expansions

$$\beta_1 + \beta_2 = \sum_{r=0}^{\infty} \dot{\varphi}_r z_r \quad \text{and} \quad \beta_3 = \sum_{r=0}^{\infty} \ddot{\varphi}_r z_r \quad \text{with} \quad \varphi_{i,r} \in \mathbb{F}_b \quad i = 1, 2, \ r \geq 0.$$

Analogously to (4.42), we obtain

(4.49) $$\dot{\varphi}_r + \ddot{\varphi}_r = 0 \quad \text{for} \quad r \in [0, m+g].$$

Using (4.44), (4.46) and (4.48), we get

$$\nu_{P_\infty}(k_{s,j}) \geq \eta_2 j \quad \text{for} \quad j \geq d_{s,1} \geq m/2 - t, \quad \text{and} \quad \ddot{\varphi}_r = 0 \text{ for } r \leq [\eta_2 d_{s,1}] - 1.$$

Therefore $\dot{\varphi}_r = 0$ for $r \leq [\eta_2 d_{s,1}] - 1$. Hence

$$\nu_{P_\infty}(\beta_1 + \beta_2) \geq [\eta_2 d_{s,1}].$$

By (4.48), we obtain

$$\beta_1 + \beta_2 \in \mathcal{L}(G_2) \quad \text{with} \quad G_2 = G + \sum_{i=1}^{s-1} [(d_i - 1)/e_i + 1] P_i - [\eta_2 d_{s,1}] P_\infty.$$

According to (4.45) and (4.46), we have

$$\deg(G_2) = 2g + \sum_{i=1}^{s-1} d_i - [\eta_2 d_{s,1}] = 2g + (s-1) d_0 e [m\epsilon] - [\eta_2 (m - t + 1 - (s-1) d_0 e [m\epsilon])]$$
$$\leq 2g + (s-1) d_0 e [m\epsilon] - \eta_2 (m - t + 1 - (s-1) d_0 e [m\epsilon]) + 1 = (1+\eta_2)(s-1) d_0 e [m\epsilon]$$
$$-m\eta_2 + 2g + 1 + \eta_2 (t-1) \leq m\eta_2((1+\eta_2)/2 - 1) + 1 + 2g + \eta_2 t < 0$$

for $m > 2(1 + 2g + \eta_2 t)\eta_2^{-1}(1-\eta_2)^{-1}$. Hence $\beta_1 + \beta_2 = 0$.

By [DiPi, Lemma 8.10] (or Lemma 5), we get that $b_{i,j} = 0$ for all $j \in [1, d_i]$, $i \in [1, s-1]$ and $b_{0,j} = 0$ for $j \in [0, g]$.
From (4.49) we have $\ddot{\varphi}_r = 0$ for $r \in [0, m+g]$. Thus $\nu_{P_\infty}(\beta_3) \geq m + g + 1$.
Applying (4.48), we derive

$$\beta_3 \in \mathcal{L}(G_3) \quad \text{with} \quad G_3 = G + [(d_{s,2} - 1)/e_s + 1] P_s - (m + g + 1) P_\infty.$$

By (4.46), we obtain

$$\deg(G_3) = 2g + m - t - (s-2)d_0e[m\epsilon] + e_s - m - g - 1 \leq g - t - 1 + e_s - (s-2)d_0e[m\epsilon] < 0$$

for $m \geq \epsilon^{-1}$ and $s \geq 3$. Hence $\beta_3 = 0$. Using (3.2) and (4.48), we get that $b_{s,j} = 0$ for all $j \in [d_{s,1}, d_{s,2}]$.

By (4.47), we have a contradiction. Thus assertions (4.45) and (3.9) are true. Therefore Theorem 2 is proved. □

4.3. **Niederreiter-Özbudak nets. Proof of Theorem 3.** Let

$$m = m_i e_i + r_i, \quad \text{with} \quad 0 \leq r_i < e_i,\ 1 \leq i \leq s \text{ and } \tilde{r}_0 = \sum_{i=1}^{s-1} r_i, \quad r_0 = \sum_{i=1}^{s} r_i. \tag{4.50}$$

Lemma 7. *There exists a divisor $\tilde{G}$ of $F/\mathbb{F}_b$ with $\deg(\tilde{G}) = g - 1 + \tilde{r}_0$, such that $\nu_{P_i}(\tilde{G}) = 0$ for $1 \leq i \leq s$, and*

$$\mathcal{N}_m(P_1, ..., P_s; G) = \mathcal{N}_m(P_1, ..., P_s; \hat{G}), \quad \text{where} \quad \hat{G} = m_1P_1 + ... + m_{s-1}P_{s-1} + \tilde{G}.$$

Proof. We have $\nu_{P_i}(G) = a_i$ and $\nu_{P_i}(t_i) = 1$ for $1 \leq i \leq s$. Using the Approximation Theorem, we obtain that there exists $y \in F$, such that

$$\nu_{P_i}(y - t_i^{a_i - m_i}) = a_i + 1, \quad \text{for} \quad 1 \leq i \leq s-1, \quad \nu_{P_s}(y - t_s^{a_s}) = a_s + m_s + 1. \tag{4.51}$$

Let $\dot{f} = fy$ and $\hat{G} = G - \mathrm{div}(y)$. We note

$$f \in \mathcal{L}(G) \Leftrightarrow \mathrm{div}(f) + G \geq 0 \Leftrightarrow \mathrm{div}(fy) + G - \mathrm{div}(y) \geq 0 \Leftrightarrow \dot{f} = fy \in \mathcal{L}(\hat{G}). \tag{4.52}$$

It is easy to see that $\nu_{P_i}(\hat{G}) = m_i$ $(1 \leq i \leq s-1)$, $\nu_{P_s}(\hat{G}) = 0$ and $\deg(\hat{G}) = \deg(G) = m(s-1) + g - 1$. Let $\tilde{G} = \hat{G} - m_1P_1 - ... - m_{s-1}P_{s-1}$. We get $\nu_{P_i}(\tilde{G}) = 0$ for $1 \leq i \leq s$. Hence

$$\deg(\tilde{G}) = m(s-1) + g - 1 - e_1m_1 - ... - e_{s-1}m_{s-1} = g - 1 + \tilde{r}_0.$$

Let $\dot{f}_{i,j} = S_j(t_i, \dot{f})$ (see (3.10)). By (4.51), we have

$$\dot{f}_{i,-j} = f_{i,-a_i+m_i-j} \ \ 1 \leq i \leq s-1, \quad \text{and} \quad \dot{f}_{s,m_s-j} = f_{s,-a_s+m_s-j} \ \text{ with } \ 1 \leq j \leq m_s.$$

Using notations (3.11), we get

$$\theta_i^{(\hat{G})}(\dot{f}) = (\mathbf{0}_{r_i}, \vartheta_i(\dot{f}_{i,-1}), ..., \vartheta_i(\dot{f}_{i,-m_i})) = (\mathbf{0}_{r_i}, \vartheta_i(f_{i,-a_i+m_i-1}), ..., \vartheta_i(f_{i,-a_i})) = \theta_i^{(G)}(f)$$

for $1 \leq i \leq s-1$, and

$$\theta_s^{(\hat{G})}(\dot{f}) = (\mathbf{0}_{r_s}, \vartheta_s(\dot{f}_{s,m_s-1}), ..., \vartheta_s(\dot{f}_{s,0})) = (\mathbf{0}_{r_s}, \vartheta_s(f_{s,-a_s+m_s-1}), ..., \vartheta_s(f_{s,-a_s})) =$$

$\theta_s^{(G)}(f)$. By (3.12), we have

$$\theta^{(\hat{G})}(\dot{f}) := (\theta_1^{(\hat{G})}(\dot{f}), ..., \theta_s^{(\hat{G})}(\dot{f})) = (\theta_1^{(G)}(f), ..., \theta_s^{(G)}(f)) = \theta^{(G)}(f)$$

for all $f \in \mathcal{L}(G)$. From (3.13) and (4.52) , we obtain the assertion of Lemma 7. $\square$

By Lemma 7, we can take $\hat{G}$ instead of G. Hence

$$G = m_1 P_1 + ... + m_{s-1} P_{s-1} + \tilde{G}, \quad \text{and} \quad a_i = m_i, \ 1 \le i \le s-1, \quad a_s = 0. \tag{4.53}$$

Let $\vartheta_i = (\vartheta_{i,1}, ..., \vartheta_{i,e_i})$. From (3.11), we get for $0 \le \check{j}_i \le m_i - 1$, $1 \le \hat{j}_i \le e_i$, that

$$\theta_i^{(G)}(f) = (\theta_{i,1}(f), ..., \theta_{i,m}(f)) = (\mathbf{0}_{r_i}, \vartheta_i(f_{i,-1}), ..., \vartheta_i(f_{i,-m_i})), \ 1 \le i \le s-1,$$

with $\theta_{i,r_i+\check{j}_i e_i+\hat{j}_i}(f) = \vartheta_{i,\hat{j}_i}(f_{i,-\check{j}_i-1})$, and

$$\theta_s^{(G)}(f) = (\theta_{s,1}(f), ..., \theta_{s,m}(f)) = (\mathbf{0}_{r_s}, \vartheta_s(f_{s,m_s-1}), ..., \vartheta_s(f_{s,0})), \tag{4.54}$$

with $\theta_{s,r_s+\check{j}_s e_s+\hat{j}_i}(f) = \vartheta_{s,\hat{j}_s}(f_{s,m_s-\check{j}_s-1})$.

Lemma 8. *Let $\vartheta_i = (\vartheta_{i,1}, ..., \vartheta_{i,e_i}) \ : \ F_{P_i} \to \mathbb{F}_b^{e_i}$ be an $\mathbb{F}_b$-linear vector space isomorphism. Then there exists an $\mathbb{F}_b$-linear vector space isomorphism $\vartheta_i^{\perp} = (\vartheta_{i,1}^{\perp}, ..., \vartheta_{i,e_i}^{\perp})$: $F_{P_i} \to \mathbb{F}_b^{e_i}$ such that*

$$\mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(\dot{x}\ddot{x}) = \sum_{j=1}^{e_i} \vartheta_{i,j}(\dot{x})\vartheta_{i,j}^{\perp}(\ddot{x}) \quad \textit{for all} \quad \dot{x}, \ddot{x} \in F_{P_i}, \quad 1 \le i \le s.$$

Proof. Using Theorem F, we get that there exists $\beta_{i,j} \in F_{P_i}$ such that

$$\vartheta_{i,j}(y) = \mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(y\beta_{i,j}) \quad \text{for} \quad 1 \le j \le e_i, \tag{4.55}$$

and $(\beta_{i,1}, ..., \beta_{i,e_i})$ is the basis of F_{P_i} over $\mathbb{F}_b$ ($1 \le i \le s$). Applying Theorem G, we obtain that there exists a basis $(\beta_{i,1}^{\perp}, ..., \beta_{i,e_i}^{\perp})$ of F_{P_i} over $\mathbb{F}_b$ such that

$$\mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(\beta_{i,j_1}\beta_{i,j_2}^{\perp}) = \delta_{j_1,j_2} \quad \text{with} \quad 1 \le j_1, j_2 \le e_i.$$

Let $\dot{x} = \sum_{j=1}^{e_i} \dot{\gamma}_j \beta_{i,j}^{\perp}$, $\ddot{x} = \sum_{j=1}^{e_i} \ddot{\gamma}_j \beta_{i,j}$ and let

$$\vartheta_{i,j}^{\perp}(\ddot{x}) := \ddot{\gamma}_j = \mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(\ddot{x}\beta_{i,j}^{\perp}). \tag{4.56}$$

By (4.55), we have $\dot{\gamma}_j = \vartheta_{i,j}(\dot{x})$. Now, we get

$$\mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(\dot{x}\ddot{x}) = \sum_{j_1,j_2=1}^{e_i} \dot{\gamma}_{j_1}\ddot{\gamma}_{j_2}\mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(\beta_{i,j_1}^{\perp}\beta_{i,j_2}) = \sum_{j=1}^{e_i} \dot{\gamma}_j\ddot{\gamma}_j = \sum_{j=1}^{e_i} \vartheta_{i,j}(\dot{x})\vartheta_{i,j}^{\perp}(\ddot{x}).$$

Hence Lemma 8 is proved. $\square$

We consider the H-differential dt_s. Let ω be the corresponding Weil differential, $\mathrm{div}(\omega)$ the divisor of ω, and $W := \mathrm{div}(dt_s) = \mathrm{div}(\omega)$. By (2.4) and (2.6), we have

$$\deg(W) = 2g - 2 \quad \text{and} \quad \nu_{P_s}(W) = \nu_{P_s}(dt_s) = \nu_{P_s}(dt_s/dt_s) = 0. \tag{4.57}$$

Using notations of Lemma 7, we define

$$G^{\perp} = m_s P_s - \tilde{G} + W, \quad \text{where} \quad \deg(\tilde{G}) = g - 1 + \tilde{r}_0 \quad \text{and} \quad \nu_{P_i}(\tilde{G}) = 0 \tag{4.58}$$

for $1 \leq i \leq s$. Let $a_i^{\perp} := \nu_{P_i}(G^{\perp} - W)$ for $1 \leq i \leq s$. We obtain from (4.58) that $a_i^{\perp} = 0$ for $1 \leq i \leq s-1$ and $a_s^{\perp} = m_s$. Let $f^{\perp} \in \mathcal{L}(G^{\perp})$, then $\operatorname{div}(f^{\perp}) + W + G^{\perp} - W \geq 0$ and $\nu_{P_i}(\operatorname{div}(f^{\perp}) + W) \geq -\nu_{P_i}(G^{\perp} - W)$. Applying (2.6), we get

$$\nu_{P_i}(f^{\perp}\mathrm{d}t_s) = \nu_{P_i}(f^{\perp}) + \nu_{P_i}(W) \geq -\nu_{P_i}(G^{\perp} - W) = -a_i^{\perp}, \text{ with } a_i^{\perp} = 0, \tag{4.59}$$

$1 \leq i \leq s-1$, and $a_s^{\perp} = m_s$ for $f^{\perp} \in \mathcal{L}(G^{\perp})$. According to Proposition A, we have that there exists $\tau_i \in F$, such that

$$\mathrm{d}t_s = \tau_i \mathrm{d}t_i, \qquad 1 \leq i \leq s. \tag{4.60}$$

From (2.4) and (4.59), we get

$$\nu_{P_i}(f^{\perp}\tau_i) = \nu_{P_i}(f^{\perp}\tau_i \mathrm{d}t_i) = \nu_{P_i}(f^{\perp}\mathrm{d}t_s) \geq -a_i^{\perp}, \qquad 1 \leq i \leq s.$$

By (2.2), we have the local expansions

$$f^{\perp}\tau_i := \sum_{j=-a_i^{\perp}}^{\infty} S_j(t_i, f^{\perp}\tau_i) t_i^j, \quad \text{where all} \quad S_j(t_i, f^{\perp}\tau_i) \in F_{P_i} \tag{4.61}$$

for $1 \leq i \leq s$ and $f^{\perp} \in \mathcal{L}(G^{\perp})$. We denote $S_j(t_i, f^{\perp}\tau_i)$ by $f_{i,j}^{\perp}$.

Using (2.7), (2.8) and (4.56), we denote

$$\vartheta_{i,\hat{j}_i}^{\perp}(f_{i,\check{j}_i}^{\perp}) := \operatorname{Tr}_{F_{P_i}/\mathbb{F}_b}(\beta_{i,\hat{j}_i}^{\perp} f_{i,\check{j}_i}^{\perp}) = \operatorname*{Res}_{P_i,t_i}(\beta_{i,\hat{j}_i}^{\perp} t_i^{-\check{j}_i-1} f^{\perp}\tau_i) \tag{4.62}$$

and $\vartheta_i^{\perp} = (\vartheta_{i,1}^{\perp}, ..., \vartheta_{i,e_i}^{\perp})$ with $1 \leq \hat{j}_i \leq e_i$, $-a_i^{\perp} \leq \check{j}_i \leq -a_i^{\perp} + m_i - 1$, $1 \leq i \leq s$.
For $f^{\perp} \in \mathcal{L}(G^{\perp})$, the image of $f^{\perp}$ under $\dot{\theta}_i^{\perp}$, for $1 \leq i \leq s$, is defined as

$$\dot{\theta}_i^{\perp}(f^{\perp}) = (\dot{\theta}_{i,1}^{\perp}(f^{\perp}), ..., \dot{\theta}_{i,m}^{\perp}(f^{\perp})) := (\vartheta_i^{\perp}(f_{i,-a_i^{\perp}}^{\perp}), ..., \vartheta_i^{\perp}(f_{i,-a_i^{\perp}+m_i-1}^{\perp}), \mathbf{0}_{r_i}) \in \mathbb{F}_b^m,$$

It is easy to verify that

$$\dot{\theta}_{i,\check{j}_i e_i + \hat{j}_i}^{\perp}(f^{\perp}) = \vartheta_{i,\hat{j}_i}^{\perp}(f_{i,\check{j}_i}^{\perp}), \quad \text{for} \quad 1 \leq \hat{j}_i \leq e_i,\ 0 \leq \check{j}_i \leq m_i - 1, \tag{4.63}$$

$$1 \leq i \leq s-1 \quad \text{and} \quad \dot{\theta}_{s,\check{j}_s e_s + \hat{j}_s}^{\perp}(f^{\perp}) = \vartheta_{s,\hat{j}_s}^{\perp}(f_{s,-m_s+\check{j}_s}^{\perp}),\ 0 \leq \check{j}_s \leq m_s - 1. \tag{4.64}$$

Let

$$\dot{\theta}^{(G,\perp)}(f^{\perp}) := \big(\dot{\theta}_1^{\perp}(f^{\perp}), ..., \dot{\theta}_s^{\perp}(f^{\perp})\big) \in \mathbb{F}_b^{ms}. \tag{4.65}$$

Let $\boldsymbol{\varphi}_i = (\varphi_{i,1}, ..., \varphi_{i,r_i})$ with $\varphi_{i,j} \in \mathbb{F}_b$ $(1 \leq j \leq r_i,\ 1 \leq i \leq s)$, and let

$$\Phi = \{\boldsymbol{\varphi} = (\boldsymbol{\varphi}_1, ..., \boldsymbol{\varphi}_s) \mid \boldsymbol{\varphi}_i \in \mathbb{F}_b^{r_i},\ i = 1, ..., s\} \text{ with } \dim(\Phi) = r_0 = \sum_{i=1}^{s} r_i. \tag{4.66}$$

Now, we set

$$\theta^{(G,\perp)}(f^{\perp}, \boldsymbol{\varphi}) := \big(\theta_1^{\perp}(f^{\perp}, \boldsymbol{\varphi}), ..., \theta_s^{\perp}(f^{\perp}, \boldsymbol{\varphi})\big) \in \mathbb{F}_b^{ms}, \tag{4.67}$$

where

$$\theta_i^{\perp}(f^{\perp},\boldsymbol{\varphi}) = (\theta_{i,1}^{\perp}(f^{\perp},\boldsymbol{\varphi}),...,\theta_{i,m}^{\perp}(f^{\perp},\boldsymbol{\varphi})) := (\boldsymbol{\varphi}_i,\dot{\theta}_{i,1}^{\perp}(f^{\perp}),...,\dot{\theta}_{i,m-r_i}^{\perp}(f^{\perp})) \in \mathbb{F}_b^m.$$

We define the $\mathbb{F}_b$-linear maps

$$\theta^{(G,\perp)} \ : \ \big(\mathcal{L}(G^{\perp}),\Phi\big) \to \mathbb{F}_b^{ms}, \quad (f^{\perp},\boldsymbol{\varphi}) \mapsto \theta^{(G,\perp)}(f^{\perp},\boldsymbol{\varphi}) \tag{4.68}$$

$$\text{and} \qquad \dot{\theta}^{(G,\perp)} \ : \ \mathcal{L}(G^{\perp}) \to \mathbb{F}_b^{ms}, \quad f^{\perp} \mapsto \dot{\theta}^{(G,\perp)}(f^{\perp}).$$

The images of $\theta^{(G,\perp)}$ and $\dot{\theta}^{(G,\perp)}$ are denoted by

$$\Xi_m := \{\theta^{(G,\perp)}(f^{\perp},\boldsymbol{\varphi}) \mid f^{\perp} \in \mathcal{L}(G^{\perp}),\ \boldsymbol{\varphi} \in \Phi\} \tag{4.69}$$

$$\text{and} \qquad \dot{\Xi}_m := \{\dot{\theta}^{(G,\perp)}(f^{\perp}) \mid f^{\perp} \in \mathcal{L}(G^{\perp})\}.$$

Lemma 9 *With notation as above, we have* $\ker(\theta^{(G,\perp)}) = \mathbf{0}$ *and*

$$\delta_m^{\perp}(\dot{\Xi}_m) \le m + g - 1 + e_0 - r_0.$$

Proof. Consider (4.57)-(4.60). Let $f^{\perp} \in \mathcal{L}(G^{\perp}) \setminus \{0\}$, and let

$$\nu_{P_i}(f^{\perp}\tau_i) = d_i \quad \text{for} \quad 1 \le i \le s-1, \qquad \nu_{P_s}(f^{\perp}) = d_s - m_s. \tag{4.70}$$

We see that

$$\operatorname{div}(f^{\perp}) + G^{\perp} \ge 0, \quad \text{with} \quad G^{\perp} = m_s P_s - \tilde{G} + W \quad \text{and} \quad W = (\mathrm{d}t_s). \tag{4.71}$$

Hence

$$\nu_P\big(\operatorname{div}(f^{\perp}) + m_s P_s - \tilde{G} + W\big) \ge 0, \quad \text{for all} \quad P \in \mathbb{P}_F. \tag{4.72}$$

By (2.4) and (2.6), we obtain $\nu_{P_i}(W) = \nu_{P_i}(dt_s) = \nu_{P_i}(\tau_i)$, $1 \le i \le s$.
Bearing in mind (4.70) and that $\nu_{P_i}(\tilde{G}) = 0$ for $i \in [1,s]$, we get

$$\nu_{P_i}(\operatorname{div}(f^{\perp}) + m_s P_s - \tilde{G} + W) = d_i \ge 0, \qquad 1 \le i \le s.$$

Therefore

$$\nu_{P_i}(\operatorname{div}(f^{\perp}) + \dot{G}) \ge 0 \ \text{ for } \ f^{\perp} \in \mathcal{L}(G^{\perp}) \setminus \{0\}, \text{ where } \dot{G} = G^{\perp} - \sum_{i=1}^{s} d_i P_i$$

and $G^{\perp} = m_s P_s - \tilde{G} + W$. Taking into account that $f^{\perp} \in \mathcal{L}(G^{\perp}) \setminus \{0\}$, we obtain

$$0 \le \deg(\dot{G}) = \deg\Big(G^{\perp} - \sum_{i=1}^{s} d_i P_i\Big) = \deg(G^{\perp}) - \sum_{i=1}^{s} d_i e_i.$$

By (4.57), (4.58) and (4.50), we get

$$\sum_{i=1}^{s} d_i e_i \le \deg(m_s P_s - \tilde{G} + W) = m_s e_s - (g - 1 + \tilde{r}_0) + 2g - 2 = m - r_0 + g - 1.$$

According to (4.61), (4.62) and (4.70), we obtain

$$f^{\perp}_{i,a^{\perp}_i+j}=0 \quad \text{for} \quad 0\le j<d_i \quad \text{and} \quad f^{\perp}_{i,a^{\perp}_i+d_i}\neq 0, \quad 1\le i\le s.$$

From (2.22), (4.64) and Lemma 8, we have

$$v^{\perp}_m(\dot{\theta}^{\perp}_i(f^{\perp}))\le (d_i+1)e_i \quad \text{for} \quad 1\le i\le s.$$

Applying (4.65) and (2.23), we derive

$$V^{\perp}_m\big(\dot{\theta}^{(G,\perp)}(f^{\perp})\big)\le \sum_{i=1}^{s}(d_i+1)e_i\le m+g-1+e_0-r_0.$$

By (2.24), $\delta^{\perp}_m(\dot{\Xi}_m)\le m+g-1+e_0-r_0$. Taking into account (2.22) and that $s\ge 3$, we get $\ker(\theta^{(G,\perp)})=\mathbf{0}$.

Therefore Lemma 9 is proved. □

Lemma 10. *With notation as above, we have that* $\dim(\Xi_m)=m$.

Proof. By (4.57) and (4.58), we have

$$\deg(G^{\perp})=\deg(m_sP_s-\tilde{G}+W)=m_se_s-\deg(\tilde{G})+2g-2=m-r_s+2g-2-\tilde{r}_0-g+1.$$

Using (4.50) and the Riemann-Roch theorem, we obtain for $m\ge g+e_0-1\ge g+r_0$ that

$$\dim(\mathcal{L}(G^{\perp}))=\deg(m_sP_s-\tilde{G}+W)-g+1=m-r_0+2g-2-2g+2=m-r_0.$$

From (4.66), we have $\dim(\Phi)=r_0$. Hence

$$\dim\big((\mathcal{L}(G^{\perp}),\Phi)\big)=\dim(\mathcal{L}(G^{\perp}))+\dim(\Phi)=m-r_0+r_0=m.$$

By Lemma 9, we get $\ker(\theta^{(G,\perp)})=\mathbf{0}$. Bearing in mind that $\theta^{(G,\perp)}\big((\mathcal{L}(G^{\perp}),\Phi)\big)=\Xi_m$, we obtain the assertion of Lemma 10. □

Lemma 11. *Let* $f\in\mathcal{L}(G)$, *and* $f^{\perp}\in\mathcal{L}(G^{\perp})$. *Then*

$$\sum_{i=1}^{s}\operatorname*{Res}_{P_i}(ff^{\perp}\mathrm{d}t_s)=0, \tag{4.73}$$

$$\operatorname*{Res}_{P_i}(ff^{\perp}\mathrm{d}t_s)=\sum_{j=0}^{m_i-1}\mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}\big(f_{i,-j-1}\,f^{\perp}_{i,j}\big), \qquad 1\le i\le s-1 \tag{4.74}$$

$$\text{and} \quad \operatorname*{Res}_{P_s}(ff^{\perp}\mathrm{d}t_s)=\sum_{j=0}^{m_s-1}\mathrm{Tr}_{F_{P_s}/\mathbb{F}_b}\big(f_{s,m_s-j-1}\,f^{\perp}_{s,-m_s+j}\big). \tag{4.75}$$

Proof. By (4.53) and (4.58), we have

$$G=m_1P_1+...+m_{s-1}P_{s-1}+\tilde{G}, \quad \text{and} \quad G^{\perp}=m_sP_s-\tilde{G}+W.$$

Bearing in mind that $\mathrm{div}(f)+G \geq 0$, $\mathrm{div}(f^{\perp})+G^{\perp} \geq 0$ and that $W=\mathrm{div}(\mathrm{d}t_s)$, we obtain

$$\mathrm{div}(f)+\sum_{i=1}^{s} m_i P_i + \tilde{G} + \mathrm{div}(f^{\perp}) - \tilde{G} + W = \mathrm{div}(f)+\mathrm{div}(f^{\perp})+\sum_{i=1}^{s} m_i P_i + \mathrm{div}(\mathrm{d}t_s) \geq 0.$$

From (2.6), we derive

$$\nu_P(ff^{\perp}\mathrm{d}t_s) = \nu_P(ff^{\perp}) + \nu_P(\mathrm{div}(\mathrm{d}t_s)) \geq 0 \quad \text{and} \quad \operatorname*{Res}_P(ff^{\perp}\mathrm{d}t_s)=0$$

for all $P \in \mathbb{P}_f \setminus \{P_1,...,P_s\}$.
Applying the Residue Theorem, we get assertion (4.73).
By (3.10) and (4.61), we derive

$$\begin{aligned}
\operatorname*{Res}_{P_s}(ff^{\perp}\mathrm{d}t_s) &= \operatorname*{Res}_{P_s}\Big(\sum_{j_1=0}^{\infty} S_{j_1}(t_s,f)t_s^{j_1} \sum_{j_2=-m_s}^{\infty} S_{j_2}(t_s,f^{\perp})t_s^{j_2}\mathrm{d}t_s\Big) \\
&= \sum_{j_1=0}^{\infty}\sum_{j_2=-m_s}^{\infty} \operatorname*{Res}_{P_s}\big(S_{j_1}(t_s,f)\ S_{j_2}(t_s,f^{\perp})t_s^{j_1+j_2}\mathrm{d}t_s\big) \\
&= \sum_{0\leq j_1\leq m_s-1,\ j_1+j_2=-1} \mathrm{Tr}_{F_{P_s}/\mathbb{F}_b}\big(S_{j_1}(t_s,f)\ S_{j_2}(t_s,f^{\perp})\big)
\end{aligned}$$

$$= \sum_{j=0}^{m_s-1} \mathrm{Tr}_{F_{P_s}/\mathbb{F}_b}\big(S_{m_s-j-1}(t_s,f)\ S_{-m_s+j}(t_s,f^{\perp})\big) = \sum_{j=0}^{m_s-1} \mathrm{Tr}_{F_{P_s}/\mathbb{F}_b}\big(f_{s,m_s-j-1}\ f^{\perp}_{s,-m_s+j}\big).$$

Hence assertion (4.75) is proved.
Analogously, using (4.60), we have

$$\begin{aligned}
\operatorname*{Res}_{P_i}(ff^{\perp}\mathrm{d}t_s) = \operatorname*{Res}_{P_i}(ff^{\perp}\tau_i\mathrm{d}t_i) &= \operatorname*{Res}_{P_i}\Big(\sum_{j_1=-m_i}^{\infty} S_{j_1}(t_i,f)t_i^{j_1} \sum_{j_2=0}^{\infty} S_{j_2}(t_i,f^{\perp}\tau_i)t_i^{j_2}\mathrm{d}t_i\Big) \\
&= \sum_{0\leq j_2\leq m_i-1,\ j_1+j_2=-1} \mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}\big(S_{j_1}(t_i,f)\ S_{j_2}(t_i,f^{\perp}\tau_i)\big), \\
&= \sum_{j=0}^{m_i-1} \mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}\big(f_{i,-j-1}\ f^{\perp}_{i,j}\big), \quad \text{for} \quad 1 \leq i \leq s-1.
\end{aligned}$$

Thus Lemma 11 is proved. □

Lemma 12. *With notation as above, we have* $\Xi_m = \mathcal{N}^{\perp}(P_1,...,P_s,G)$.

Proof. Using (3.14) and Lemma 10, we have

$$\dim_{\mathbb{F}_b}(\mathcal{N}_m) = ms - m \quad \text{and} \quad \dim_{\mathbb{F}_b}(\Xi_m) = m.$$

From (3.13), (4.68) and (4.69), we get that $\mathcal{N}_m, \Xi_m \subset \mathbb{F}_b^{ms}$.

By (2.19), in order to obtain the assertion of the lemma, it is sufficient to prove that $A \cdot B = 0$ for all $A \in \mathcal{N}_m$ and $B \in \Xi_m$.

According to (3.11), (3.13), (4.54) and (4.64) - (4.69), it is enough to verify that

$$A \cdot B = \sum_{i=1}^{s} \eth_i = 0 \quad \text{with} \quad \eth_i = \sum_{j=1}^{m} \theta_{i,j}(f)\theta^{\perp}_{i,j}((f^{\perp},\boldsymbol{\varphi})) \quad \text{for all} \quad f \in \mathcal{L}(G), \tag{4.76}$$

and $(f^{\perp},\boldsymbol{\varphi}) \in (\mathcal{L}(G^{\perp}),\Phi)$. From (4.54) and (4.62) - (4.64), we derive

$$\eth_i = \sum_{\check{j}_i=0}^{m_i-1} \varkappa_{i,j_1} \quad \text{with} \quad \varkappa_{i,\check{j}_i} = \sum_{\hat{j}_i=1}^{e_i} \theta_{i,r_i+\check{j}_i e_i+\hat{j}_i}(f) \;\; \theta^{\perp}_{i,r_i+\check{j}_i e_i+\hat{j}_i}((f^{\perp},\boldsymbol{\varphi})). \tag{4.77}$$

Using (4.54) and (4.64)-(4.67), we have for $\check{j}_i \in [0, m_i-1]$, $\hat{j}_i \in [1, e_i]$

$$\theta_{s,r_s+\check{j}_s e_s+\hat{j}_s}(f) = \vartheta_{s,\hat{j}_s}(f_{s,m_s-\check{j}_s-1}) \quad \text{and} \quad \theta^{\perp}_{s,r_s+\check{j}_s e_s+\hat{j}_s}((f^{\perp},\boldsymbol{\varphi})) = \vartheta^{\perp}_{s,\hat{j}_s}(f^{\perp}_{s,-m_s+\check{j}_s}),$$

$$\theta_{i,r_i+\check{j}_i e_i+\hat{j}_i}(f) = \vartheta_{i,\hat{j}_i}(f_{i,-\check{j}_i-1}) \;\text{ and }\; \theta^{\perp}_{i,r_1+\check{j}_i e_i+\hat{j}_i}((f^{\perp},\boldsymbol{\varphi})) = \vartheta^{\perp}_{i,\hat{j}_i}(f^{\perp}_{i,\check{j}_i}),\; 1 \leq i \leq s-1.$$

By Lemma 8 and (4.77), we obtain

$$\varkappa_{s,\check{j}_s} = \sum_{\hat{j}_i=s}^{e_s} \vartheta_{s,\hat{j}_s}(f_{s,m_s-\check{j}_s-1}) \; \vartheta^{\perp}_{s,\hat{j}_s}(f^{\perp}_{s,-m_s+\check{j}_s}) = \mathrm{Tr}_{F_{P_s}/\mathbb{F}_b}\big(f_{s,m_s-\check{j}_s-1}\; f^{\perp}_{s,-m_s+\check{j}_s}\big)$$

and

$$\varkappa_{i,\check{j}_i} = \sum_{\hat{j}_i=1}^{e_i} \vartheta_{i,\hat{j}_i}(f_{i,-\check{j}_i-1}) \; \vartheta^{\perp}_{i,\hat{j}_i}(f^{\perp}_{i,\check{j}_i}) = \mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}\big(f_{i,-\check{j}_i-1} f^{\perp}_{i,\check{j}_i}\big) \quad \text{for} \quad 1 \leq i \leq s-1.$$

From (4.74), (4.75) and (4.77), we get

$$\eth_i = \operatorname*{Res}_{P_i}(f f^{\perp} \mathrm{d}t_s) \quad \text{for} \quad 1 \leq i \leq s.$$

Applying Lemma 11, we get assertion (4.76). Hence Lemma 12 is proved. □

Let

$$G_i = \tilde{G} + q_i P_i - q_s P_s \text{ with } q_s = [\frac{g+\tilde{r}_0}{e_s}] + 1 \quad \text{and} \quad q_i = [\frac{g-\tilde{r}_0+q_s e_s}{e_i}] + 1 \tag{4.78}$$

for $i \in [1, s-1]$. By (4.58), we have $\deg(\tilde{G}) = g - 1 + \tilde{r}_0$ and $\nu_{P_i}(\tilde{G}) = 0$, $i \in [1, s]$. It is easy to see that $\deg(G_i) \geq 2g-1$, $i \in [1, s-1]$. Let $z_i = \dim(\mathcal{L}(G_i))$, and let $u^{(i)}_1, ..., u^{(i)}_{z_i}$ be a basis of $\mathcal{L}(G_i)$ over $\mathbb{F}_b$, $i \in [1, s-1]$.

For each $i \in [1, s-1]$, we consider the chain

$$\mathcal{L}(G_i) \subset \mathcal{L}(G_i + P_i) \subset \mathcal{L}(G_i + 2P_i) \subset ...$$

of vector spaces over $\mathbb{F}_b$. By starting from the basis $u^{(i)}_1, ..., u^{(i)}_{z_i}$ of $\mathcal{L}(G_i)$ and successively adding basis vectors at each step of the chain, we obtain for each $n \geq q_i$ a basis

$$\{u^{(i)}_1, ..., u^{(i)}_{z_i}, k^{(i)}_{q_i,1}, ..., k^{(i)}_{q_i,e_i}, ..., k^{(i)}_{n,1}, ..., k^{(i)}_{n,e_i}\} \tag{4.79}$$

of $\mathcal{L}(G_i+(n-q_i+1)P_i)$. We note that we then have

$$k^{(i)}_{j_1,j_2} \in \mathcal{L}(G_i+(j_1-q_i+1)P_i) \text{ and } \nu_{P_i}(k^{(i)}_{j_1,j_2})=-j_1-1,\ \nu_{P_s}(k^{(i)}_{j_1,j_2}) \geq q_s \quad (4.80)$$

for $j_1 \geq q_i$, $1 \leq j_2 \leq e_i$, $1 \leq i \leq s-1$.
Let $\check{G} = \tilde{G} + gP_s$. We see that $\deg(\check{G}) = g-1+\tilde{r}_0+ge_s \geq 2g-1$. Let $u^{(0)}_1,...,u^{(0)}_{z_0}$ be a basis of $\mathcal{L}(\check{G})$ over $\mathbb{F}_b$. In a similar way, we construct a basis
$\{u^{(0)}_1,...,u^{(0)}_{z_0},k^{(i)}_{0,1},...,k^{(i)}_{0,e_i},...,k^{(i)}_{(q_i-1),1},...,k^{(i)}_{(q_i-1),e_i}\}$ of $\mathcal{L}(\check{G}+q_iP_i)$ with

$$k^{(i)}_{j_1,j_2} \in \mathcal{L}(\check{G}+(j_1+1)P_i) \text{ and } \nu_{P_i}(k^{(i)}_{j_1,j_2})=-j_1-1 \ \text{ for } \ j_1\in[0,q_i), \quad (4.81)$$

$1 \leq j_2 \leq e_i$, $1 \leq i \leq s-1$.

Now, consider the chain

$$\mathcal{L}(q_sP_s-\tilde{G}+W) \subset \mathcal{L}((q_s+1)P_s-\tilde{G}+W) \subset ... \subset \mathcal{L}(G^{\perp}-P_s) \subset \mathcal{L}(G^{\perp}),$$

where $G^{\perp}=m_sP_s-\tilde{G}+W$ and $q_s=[(g+\tilde{r}_0)/e_s]+1$. By (4.57) and (4.58), we have $\deg(\tilde{G})=g-1+\tilde{r}_0$, $\deg(W)=2g-2$ and $\nu_{P_s}(\tilde{G})=\nu_{P_s}(W)=0$. Hence $\deg(q_sP_s-\tilde{G}+W)\geq 2g-1$. Let $u^{(s)}_1,...,u^{(s)}_{z_s}$ be a basis of $\mathcal{L}(q_sP_s-\tilde{G}+W)$ over $\mathbb{F}_b$. In a similar way, we construct a basis $\{u^{(s)}_1,...,u^{(s)}_{z_s},k^{(s)}_{q_s,1},...,k^{(s)}_{q_s,e_s},...,k^{(s)}_{n,1},...,k^{(i)}_{n,e_s}\}$ of $\mathcal{L}((n+1)P_s-\check{G}+W)$ with

$$k^{(s)}_{j_1,j_2} \in \mathcal{L}((j_1+1)P_s-\check{G}+W) \ \text{ and } \ \nu_{P_s}(k^{(s)}_{j_1,j_2})=-j_1-1 \ \text{ for } \ j_1\geq q_s \quad (4.82)$$

and $j_2\in[1,e_s]$. By (4.79)-(4.81), we have the following local expansions

$$k^{(i)}_{j_1,j_2} := \sum_{r=-j_1}^{\infty} \varkappa^{(i,j_2)}_{j_1,r} t_i^{r-1} \quad \text{for} \quad \varkappa^{(i,j_2)}_{j_1,r} \in F_{P_i}, \quad i\in[1,s]. \quad (4.83)$$

Lemma 13. *Let $j_i \geq 0$ for $i\in[1,s-1]$ and let $j_s\geq q_s$. Then $\{\varkappa^{(i,1)}_{j_i,-j_i},...,\varkappa^{(i,e_i)}_{j_i,-j_i}\}$ is a basis of F_{P_i} over $\mathbb{F}_b$ for $i\in[1,s]$.*

Proof. Let $i\in[1,s-1]$ and let $j_i\geq q_i$. Suppose that there exist $a_1,...,a_{e_i}\in\mathbb{F}_b$, such that $\sum_{1\leq j\leq e_i} a_i\varkappa^{(i,j)}_{j_i,-j_i}=0$ and $(a_1,...,a_{e_i})\neq(\bar{0},...,\bar{0})$. By (4.83), we get $\nu_{P_i}(\alpha)\geq -j_i$, where $\alpha:=\sum_{1\leq j_2\leq e_i} a_ik^{(i)}_{j_i,j_2}$. Hence $\alpha\in\mathcal{L}(G_i+(j_i-q_i)P_i)$. We have a contradiction with the construction of the basis vectors (4.79).

Similarly, we can consider the cases $i\in[1,s-1]$, $j_i\in[0,q_i-1]$ and $i=s$. Therefore Lemma 13 is proved. □

Lemma 14. *Let $d_i \geq 1$ be an integer $(i = 1, ..., s-1)$ and $f^{\perp} \in G^{\perp}$. Suppose that* $\mathrm{Res}_{P_s,t_s}(f^{\perp} k^{(i)}_{j_1,j_2}) = 0$ *for $j_1 \in [0, d_i - 1], j_2 \in [1, e_i]$ and $i \in [1, s-1]$. Then*

$$\vartheta^{\perp}_{i,j_2}(f^{\perp}_{i,j_1}) = 0 \quad \text{for} \quad j_1 \in [0, d_i - 1],\ j_2 \in [1, e_i] \text{ and } i \in [1, s-1]. \tag{4.84}$$

Proof. By (4.71), (4.72), (4.78), (4.80) and (4.81), we have $\nu_P\big(\mathrm{div}(f^{\perp}) + m_s P_s - \tilde{G} + W\big) \geq 0$, for all $P \in \mathbb{P}_F$ and $k^{(i)}_{j_1,j_2} \in \mathcal{L}(\tilde{G} + a_{j_1} P_s + (j_1+1) P_i)$ with some integer a_{j_1}.
From (2.4), (2.6) and (2.7), we derive

$$\nu_P(f^{\perp} k^{(i)}_{j_1,j_2} \mathrm{d}t_s) \geq 0 \quad \text{and} \quad \mathop{\mathrm{Res}}_{P}(f^{\perp} k^{(i)}_{j_1,j_2} \mathrm{d}t_s) = 0 \quad \text{for all} \quad P \in \mathbb{P}_F \setminus \{P_i, P_s\}.$$

Applying (4.60) and the Residue Theorem, we get

$$\mathop{\mathrm{Res}}_{P_i,t_i}(f^{\perp} \tau_i k^{(i)}_{j_1,j_2}) = \mathop{\mathrm{Res}}_{P_i}(f^{\perp} k^{(i)}_{j_1,j_2} \mathrm{d}t_s) = -\mathop{\mathrm{Res}}_{P_s}(f^{\perp} k^{(i)}_{j_1,j_2} \mathrm{d}t_s) = -\mathop{\mathrm{Res}}_{P_s,t_s}(f^{\perp} k^{(i)}_{j_1,j_2})$$

for all $0 \leq j_1,\ 1 \leq j_2 \leq e_i,\ 1 \leq i \leq s-1$.
By (4.61), (4.83) and the conditions of the lemma, we obtain

$$-\mathop{\mathrm{Res}}_{P_s,t_s}(f^{\perp} k^{(i)}_{j_1,j_2}) = \mathop{\mathrm{Res}}_{P_i,t_i}(f^{\perp} \tau_i k^{(i)}_{j_1,j_2}) = \mathop{\mathrm{Res}}_{P_i,t_i}\Big(\sum_{j=0}^{\infty} f^{\perp}_{i,j} t_i^j \sum_{r=-j_1}^{\infty} \varkappa^{(i,j_2)}_{j_1,r} t_i^{r-1}\Big)$$

$$= \sum_{j=0}^{\infty} \sum_{r=-j_1}^{\infty} \mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(f^{\perp}_{i,j} \varkappa^{(i,j_2)}_{j_1,r}) \delta_{j,-r} = \sum_{j=0}^{j_1} \mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(f^{\perp}_{i,j} \varkappa^{(i,j_2)}_{j_1,-j}) = 0 \tag{4.85}$$

for $0 \leq j_1 \leq d_i - 1,\ 1 \leq j_2 \leq e_i$, and $1 \leq i \leq s-1$.
Consider (4.85) for $j_1 = 0$. We have $\mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(f^{\perp}_{i,0} \varkappa^{(i,j_2)}_{0,0}) = 0$ for all $j_2 \in [1.e_i]$. By Lemma 13, we obtain that $f^{\perp}_{i,0} = 0$. Suppose that $f^{\perp}_{i,j} = 0$ for $0 \leq j < j_0$. Consider (4.85) for $j_1 = j_0$. We get $\mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(f^{\perp}_{i,j_0} \varkappa^{(i,j_2)}_{j_0,-j_0}) = 0$ for all $j_2 \in [1.e_i]$. Applying Lemma 13, we have that $f^{\perp}_{i,j_0} = 0$. By induction, we obtain that $f^{\perp}_{i,j} = 0$ for all $j \in [0, d_i - 1]$ and $i \in [1, s-1]$. Now, using (4.62), we get that assertion (4.84) is true. Hence Lemma 14 is proved. □

Lemma 15. *Let $s \geq 3$, $\{\beta^{\perp}_{s,1}, ..., \beta^{\perp}_{s,e_s}\}$ be a basis of $F_{P_s}/\mathbb{F}_b$,*

$$\Lambda_1 = \Big\{ \big(\mathop{\mathrm{Res}}_{P_s,t_s}(f^{\perp} k^{(i)}_{j_1,j_2})\big)_{d_{i,1} \leq j_1 \leq d_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s-1},$$
$$\big(\mathop{\mathrm{Res}}_{P_s,t_s}(\beta^{\perp}_{s,j_2} f^{\perp} t_s^{m_s - j_1 - 1})\big)_{d_{s,1} \leq j_1 \leq d_{s,2}, 1 \leq j_2 \leq e_s} \mid f^{\perp} \in \mathcal{L}(G^{\perp}) \Big\}$$

with $d_{s,1} = m_s + 1 - [t/e_s] - (s-1) d_0 \dot{m} e/e_s$, $\dot{m} = [\tilde{m}\epsilon]$, $\tilde{m} = m - r_0$,

$$d_{s,2} = m_s - 2 - [t/e_s] - (s-2) d_0 \dot{m} e/e_s, \quad d_{i,1} = q_i,\ d_{i,2} = d_0 \dot{m}] e/e_i - 1, \tag{4.86}$$

$i \in [1, s-1]$, $d_0 = d+t$, $e = e_1 e_2 \cdots e_s$, $\epsilon = \eta(2(s-1)d_0 e)^{-1}$, $\eta = (1 + \deg((t_s)_\infty))^{-1}$. *Then*

$$\Lambda_1 = \mathbb{F}_b^{\chi}, \text{ with } \chi = \sum_{i=1}^{s} (d_{i,2} - d_{i,1} + 1) e_i \textit{ for } m > 2(g-1+e_0)e_s + 2t(\eta^{-1} - 1). \tag{4.87}$$

Proof. Suppose that (4.87) is not true. Then there exists $b^{(i)}_{j_1,j_2} \in \mathbb{F}_b$ $(i, j_1, j_2 \geq 1)$ such that

$$\sum_{i=1}^{s} \sum_{j_1=d_{i,1}}^{d_{i,2}} \sum_{j_2=1}^{e_i} |b^{(i)}_{j_1,j_2}| > 0 \tag{4.88}$$

and

$$\sum_{i=1}^{s-1} \sum_{j_1=d_{i,1}}^{d_{i,2}} \sum_{j_2=1}^{e_i} b^{(i)}_{j_1,j_2} \operatorname*{Res}_{P_s,t_s}(f^{\perp} k^{(i)}_{j_1,j_2}) + \sum_{j_1=d_{s,1}}^{d_{s,2}} \sum_{j_2=1}^{e_s} b^{(s)}_{j_1,j_2} \operatorname*{Res}_{P_s,t_s}(\beta^{\perp}_{s,j_2} f^{\perp} t_s^{m_s - j_1 - 1}) = 0 \tag{4.89}$$

for all $f^{\perp} \in \mathcal{L}(G^{\perp})$. Let $\alpha = \alpha_1 + \alpha_2$ with

$$\alpha_1 = \sum_{i=1}^{s-1} \sum_{j_1=d_{i,1}}^{d_{i,2}} \sum_{j_2=1}^{e_i} b^{(i)}_{j_1,j_2} k^{(i)}_{j_1,j_2} \quad \text{and} \quad \alpha_2 = \sum_{j_1=d_{s,1}}^{d_{s,2}} \sum_{j_2=1}^{e_s} b^{(s)}_{j_1,j_2} \beta^{\perp}_{s,j_2} t_s^{m_s - j_1 - 1}. \tag{4.90}$$

By (4.89), we have

$$\operatorname*{Res}_{P_s,t_s}(f^{\perp}\alpha) = 0 \quad \text{for all} \quad f^{\perp} \in \mathcal{L}(G^{\perp}). \tag{4.91}$$

From (4.80), we get $\nu_{P_s}(\alpha) \geq q_s$. Consider the local expansion

$$\alpha = \sum_{r=q_s}^{\infty} \varphi_r t_s^r \quad \text{with} \quad \varphi_r \in F_{P_s} \quad \text{for} \quad r \geq q_s.$$

Suppose that $m_s > j_0 := \nu_{P_s}(\alpha)$. Therefore $\varphi_{j_0} \neq 0$. From (4.82), we obtain that $k^{(s)}_{j_0,j_2} \in \mathcal{L}(G^{\perp})$ for all $j_2 \in [1, e_s]$. Applying (4.83) and (4.91), we derive

$$\operatorname*{Res}_{P_s,t_s}(k^{(s)}_{j_0,j_2}\alpha) = \operatorname*{Res}_{P_s,t_s}\Big(\sum_{j=-j_0}^{\infty} \varkappa^{(s,j_2)}_{j_0,j} t_s^{j-1} \sum_{r=j_0}^{\infty} \varphi_r t_s^r \Big) = \mathrm{Tr}_{F_{P_s}/\mathbb{F}_b}(\varkappa^{(s,j_2)}_{j_0,-j_0} \varphi_{j_0}) = 0$$

for all $j_2 \in [1, e_s]$. By Lemma 13, $\{\varkappa^{(s,1)}_{j_0,-j_0}, ..., \varkappa^{(s,e_s)}_{j_0,-j_0}\}$ is a basis of F_{P_s}. Hence $\varphi_{j_0} = 0$. We have a contradiction. Thus $\nu_{P_s}(\alpha) \geq m_s$.

We consider the compositum field $F' = F F_{P_s}$. Let $\mathfrak{B}_1, ..., \mathfrak{B}_\mu$ be all the places of F'/F_{P_s} lying over P_s. From (2.11), we get

$$\nu_{\mathfrak{B}_i}(\alpha) \geq m_s \quad \text{for} \quad i = 1, ..., \mu. \tag{4.92}$$

According to (4.78) and (4.80), we obtain

$$\alpha_1 \in \mathcal{L}_F(A_1) = \mathcal{L}(A_1), \quad \text{with} \quad A_1 := \tilde{G} - q_s P_s + \sum_{i=1}^{s-1} (d_{i,2}+1) P_i.$$

Applying Theorem D(d), we have

$$\alpha_1 \in \mathcal{L}_{F'}(\mathrm{Con}_{F'/F}(A_1)).$$

By (4.90), we derive

$$\alpha_2 \in \mathcal{L}_{F'}(A_2), \quad \text{with} \quad A_2 = ((t_s)_\infty^{F'})^{m_s - d_{s,1}-1}.$$

Using (4.92), we get

$$\alpha \in \mathcal{L}_{F'}(A_1 + A_2 - m_s \sum_{i=1}^{\mu} \mathfrak{B}_i).$$

From (2.9), Theorem D(a) and Theorem E, we derive $\mathrm{Con}_{F'/F}(P_s) = \sum_{i=1}^{\mu} \mathfrak{B}_i$, $\mathrm{Con}_{F'/F}((t_s)_\infty^F) = (t_s)_\infty^{F'}$ and

$$\alpha \in \mathcal{L}_{F'}(A_3), \quad \text{with} \quad A_3 = \mathrm{Con}_{F'/F}\big(A_1 + (m_s - d_{s,1} - 1)(t_s)_\infty^F - m_s P_s\big).$$

Applying Theorem D(c) and (4.78), we have

$$\begin{aligned}
&\deg(A_3) = \deg\Big(\tilde{G} + \sum_{i=1}^{s-1}(d_{i,2}+1)P_i + (m_s - d_{s,1} - 1)(t_s)_\infty^F - m_s P_s\Big) \\
&\le g - 1 + \tilde{r}_0 + (s-1)d_0 e\dot{m} + (m_s - d_{s,1} - 1)\deg((t_s)_\infty) - m_s e_s \\
&\le g - 1 + e_0 - e_s + (s-1)d_0 e\dot{m} + ([t/e_s] + (s-1)d_0 \dot{m} e/e_s - 2)(\eta^{-1} - 1) \\
&- m_s e_s \le g - 1 + e_0 + (t/e_s - 2)(\eta^{-1} - 1) + (s-1)d_0 e\dot{m}(1 + (\eta^{-1} - 1)/e_s) - m \\
&\le g - 1 + e_0 + t(\eta^{-1} - 1)/e_s - m\big((2e_s)^{-1} + (1 - \eta/2)(1 - 1/e_s)\big) \le \beta - m/(2e_s) < 0
\end{aligned}$$

for $m > 2e_s\beta$, with $\beta = g - 1 + e_0 + t(\eta^{-1} - 1)/e_s$ and $\epsilon = \eta(2(s-1)d_0 e)^{-1}$. Hence $\alpha = 0$.

Suppose that $\sum_{i=1}^{s-1} \sum_{j_1=d_{i,1}}^{d_{i,2}} \sum_{j_2=1}^{e_i} |b_{j_1,j_2}^{(i)}| = 0$. Then $\alpha_2 = 0$ and $\sum_{j_2=1}^{e_s} b_{j_1,j_2}^{(s)} \beta_{s,j_2}^{\perp} = 0$ for all $j_1 \in [d_{s,1}, d_{s,2}]$. Bearing in mind that $(\beta_{s,j_2}^{\perp})_{1 \le j_2 \le e_2}$ is a basis of $F_{P_s}/\mathbb{F}_b$, we get $\sum_{j_1=d_{s,1}}^{d_{s,2}} \sum_{j_2=1}^{e_s} |b_{j_1,j_2}^{(s)}| = 0$. By (4.88), we have a contradiction.

Therefore there exists $h \in [1, s-1]$ with

$$\sum_{j_1=d_{h,1}}^{d_{h,2}} \sum_{j_2=1}^{e_h} |b_{j_1,j_2}^{(h)}| > 0. \tag{4.93}$$

Let $\mathfrak{B}_{h,1},...,\mathfrak{B}_{h,\mu_h}$ be all the places of F'/F_{P_s} lying over P_h. Let

$$\alpha_{1,i} = \sum_{j_1=d_{i,1}}^{d_{i,2}} \sum_{j_2=1}^{e_i} b^{(i)}_{j_1,j_2} k^{(i)}_{j_1,j_2}, \quad i=1,...,s-1.$$

Let $\nu_{P_h}(t_s) \geq 0$ or $\alpha_2 = 0$. Therefore $\nu_{\mathfrak{B}_{h,j}}(\alpha_2) \geq 0$ for $1 \leq j \leq \mu_h$. Taking into account that $\alpha_1 = -\alpha_2$, we get $\nu_{\mathfrak{B}_{h,j}}(\alpha_1) \geq 0$ for $1 \leq j \leq \mu_h$, and $\nu_{P_h}(\alpha_1) \geq 0$. Using (4.58), (4.78), (4.80) and (4.86), we obtain $\nu_{P_h}(\alpha_{1,i}) \geq 0$ for $1 \leq i \leq s-1, i \neq h$. Bearing in mind (4.93) and that $\{u^{(h)}_1,...,u^{(h)}_{z_h}, k^{(h)}_{q_h,1},...,k^{(h)}_{q_h,e_h},...,k^{(h)}_{n,1},...,k^{(h)}_{n,e_h}\}$ is a basis of $\mathcal{L}(G_h + (n-q_h+1)P_h)$, we get

$$\alpha_{1,h} \in \mathcal{L}(G_h + (j-q_h+1)P_h) \setminus \mathcal{L}(G_h + (j-q_h)P_h) \text{ with some } j \geq q_h.$$

By (4.78) and (4.80), we get $\nu_{P_h}(\alpha_{1,h}) \leq -1$. We have a contradiction.

Now let $\nu_{P_h}(t_s) \leq -1$ and $\alpha_2 \neq 0$. We have $\nu_{P_h}(\alpha_{1,h}) \geq -d_{h,2}-1$, $\nu_{P_h}(\alpha_1) \geq -d_{h,2}-1$ and $\nu_{\mathfrak{B}_{h,j}}(\alpha_1) \geq -d_{h,2}-1$, $j=1,...,\mu_h$. On the other hand, using (4.90) and (2.11), we have $\nu_{\mathfrak{B}_{h,j}}(\alpha_2) \leq -(m_s - d_{s,2} - 1)$, $j=1,...,\mu_h$. According to (3.17) and (4.86), we obtain $s \geq 3$, $e_h \geq e_s$ and

$$m_s - d_{s,2} - 1 - d_{h,2} - 1 = [t/e_s] + 1 + (s-2)d_0 e\dot{m}/e_s - d_0 \dot{m}e/e_h \geq 1.$$

We have a contradiction. Thus assertion (4.89) is not true. Hence (4.87) is true and Lemma 15 follows. □

End of the proof of Theorem 3.
Using (2.15), (3.15), (4.67)-(4.69) and Lemma 12, we have

$$\mathcal{P}_1 = \{\tilde{\mathbf{x}}(f^\perp, \boldsymbol{\varphi}) = (\tilde{x}_1(f^\perp, \boldsymbol{\varphi}),...,\tilde{x}_s(f^\perp, \boldsymbol{\varphi})) \mid f^\perp \in \mathcal{L}(G^\perp), \boldsymbol{\varphi} \in \Phi\} \tag{4.94}$$

with

$$\tilde{x}_i(f^\perp, \boldsymbol{\varphi}) = \sum_{j=1}^{m} \phi^{-1}(\theta^\perp_{i,j}(f^\perp, \boldsymbol{\varphi}))b^{-j} = \sum_{j=1}^{r_i} \phi^{-1}(\varphi_{i,j})b^{-j} + b^{-r_i} \sum_{j=1}^{m-r_i} \phi^{-1}(\dot{\theta}^\perp_{i,j}(f^\perp))b^{-j}.$$

By (3.16), we have

$$\mathcal{P}_2 = \{\dot{\mathbf{x}}(f^\perp) = (\dot{x}_1(f^\perp),...,\dot{x}_s(f^\perp)) \mid f^\perp \in \mathcal{L}(G^\perp)\} \tag{4.95}$$

with

$$\dot{x}_i(f^\perp) = \sum_{j=1}^{m-r_i} \phi^{-1}(\dot{\theta}^\perp_{i,j}(f^\perp))b^{-j}, \quad 1 \leq i \leq s. \tag{4.96}$$

Lemma 16. *With notation as above, $\mathcal{P}_2$ is a $d-$admissible $(t, m-r_0, s)$-net in base b with $d = g + e_0$, and $t = g + e_0 - s$.*

Proof. Let $J = \prod_{i=1}^{s}[A_i/b^{d_i},(A_i+1)/b^{d_i})$ with $d_i \geq 0$, and $0 \leq A_i < b^{d_i}$, $1 \leq i \leq s$, and let $J_{\boldsymbol{\psi}} = \prod_{i=1}^{s}[\psi_i/b^{r_i} + A_i/b^{r_i+d_i}, \psi_i/b^{r_i} + (A_i+1)/b^{r_i+d_i})$ with $\psi_i/b^{r_i} = \psi_{i,1}/b + ... + \psi_{i,r_i}/b^{r_i}$, $\psi_{i,j} \in Z_b$, $1 \leq i \leq s$, $d_1 + ... + d_s = m - r_0 - t$.
It is easy to see, that

$$\dot{\mathbf{x}}(f^{\perp}) \in J \Longleftrightarrow \tilde{\mathbf{x}}(f^{\perp},\boldsymbol{\varphi}) \in J_{\boldsymbol{\psi}} \quad \text{with} \quad \psi_{i,j} = \phi^{-1}(\varphi_{i,j}),\ 1 \leq j \leq r_i,\ 1 \leq i \leq s.$$

Bearing in mind that $\mathcal{P}_1$ is a (t,m,s) net with $t = g + e_0 - s$, we have

$$\sum_{f^{\perp} \in \mathcal{L}(G^{\perp})} \mathbb{1}(J, \dot{\mathbf{x}}(f^{\perp})) = \sum_{f^{\perp} \in \mathcal{L}(G^{\perp}), \boldsymbol{\varphi} \in \Phi} \mathbb{1}(J_{\boldsymbol{\psi}}, \mathbf{x}(f^{\perp},\boldsymbol{\varphi})) = b^t.$$

Therefore $\mathcal{P}_2$ is a $(t, m - r_0, s)$-net in base b with $t = g + e_0 - s$.

Using (4.69), Definition 5 and Definition 10, we can get d from the following equation $-\delta_m^{\perp}(\dot{\Xi}_m) = -(m - r_0) - d + 1$. Applying Lemma 9, we obtain $-(m + g - 1 + e_0 - r_0) \leq -(m - r_0) - d + 1$. Hence $d \leq g + e_0$. Thus Lemma 16 is proved. □

Let $V_i \subseteq \mathbb{F}_b^{\mu_i}$ be a vector space over $\mathbb{F}_b$, $\mu_i \geq 1$, $i = 1, 2$. Consider a linear map $h:\ V_1 \to V_2$. By the first isomorphism theorem, we have

$$\dim_{\mathbb{F}_b}(V_1) = \dim_{\mathbb{F}_b}(\ker(h)) + \dim_{\mathbb{F}_b}(\mathrm{im}(h)). \tag{4.97}$$

Let

$$\Lambda_1' = \Big\{ \big(\operatorname*{Res}_{P_s,t_s}(f^{\perp} k_{j_1,j_2}^{(i)})\big)_{0 \leq j_1 \leq d_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s-1}, \big(\operatorname*{Res}_{P_s,t_s}(\beta_{s,j_2}^{\perp} f^{\perp} t_s^{m_s - j_1 - 1})\big)_{d_{s,1} \leq j_1 \leq d_{s,2}, 1 \leq j_2 \leq e_s} \mid f^{\perp} \in \mathcal{L}(G^{\perp}) \Big\}$$

and

$$\Lambda_2 = \Big\{ \big(\operatorname*{Res}_{P_s,t_s}(\beta_{s,j_2}^{\perp} f^{\perp} t_s^{m_s - j_1 - 1})\big)_{d_{s,1} \leq j_1 \leq d_{s,2}, 1 \leq j_2 \leq e_s} \mid \operatorname*{Res}_{P_s,t_s}(f^{\perp} k_{j_1,j_2}^{(i)}) = 0 \quad \text{for} \quad 0 \leq j_1 \leq d_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s-1,\ f^{\perp} \in \mathcal{L}(G^{\perp}) \Big\}$$

with $d_{s,1} = m_s + 1 - [t/e_s] - (s-1)d_0\dot{m}e/e_s$,

$$d_{s,2} = m_s - 2 - [t/e_s] - (s-2)d_0\dot{m}e/e_s, \quad d_{i,1} = q_i,\ d_{i,2} = d_0\dot{m}e/e_i - 1, \tag{4.98}$$

$i \in [1, s-1]$, $d_0 = d + t$, $e = e_1 e_2 \cdots e_s$, $\epsilon = \eta(2(s-1)d_0 e)^{-1}$, $\eta = (1 + \deg((t_s)_{\infty}))^{-1}$, $\dot{m} = [\tilde{m}\epsilon]$, $\tilde{m} = m - r_0$, $m > 2(g - 1 + e_0)e_s + 2t(\eta^{-1} - 1)$, $d = g + e_0$ and $t = g + e_0 - s$.

By (4.97), (4.98) and Lemma 15, we have $\dim_{\mathbb{F}_b}(\Lambda_1') \geq \dim_{\mathbb{F}_b}(\Lambda_1)$ and

$$\dim_{\mathbb{F}_b}(\Lambda_2) = \dim_{\mathbb{F}_b}(\Lambda_1') - \dim_{\mathbb{F}_b}\Big(\Big\{ \big(\operatorname*{Res}_{P_s,t_s}(f^{\perp} k_{j_1,j_2}^{(i)})\big)_{\substack{0 \leq j_1 \leq d_{i,2}, 1 \leq j_2 \leq e_i \\ 1 \leq i \leq s-1}} | f^{\perp} \in \mathcal{L}(G^{\perp} \Big\} \Big)$$

$$\geq \dim_{\mathbb{F}_b}(\Lambda_1) - \sum_{i=1}^{s-1}(d_{i,2}+1)e_i \geq (d_{s,2}-d_{s,1}+1)e_s - \sum_{i=1}^{s-1} q_i e_i = d_0 e \dot{m} - 2e_s - \sum_{i=1}^{s-1} q_i e_i.$$

Let

$$\Lambda_3 = \Big\{ \big(\operatorname*{Res}_{P_s,t_s}(\beta^{\perp}_{s,j_2} f^{\perp} t_s^{m_s-j_1-1})\big)_{d_{s,1}\leq j_1\leq d_{s,2}, 1\leq j_2\leq e_s} \mid \vartheta^{\perp}_{i,j_2}(f^{\perp}_{i,j_1}) = 0$$

$$\text{for } 0 \leq j_1 \leq d_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s-1 \mid f^{\perp} \in \mathcal{L}(G^{\perp}) \Big\}.$$

Using Lemma 14, we get $\Lambda_3 \supseteq \Lambda_2$ and $\dim_{\mathbb{F}_b}(\Lambda_3) \geq \dim_{\mathbb{F}_b}(\Lambda_2)$. Let

$$\Lambda_4 = \Big\{ \big(\vartheta^{\perp}_{i,j_2}(f^{\perp}_{i,j_1})\big)_{0\leq j_1\leq d_{i,2}, 1\leq j_2\leq e_i, 1\leq i\leq s-1} \mid f^{\perp} \in \mathcal{L}(G^{\perp}) \Big\}.$$

Taking into account that $\mathcal{P}_2$ is a $(t, m-r_0, s)$-net in base b, we get from (4.95) that $\dim_{\mathbb{F}_b}(\Lambda_4) = (s-1)d_0 e \dot{m}$. Let

$$\Lambda_5 = \Big\{ \Big(\vartheta^{\perp}_{i,j_2}(f^{\perp}_{i,j_1})\Big)_{\substack{0\leq j_1\leq d_{i,2}, 1\leq j_2\leq e_i,\\ 1\leq i\leq s-1}}, \Big(\operatorname*{Res}_{P_s,t_s}(\beta^{\perp}_{s,j_2} f^{\perp} t_s^{m_s-j_1-1})\Big)_{\substack{d_{s,1}\leq j_1\leq d_{s,2}\\ 1\leq j_2\leq e_s}} \Big| f^{\perp} \in \mathcal{L}(G^{\perp}) \Big\}.$$

By (4.78)and (4.97), we have

$$\dim_{\mathbb{F}_b}(\Lambda_5) = \dim_{\mathbb{F}_b}(\Lambda_3) + \dim_{\mathbb{F}_b}(\Lambda_4) \geq s d_0 e \dot{m} - 2e_s - 2(s-1)(g+e_0).$$

Let $\dot{m}_1 = d_0 e \dot{m}$, $\dot{m} = [\tilde{m}\epsilon]$, $\ddot{m}_i = 0$, $i \in [1, s-1]$ and $\ddot{m}_s = m - t - (s-1)\dot{m}_1$. Bearing in mind that $\dot{\theta}^{\perp}_{i,\check{j}_i e_i + \hat{j}_i}(f^{\perp}) = \vartheta^{\perp}_{i,\hat{j}_i}(f^{\perp}_{i,\check{j}_i})$ for $1 \leq \hat{j}_i \leq e_i$, $0 \leq \check{j}_i \leq m_i - 1$, $i \in [1, s-1]$ (see (4.63)), we obtain

$$\Big(\dot{\theta}^{\perp}_{i,\ddot{m}_i+j}(f^{\perp})\Big)_{1\leq j\leq \dot{m}_1, 1\leq i\leq s-1} \supseteq \Big(\vartheta^{\perp}_{i,j_2}(f^{\perp}_{i,j_1})\Big)_{0\leq j_1\leq d_{i,2}, 1\leq j_2\leq e_i, 1\leq i\leq s-1}. \tag{4.99}$$

From (4.98), we have $\ddot{m}_s < d_{s,1} e_s$ and $(d_{s,2}+1)e_s < \ddot{m}_s + \dot{m}_1$. Taking into account that

$$\dot{\theta}^{\perp}_{s,j_1 e_s + j_2}(f^{\perp}_{s,-m_s+j_1}) = \vartheta^{\perp}_{s,j_2}(f^{\perp}) = \operatorname{Res}_{P_s,t_s}(\beta^{\perp}_{s,j_2} f^{\perp} t_s^{m_s-j_1-1})$$

(see (4.62) and (4.64)), we get

$$\Big(\dot{\theta}^{\perp}_{s,\ddot{m}_s+j}(f^{\perp})\Big)_{1\leq j\leq \dot{m}_1} \supseteq \Big(\operatorname*{Res}_{P_s,t_s}(\beta^{\perp}_{s,j_2} f^{\perp} t_s^{m_s-j_1-1})\Big)_{d_{s,1}\leq j_1\leq d_{s,2}, 1\leq j_2\leq e_s}. \tag{4.100}$$

Let

$$\Lambda_6 = \Big\{ \Big(\Big(\dot{\theta}^{\perp}_{i,\ddot{m}_i+j}(f^{\perp})\Big)_{1\leq j\leq \dot{m}_1, 1\leq i\leq s} \Big) \Big| f^{\perp} \in \mathcal{L}(G^{\perp}) \Big\}.$$

By (4.99) and (4.100), we derive

$$\dim_{\mathbb{F}_b}(\Lambda_6) \geq \dim_{\mathbb{F}_b}(\Lambda_5) \geq s d_0 e \dot{m} - 2e_s - 2(s-1)(g+e_0).$$

Applying (2.15), (3.16), (4.95) and Lemma 2, we get that there exists
$B_i \in \{0, ..., \dot{m}-1\}$, $1 \leq i \leq s$ such that

$$\Lambda_7 = \mathbb{F}_b^{s d_0 e \dot{m} - d_0 e B} \qquad \text{for} \quad \dot{m} \geq 1, \tag{4.101}$$

where $B = \#B_1 + ... + \#B_s \leq 4(s-1)(g+e_0)$ and

$$\Lambda_7 = \Big\{ \Big(\dot{\theta}^{\perp}_{i,\dot{m}_i+\dot{j}_i d_0 e + \ddot{j}_i}(f^{\perp}) \mid \dot{j}_i \in \bar{B}_i,\ \ddot{j}_i \in [1, d_0 e],\ i \in [1,s] \Big) \mid f^{\perp} \in \mathcal{L}(G^{\perp}) \Big\}$$

with $\bar{B}_i = \{0, ..., \dot{m}-1\} \setminus B_i$.
From (4.96), we have

$$\Big\{ \Big(\dot{x}_{i,\dot{m}_i+\dot{j}_i d_0 e+\ddot{j}_i}(f^{\perp}) | \dot{j}_i \in \bar{B}_i, \ddot{j}_i \in [1, d_0 e], i \in [1,s] \Big) \mid f^{\perp} \in \mathcal{L}(G^{\perp}) \Big\} = Z_b^{s d_0 e \dot{m} - d_0 e B}.$$

We apply Corollary 2 with $\dot{s} = s$, $\tilde{r} = r_0$, $\tilde{m} = m - r_0$, $\epsilon = \eta(2(s-1)d_0 e)^{-1}$ and $\hat{e} = e = e_1 e_2 \cdots e_s$.

Let $\dot{\gamma}(f^{\perp}, \dot{\mathbf{w}}) = \dot{\gamma} = (\dot{\gamma}^{(1)}, ..., \dot{\gamma}^{(\dot{s})})$ with $\dot{\gamma}^{(i)} := [(\dot{\mathbf{x}}(f^{\perp}) \oplus \dot{\mathbf{w}})^{(i)}]_{\dot{m}_i}$, $i \in [1,s]$. Using (4.96) and (4.101), we get that there exists $f^{\perp} \in G^{\perp}$ such that $\dot{\gamma}(f^{\perp}, \dot{\mathbf{w}})$ satisfy (2.36). Bearing in mind Lemma 16, we get from Corollary 2 that

$$\Big| \Delta((\dot{\mathbf{x}}(f^{\perp}) \oplus \dot{\mathbf{w}})_{f^{\perp} \in G^{\perp}}, J_{\dot{\gamma}}) \Big| \geq 2^{-2} b^{-d} K_{d,t,s}^{-s+1} \eta^{s-1} m^{s-1}. \tag{4.102}$$

for $m \geq 2^{2s+3} b^{d+t+s} (d+t)^s (s-1)^{2s-1} (g+e_0) e \eta^{-s+1}$.

Taking into account (1.2), and that $\dot{\mathbf{w}} \in E^s_{m-r_0}$ is arbitrary, we get the second assertion in Theorem 3.

Consider the first assertion in Theorem 3.
Let $\tilde{\gamma} = (\tilde{\gamma}^{(1)}, ..., \tilde{\gamma}^{(s)})$ with $\tilde{\gamma}^{(i)} = b^{-r_i} \dot{\gamma}^{(i)}$, $i \in [1,s]$, and let $\tilde{\mathbf{w}} = (\tilde{w}^{(1)}, ..., \tilde{w}^{(s)}) \in E^s_m$ with $\tilde{w}^{(i)}_{j+r_i} = \dot{w}^{(i)}_j$ for $j \in [1, m-r_0]$, $i \in [1,s]$. By (4.94) and (4.95), we have

$$\tilde{x}_i(f^{\perp}, \boldsymbol{\varphi}) \oplus \tilde{w}^{(i)} \in [0, \tilde{\gamma}_i) \Longleftrightarrow \dot{x}_i(f^{\perp}) \oplus \dot{w}^{(i)} \in [0, \dot{\gamma}_i) \ \text{ and } \ \phi^{-1}(\varphi_{i,j}) \oplus \tilde{w}_{i,j} = 0$$

for $j \in [1, r_i]$, $i \in [1,s]$. Hence

$$\sum_{\boldsymbol{\varphi} \in \Phi} (\mathbb{1}([\mathbf{0}, \tilde{\gamma}), \tilde{\mathbf{x}}(f^{\perp}, \boldsymbol{\varphi}) \oplus \tilde{\mathbf{w}}) - \tilde{\gamma}_0) = \mathbb{1}([\mathbf{0}, \dot{\gamma}), \dot{\mathbf{x}}(f^{\perp}) \oplus \dot{\mathbf{w}}) - \dot{\gamma}_0,$$

where $[\mathbf{0}, \dot{\gamma}) = \prod_{i=1}^s [0, \dot{\gamma}^{(i)})$, $[\mathbf{0}, \tilde{\gamma}) = \prod_{i=1}^s [0, \tilde{\gamma}^{(i)})$, $\tilde{\gamma}_0 = \tilde{\gamma}^{(1)} ... \tilde{\gamma}^{(s)}$ and $\dot{\gamma}_0 = \dot{\gamma}^{(1)} ... \dot{\gamma}^{(s)}$. Therefore

$$\sum_{f^{\perp} \in \mathcal{L}(G^{\perp}), \boldsymbol{\varphi} \in \Phi} \big(\mathbb{1}([\mathbf{0}, \tilde{\gamma}), \tilde{\mathbf{x}}(f^{\perp}, \boldsymbol{\varphi}) \oplus \tilde{\mathbf{w}}) - \tilde{\gamma}_0 \big) = \sum_{f^{\perp} \in \mathcal{L}(G^{\perp})} \big(\mathbb{1}([\mathbf{0}, \dot{\gamma}), \dot{\mathbf{x}}(f^{\perp}) \oplus \dot{\mathbf{w}}) - \dot{\gamma}_0 \big).$$

Using (1.1), (1.2) and (4.102), we get the first assertion in Theorem 3.
Thus Theorem 3 is proved. □

4.4. **Halton-type sequences. Proof of Theorem 4.** Using (3.24) and (3.25), we define the sequence $(\mathbf{x}^{(i)}_{n,j})_{j \geq 1}$ by

$$\sum_{j_2=1}^{e_i} x^{(i)}_{n, j_1 e_i + j_2} b^{-j_2+e_i} := \sigma_{P_i}(f^{(i)}_{n,j_1}), \quad x^{(i)}_n := \sum_{j=0}^{\infty} \frac{x^{(i)}_{n,j}}{b^j} = \sum_{j_1=0}^{\infty} \sum_{j_2=1}^{e_i} \frac{x^{(i)}_{n, j_1 e_i + j_2}}{b^{j_1 e_i + j_2}}, \tag{4.103}$$

$1 \le i \le s$, with $(x_n^{(1)},...,x_n^{(s)}) = \mathbf{x}_n = \xi(f_n)$, and $n = 0, 1, ...$.

Lemma 17. *$(\mathbf{x}_n)_{n\ge 0}$ is $d-$admissible with $d = g + e_0$, where $e_0 = e_1 + ... + e_s$.*

Proof. Suppose that the assertion of the lemma is not true. By (1.4), there exists $\dot{n} > \dot{k}$ such that $\|\dot{n} \ominus \dot{k}\|_b \|\mathbf{x}_{\dot{n}} \ominus \mathbf{x}_{\dot{k}}\|_b < b^{-d}$.
Let $d_i + 1 = \dot{d}_i e_i + \ddot{d}_i$ with $1 \le \ddot{d}_i \le e_i$, $1 \le i \le s$, $n = \dot{n} \ominus \dot{k}$, $\|n\|_b = b^{m-1}$ and let $\left\|\mathbf{x}_{\dot{n}}^{(i)} \ominus \mathbf{x}_{\dot{k}}^{(i)}\right\|_b = b^{-d_i-1}$, $1 \le i \le s$. Hence $m - 1 - \sum_{i=1}^{s}(d_i+1) \le -d-1$, and

$$(4.104)\qquad m+g-1-\sum_{i=1}^{s}\dot{d}_i e_i \le m+g-1-\sum_{i=1}^{s}(d_i+1)+e_0 \le -d-1+g+e_0 < 0.$$

We have

$$(4.105)\qquad a_{m-1}(n) \ne 0,\ a_r(n) = 0,\ \text{for } r \ge m, \quad x^{(i)}_{\dot{n},d_i+1} \ne x^{(i)}_{\dot{k},d_i+1},\ x^{(i)}_{\dot{n},r} = x^{(i)}_{\dot{k},r}$$

for $r \le d_i$, $1 \le i \le s$. From (4.103), we get

$$f^{(i)}_{\dot{n},j_1} = f^{(i)}_{\dot{k},j_1} \quad \text{and} \quad f^{(i)}_{n,j_1} = 0 \quad \text{for} \quad 0 \le j_1 < \dot{d}_i,\ 1 \le i \le s.$$

Suppose that $f^{(i)}_{n,\dot{d}_i} = 0$, then $f^{(i)}_{\dot{n},\dot{d}_i} = f^{(i)}_{\dot{k},\dot{d}_i}$ and $x^{(i)}_{\dot{n},j} = x^{(i)}_{\dot{k},j}$ for $1 \le j \le (\dot{d}_i+1)e_i$. Taking into account that $d_i + 1 \le (\dot{d}_i+1)e_i$, we have a contradiction. Therefore $f^{(i)}_{n,d_i} \ne 0$, for all $1 \le i \le s$. Applying (3.23), we derive $\nu_{P_i}(f_n) = \dot{d}_i$, $1 \le i \le s$.
Using (3.18)-(3.20) and (4.105), we obtain $f_n \in \mathcal{L}((m+g-1)P_{s+1} - \sum_{i=1}^{s}\dot{d}_i P_i) \setminus \{0\}$.
By (4.104), we get

$$\deg\big((m+g-1)P_{s+1} - \sum_{i=1}^{s}\dot{d}_i P_i\big) = m+g-1-\sum_{i=1}^{s}\dot{d}_i e_i < 0.$$

Hence $f_n = 0$. We have a contradiction. Thus Lemma 17 is proved. □

Consider the $H-$differential $\mathrm{d}t_{s+1}$. By Proposition A, we have that there exists τ_i with $\mathrm{d}t_{s+1} = \tau_i \mathrm{d}t_i$, $1 \le i \le s$. Let $W = \mathrm{div}(\mathrm{d}t_{s+1})$, and let

$$(4.106)\qquad G_i = W + q_i P_i - g P_{s+1}, \qquad \text{with} \quad q_i = [(g+1)/e_i + 1],\ \ 1 \le i \le s.$$

It is easy to see that $\deg(G_i) \ge 2g - 2 + g + 1 - g = 2g - 1$, $1 \le i \le s$. Let $z_i = \dim(\mathcal{L}(G_i))$, and let $u_1^{(i)}, ..., u_{z_i}^{(i)}$ be a basis of $\mathcal{L}(G_i)$ over $\mathbb{F}_b$, $1 \le i \le s$.
For each $1 \le i \le s-1$, we consider the chain

$$\mathcal{L}(G_i) \subset \mathcal{L}(G_i + P_i) \subset \mathcal{L}(G_i + 2P_i) \subset ...$$

of vector spaces over $\mathbb{F}_b$. By starting from the basis $u_1^{(i)}, ..., u_{z_i}^{(i)}$ of $\mathcal{L}(G_i)$ and successively adding basis vectors at each step of the chain, we obtain for each

$n \geq q_i$ a basis

$$\{u_1^{(i)}, ..., u_{z_i}^{(i)}, k_{q_i,1}^{(i)}, ..., k_{q_i,e_i}^{(i)}, ..., k_{n,1}^{(i)}, ..., k_{n,e_i}^{(i)}\}$$

of $\mathcal{L}(G_i + (n - q_i + 1)P_i)$. We note that we then have

$$k_{j_1,j_2}^{(i)} \in \mathcal{L}(G_i + (j_1 - q_i + 1)P_i) \setminus \mathcal{L}(G_i + (j_1 - q_i)P_i) \tag{4.107}$$

for $q_i \leq j_1$, $1 \leq j_2 \leq e_i$ and $1 \leq i \leq s$. Hence

$$\mathrm{div}(k_{j_1,j_2}^{(i)}) + W - gP_{s+1} + (j_1 + 1)P_i \geq 0 \ \ \textit{and} \ \ \nu_{P_{s+1}}(k_{j_1,j_2}^{(i)}) + \nu_{P_{s+1}}(W) \geq g.$$

From (2.4) and (2.6), we obtain

$$\nu_{P_{s+1}}(k_{j_1,j_2}^{(i)}) = \nu_{P_{s+1}}(k_{j_1,j_2}^{(i)}\mathrm{d}t_{s+1}) = \nu_{P_{s+1}}(k_{j_1,j_2}^{(i)}) + \nu_{P_{s+1}}(W).$$

Therefore

$$\nu_{P_{s+1}}(W) = 0 \quad \text{and} \quad \nu_{P_{s+1}}(k_{j_1,j_2}^{(i)}) \geq g. \tag{4.108}$$

Now, let $\check{G}_i = W + (e_i + 1)P_{s+1} - P_i$. We see that $\deg(\check{G}_i) = 2g - 1$. Let $\dot{u}_1^{(i)}, ..., \dot{u}_{\dot{z}_i}^{(i)}$ be a basis of $\mathcal{L}(\check{G}_i)$ over $\mathbb{F}_b$. In a similar way, we construct a basis $\{\dot{u}_1^{(i)}, ..., \dot{u}_{\dot{z}_i}^{(i)}, k_{0,1}^{(i)}, ..., k_{0,e_i}^{(i)}, ..., k_{q_i-1,1}^{(i)}, ..., k_{q_i-1,e_i}^{(i)}\}$ of $\mathcal{L}(\check{G} + q_i P_i)$ with

$$k_{j_1,j_2}^{(i)} \in \mathcal{L}(\check{G} + (j_1 + 1)P_i) \setminus \mathcal{L}(\check{G} + j_1 P_i) \text{ for } j_1 \in [0, q_i), j_2 \in [1, e_i], i \in [1, s]. \tag{4.109}$$

Lemma 18. *Let* $\{\beta_1^{(i)}, ..., \beta_{e_i}^{(i)}\}$ *be a basis of* $F_{P_i}/\mathbb{F}_b$, $s \geq 2$, $d_i \geq 1$ *be integer* $(i = 1, ..., s)$ *and* $n \in [0, b^m)$. *Suppose that* $\mathrm{Res}_{P_{s+1},t_{s+1}}(f_n k_{j_1,j_2}^{(i)}) = 0$ *for* $j_1 \in [0, d_i - 1]$, $j_2 \in [1, e_i]$ *and* $i \in [1, s]$. *Then*

$$\mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(\beta_{j_2}^{(i)} f_{n,j_1}^{(i)}) = 0 \quad for \quad j_1 \in [0, d_i - 1], \ j_2 \in [1, e_i] \text{ and } i \in [1, s].$$

Proof. Using (4.107) and (4.109), we get

$$\nu_{P_i}(k_{j_1,j_2}^{(i)}) = -j_1 - 1 - \nu_{P_i}(W) \quad \text{for} \quad j_1 \geq 0, \ j_2 \in [1, e_i] \ \textit{and} \ i \in [1, s].$$

From (2.4) and (2.6), we obtain

$$\nu_{P_i}(\tau_i) = \nu_{P_i}(\tau_i \mathrm{d}t_i) = \nu_{P_i}(\mathrm{d}t_{s+1}) = \nu_{P_i}(\mathrm{div}(\mathrm{d}t_{s+1})) = \nu_{P_i}(W). \tag{4.110}$$

Hence

$$\nu_{P_i}(k_{j_1,j_2}^{(i)}\tau_i) = -j_1 - 1 \quad \text{for} \quad j_1 \geq 0, \ j_2 \in [1, e_i] \ \textit{and} \ i \in [1, s]. \tag{4.111}$$

By (4.107) and (4.109), we have

$$\mathrm{div}(k_{j_1,j_2}^{(i)}) + \mathrm{div}(\mathrm{d}t_{s+1}) + (j_1 + 1)P_i + a_{j_1}P_{s+1} \geq 0 \tag{4.112}$$

for $j_1 \geq 0,\ j_2 \in [1, e_i]$, $i \in [1, s]$ and some $a_{j_1} \in \mathbb{Z}$. According to (3.18) and (3.20), we get $f_n \in \mathcal{L}((m+g-1)P_{s+1})$. Therefore

$$\nu_P(f_n k^{(i)}_{j_1,j_2} \mathrm{d}t_{s+1}) \geq 0 \quad \text{and} \quad \operatorname*{Res}_P(f_n k^{(i)}_{j_1,j_2} \mathrm{d}t_{s+1}) = 0 \quad \text{for all} \quad P \in \mathbb{P}_f \setminus \{P_i, P_{s+1}\}.$$

Applying the Residue Theorem, we derive

$$\operatorname*{Res}_{P_i}(f_n k^{(i)}_{j_1,j_2} \mathrm{d}t_{s+1}) = -\operatorname*{Res}_{P_{s+1}}(f_n k^{(i)}_{j_1,j_2} \mathrm{d}t_{s+1}) \tag{4.113}$$

for $j_1 \geq 0,\ j_2 \in [1, e_i]$ and $i \in [1, s]$. Using (4.111), we get the following local expansion

$$\tau_i k^{(i)}_{j_1,j_2} := \sum_{r=-j_1}^{\infty} \varkappa^{(i,j_2)}_{j_1,r} t_i^{r-1}, \quad \text{where all} \quad \varkappa^{(i,j_2)}_{j_1,r} \in \mathbb{F}_b \text{ and } \varkappa^{(i,j_2)}_{j_1,j_1} \neq 0$$

for $j_1 \geq 0,\ j_2 \in [1, e_i]$ and $i \in [1, s]$. By (3.23) and (4.113), we obtain

$$-\operatorname*{Res}_{P_{s+1},t_{s+1}}(f_n k^{(i)}_{j_1,j_2}) = \operatorname*{Res}_{P_i,t_i}(f_n \tau_i k^{(i)}_{j_1,j_2}) = \operatorname*{Res}_{P_i,t_i}\Big(\sum_{j=0}^{\infty} f^{(i)}_{n,j} t_i^j \sum_{r=-j_1}^{\infty} \varkappa^{(i,j_2)}_{j_1,r} t_i^{r-1}\Big)$$

$$= \sum_{j=0}^{\infty} \sum_{r=-j_1}^{0} \mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(f^{(i)}_{n,j} \varkappa^{(i,j_2)}_{j_1,r}) \delta_{j,-r} = \sum_{j=0}^{j_1} \mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(f^{(i)}_{n,j} \varkappa^{(i,j_2)}_{j_1,-j}) = 0 \tag{4.114}$$

for $0 \leq j_1 \leq d_i - 1$, $1 \leq j_2 \leq e_i$ and $1 \leq i \leq s$. Similarly to the proof of Lemma 14, we get from (4.114) the assertion of Lemma 18. □

Lemma 19. *Let $s \geq 2$, $d_0 = d + t$, $\epsilon = \eta_1 (2sd_0 e)^{-1}$, $\eta_1 = (1 + \deg((t_{s+1})_\infty))^{-1}$,*

$$\Lambda_1 = \Big\{ \Big(\Big(\operatorname*{Res}_{P_{s+1},t_{s+1}}(f_n k^{(i)}_{j_1,j_2}) \Big)_{\substack{d_{i,1} \leq j_1 \leq d_{i,2} \\ 1 \leq j_2 \leq e_i}, 1 \leq i \leq s}, \bar{a}_{d_{s+1,1}}(n), \ldots, \bar{a}_{d_{s+1,2}}(n) \Big) | n \in [0, b^m) \Big\}$$

with $e = e_1 e_2 \cdots e_s$, $e_{s+1} = 1$, $d_{s+1,1} = t + (s-1) d_0 [m\epsilon] e$,

$$d_{s+1,2} = t - 1 + sd_0[m\epsilon]e, \quad d_{i,1} = q_i,\ d_{i,2} = d_0[m\epsilon]e/e_i - g - 1 \text{ for } i \in [1, s], \tag{4.115}$$

and $m \geq |2g - 2 + 2(t + g - 2)(\eta_1^{-1} - 1)| + 2t + 2/\epsilon$. Then

$$\Lambda_1 = \mathbb{F}_b^{\chi} \qquad \text{with} \qquad \chi = \sum_{i=1}^{s+1} (d_{i,2} - d_{i,1} + 1) e_i. \tag{4.116}$$

Proof. Suppose that (4.116) is not true. We get that there exists $b^{(i)}_{j_1,j_2} \in \mathbb{F}_b$ $(i, j_1, j_2 \geq 1)$ such that

$$\sum_{i=1}^{s} \sum_{j_1=d_{i,1}}^{d_{i,2}} \sum_{j_2=1}^{e_i} |b^{(i)}_{j_1,j_2}| + \sum_{j_1=d_{s+1,1}}^{d_{s+1,2}} |b^{(s+1)}_{j_1}| > 0 \tag{4.117}$$

and

$$\sum_{i=1}^{s}\sum_{j_1=d_{i,1}}^{d_{i,2}}\sum_{j_2=1}^{e_i} b^{(i)}_{j_1,j_2}\operatorname*{Res}_{P_{s+1},t_{s+1}}(f_n k^{(i)}_{j_1,j_2}) + \sum_{j_1=d_{s+1,1}}^{d_{s+1,2}} b^{(s+1)}_{j_1}\bar{a}_{j_1}(n) = 0 \tag{4.118}$$

for all $n \in [0, b^m)$. From (3.18)-(3.20), we obtain the following local expansion

$$f_n = \dot{f}_n + \ddot{f}_n = \sum_{r \le m+g-1} f^{(s+1)}_{n,r} t^{-r}_{s+1}, \quad \text{with} \quad \ddot{f}_n = \sum_{i=g}^{m-1} \bar{a}_i(n) v_i, \tag{4.119}$$

and $\dot{f}_n = \sum_{i=0}^{g-1} \bar{a}_i(n) v_i$, where $n \in [0, b^m)$. Let $r \ge g$.
Using (3.18)-(3.20) and (3.28), we derive that $\nu_{P_{s+1}}(\dot{f}_n) \ge -2g+1$, $\nu_{P_{s+1}}(\dot{f}_n t^{r+g-1}_{s+1})$ ≥ 0 and

$$f^{(s+1)}_{n,r+g} = \operatorname*{Res}_{P_{s+1},t_{s+1}}(f_n t^{r+g-1}_{s+1}) = \operatorname*{Res}_{P_{s+1},t_{s+1}}(\ddot{f}_n t^{r+g-1}_{s+1}) = \operatorname*{Res}_{P_{s+1},t_{s+1}}\Big(\sum_{i=g}^{m-1} \bar{a}_i(n)$$

$$\times \sum_{j \le i+g} v_{i,j} t^{-j+r+g-1}_{s+1}\Big) = \sum_{i=g}^{m-1} \bar{a}_i(n) \sum_{j \le i+g} v_{i,j}\delta_{j,r+g} = \sum_{m-1 \ge i \ge r} \bar{a}_i(n) v_{i,r+g} \text{ for } r \ge g.$$

Taking into account that $v_{i,i+g} = 1$ and $v_{i,r+g} = 0$ for $i > r \ge g$ (see (3.29)), we get

$$f^{(s+1)}_{n,r+g} = \bar{a}_r(n) \quad \text{for} \quad r \ge g \quad \text{and} \quad n \in [0, b^m). \tag{4.120}$$

By (4.118), we have

$$\sum_{i=1}^{s}\sum_{j_1=d_{i,1}}^{d_{i,2}}\sum_{j_2=1}^{e_i} b^{(i)}_{j_1,j_2}\operatorname*{Res}_{P_{s+1},t_{s+1}}(f_n k^{(i)}_{j_1,j_2}) + \sum_{j_1=d_{s+1,1}}^{d_{s+1,2}} b^{(s+1)}_{j_1}\operatorname*{Res}_{P_{s+1},t_{s+1}}(f_n t^{j_1+g-1}_{s+1}) = 0$$

for all $n \in [0, b^m)$. Hence

$$\operatorname*{Res}_{P_{s+1},t_{s+1}}(f_n \alpha) = 0 \quad \text{for all} \quad n \in [0, b^m), \quad \text{where} \quad \alpha = \alpha_1 + \alpha_2, \tag{4.121}$$

$$\alpha_1 = \sum_{i=1}^{s} \alpha_{1,i}, \;\; \alpha_{1,i} = \sum_{j_1=d_{i,1}}^{d_{i,2}}\sum_{j_2=1}^{e_i} b^{(i)}_{j_1,j_2} k^{(i)}_{j_1,j_2}, \;\; \text{and} \;\; \alpha_2 = \sum_{j_1=d_{s+1,1}}^{d_{s+1,2}} b^{(s+1)}_{j_1} t^{j_1+g-1}_{s+1}.$$

According to (4.108), we get the following local expansion

$$k^{(i)}_{j_1,j_2} := \sum_{r=g+1}^{\infty} \varkappa^{(i,j_2)}_{j_1,r} t^{r-1}_{s+1}, \quad \text{where all} \quad \varkappa^{(i,j_2)}_{j_1,r} \in \mathbb{F}_b,$$

and

$$\alpha = \sum_{r=g+1}^{\infty} \varphi_r t^{r-1}_{s+1} \quad \text{with} \quad \varphi_r \in \mathbb{F}_b, \quad r \ge g+1. \tag{4.122}$$

Using (2.12) and (4.119)-(4.121), we have

$$\operatorname*{Res}_{P_{s+1},t_{s+1}}(f_n\alpha) = \operatorname*{Res}_{P_{s+1},t_{s+1}}\Big(\sum_{j\le m+g-1} f_{n,j}^{(s+1)} t_{s+1}^{-j}\sum_{r=g+1}^{\infty}\varphi_r t_{s+1}^{r-1}\Big)$$

$$= \sum_{j\le m+g-1} f_{n,j}^{(s+1)}\sum_{r=g+1}^{\infty}\varphi_r\delta_{j,r} = \sum_{j=g+1}^{m+g-1} f_{n,j}^{(s+1)}\varphi_j = \sum_{r=g+1}^{m+g-1}\bar{a}_r(n)\varphi_r = 0.$$

for $n\in[0,b^m))$. Hence

$$\varphi_r = 0 \quad \text{for} \quad g+1\le r\le m+g-1.$$

By (4.122), we obtain

$$\nu_{P_{s+1}}(\alpha)\ge m+g-1.$$

Applying (4.106), (4.107) and (4.121), we derive

$$\alpha\in\mathcal{L}(G_1), \text{ with } G_1 = W+\sum_{i=1}^{s} d_{i,2}P_i + (d_{s+1,2}+g-1)(t_{s+1})_\infty - (m+g-1)P_{s+1}.$$

From (4.115), we have

$$\deg(G_1) = 2g-2+\sum_{i=1}^{s} d_{i,2}e_i + (d_{s+1,2}+g-1)\deg((t_{s+1})_\infty) - (m+g-1)$$

$$\le 2g-2+sd_0e[m\epsilon] + (t-1+sd_0e[m\epsilon]+g-1)(\eta_1^{-1}-1) - (m+g-1)$$

$$\le g-1+(t+g-2)(\eta_1^{-1}-1)+sd_0em\epsilon\eta_1^{-1}-m = g-1+(t+g-2)(\eta_1^{-1}-1)-m/2<0$$

for $m>2g-2+2(t+g-2)(\eta_1^{-1}-1)$. Hence $\alpha=0$.

Suppose that $\sum_{i=1}^{s}\sum_{j_1=d_{i,1}}^{d_{i,2}}\sum_{j_2=1}^{e_i}|b_{j_1,j_2}^{(i)}| = 0$. Then $\alpha_2=0$. From (4.121), we derive $b_{j_1}^{(s+1)}=0$ for all $j_1\in[d_{s+1,1},d_{s+1,2}]$. According to (4.117), we have a contradiction. Hence there exists $h\in[1,s]$ with

$$\sum_{j_1=d_{h,1}}^{d_{h,2}}\sum_{j_2=1}^{e_h}|b_{j_1,j_2}^{(h)}|>0. \tag{4.123}$$

Let $h>1$. By (3.27) and (4.121), we get $\nu_{P_h}(t_{s+1})\ge 0$ and $\nu_{P_h}(\alpha_2)\ge 0$. Applying (2.3) and (2.4), we derive $\nu_{P_h}(W)=\nu_{P_h}(\mathrm{d}t_{s+1})=\nu_{P_h}(\mathrm{d}t_{s+1}/\mathrm{d}t_h)\ge 0$.

By (4.112), we have $\nu_{P_h}(\alpha_{1,j})\ge -\nu_{P_h}(W)$ for $1\le j\le s, j\ne h$. Taking into account that $\alpha_{1,h} = -\sum_{1\le j\le s, j\ne h}\alpha_{1,j}-\alpha_2$, we get $\nu_{P_h}(\alpha_{1,h})\ge -\nu_{P_h}(W)$.

Using (4.110) and (4.111), we obtain $\nu_{P_h}(k_{j_1,j_2}^{(h)}) = -j_1-1-\nu_{P_h}(W)$. Bearing in mind (4.123) and that $\{u_1^{(i)},...,u_{z_i}^{(i)},k_{q_i,1}^{(i)},...,k_{q_i,e_1}^{(i)},...,k_{n,1}^{(i)},...,k_{n,e_1}^{(i)}\}$ is a basis of $\mathcal{L}(G_i+(n-q_i+1)P_i)$, we get

$$\alpha_{1,h}\in\mathcal{L}(G_i+(d_{i,2}-q_i+1)P_i)\setminus\mathcal{L}(G_i+(d_{i,1}-q_i)P_i).$$

From (4.115) and (4.121), we derive $\nu_{P_h}(\alpha_{1,h}) \leq -\nu_{P_h}(W) - 1$. We have a contradiction.

Now let $h = 1$ and (4.123) is not true for $h \in [2, s]$. Hence $\alpha_{1,1} = -\alpha_2$ and $\nu_{P_{s+1}}(\alpha_{1,1}) \geq d_{s+1,1} + g - 1$. By (4.106), (4.107) and (4.121), we have

$$\alpha_{1,1} \in \mathcal{L}(\dot{G}) \quad \text{with} \quad \dot{G} = W + (d_{1,2}+1)P_1 - (d_{s+1,1} + g - 1)P_{s+1}.$$

From (4.115), we get

$$\deg(\dot{G}) = 2g - 2 + d_0 e[m\epsilon] - g e_1 - (s-1) d_0 e[m\epsilon] - g + 1 \leq 2g - 2 - 2g + 1 < 0.$$

Hence $\alpha_{1,1} = 0$. Therefore (4.123) is not true for $h = 1$. We have a contradiction. Thus assertion (4.117) is not true, and Lemma 19 follows. □

End of the proof of Theorem 4.

Let $\tilde{d}_{i,2} = d_{i,2} + g = d_0[m\epsilon]e/e_i - 1$ $(1 \leq i \leq s)$,

$$\Lambda_1^{'} = \Big\{ \Big(\big(\operatorname*{Res}_{P_{s+1}, t_{s+1}} (f_n k_{j_1,j_2}^{(i)}) \big)_{0 \leq j_1 \leq \tilde{d}_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s}, \bar{a}_{d_{s+1,1}}(n), ..., \bar{a}_{d_{s+1,2}}(n) \Big) \Big| n \in [0, b^m) \Big\}$$

and

$$\Lambda_2 = \Big\{ (\bar{a}_{d_{s+1,1}}(n), ..., \bar{a}_{d_{s+1,2}}(n)) \ \Big| \operatorname*{Res}_{P_{s+1}, t_{s+1}} (f_n k_{j_1,j_2}^{(i)}) = 0$$
$$\text{for} \quad 0 \leq j_1 \leq \tilde{d}_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s, \ n \in [0, b^m) \Big\}.$$

By (4.97) and Lemma 19, we have $\dim_{\mathbb{F}_b}(\Lambda_1^{'}) \geq \dim_{\mathbb{F}_b}(\Lambda_1)$ and

$$\dim_{\mathbb{F}_b}(\Lambda_2) = \dim_{\mathbb{F}_b}(\Lambda_1^{'}) - \dim_{\mathbb{F}_b} \Big(\Big\{ \big(\operatorname*{Res}_{P_{s+1}, t_{s+1}} (f_n k_{j_1,j_2}^{(i)}) \big)_{\substack{0 \leq j_1 \leq \tilde{d}_{i,2}, 1 \leq j_2 \leq e_i \\ 1 \leq i \leq s}} \Big| n \in [0, b^m) \Big\} \Big)$$

$$\geq \dim_{\mathbb{F}_b}(\Lambda_1) - \sum_{i=1}^{s} (\tilde{d}_{i,2} + 1) e_i \geq d_{s+1,2} - d_{s+1,1} + 1 - \sum_{i=1}^{s} (q_i + g) e_i. \tag{4.124}$$

Using Lemma 18, we get $\Lambda_3 \supseteq \Lambda_2$ and $\dim_{\mathbb{F}_b}(\Lambda_3) \geq \dim_{\mathbb{F}_b}(\Lambda_2)$, where

$$\Lambda_3 = \Big\{ (\bar{a}_{d_{s+1,1}}(n), ..., \bar{a}_{d_{s+1,2}}(n)) \ \Big| \ \mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(\beta_{j_2}^{(i)} f_{n,j_1}^{(i)}) = 0$$
$$\text{for } 0 \leq j_1 \leq \tilde{d}_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s, \ n \in [0, b^m) \Big\}.$$

Taking into account that $(\mathbf{x}_n)_{0 \leq n < b^m}$ is a (t, m, s) net in base b, we get from (3.24) and (3.25) that

$$\Big\{ \Big(f_{n,j_1}^{(i)}) \Big)_{0 \leq j_1 \leq \tilde{d}_{i,2}, 1 \leq i \leq s} \ \Big| \ n \in [0, b^m) \Big\} = \prod_{i=1}^{s} F_{P_i}^{\tilde{d}_{i,2}+1}.$$

Bearing in mind that $\{\beta_1^{(i)},...,\beta_{e_i}^{(i)}\}$ is a basis of $F_{P_i}/\mathbb{F}_b$ (see Lemma 18), we obtain

$$\Lambda_4 = \Big\{ \Big(\mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(\beta_{j_2}^{(i)} f_{n,j_1}^{(i)}) \Big)_{0\le j_1\le \tilde{d}_{i,2}, 1\le j_2\le e_i, 1\le i\le s} \Bigm| n\in[0,b^m) \Big\} = \mathbb{F}_b^{sd_0e[m\epsilon]}.$$

Let

$$\Lambda_5 = \Big\{ \Big(\mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(\beta_{j_2}^{(i)} f_{n,j_1}^{(i)}) \Big)_{0\le j_1\le \tilde{d}_{i,2}, 1\le j_2\le e_i, 1\le i\le s}, \big(\bar{a}_j(n)\big)_{d_{s+1,1}\le j\le d_{s+1,2}} \Bigm| n\in[0,b^m) \Big\}.$$

By (4.124), (4.97) and (4.106), we have

$$\dim_{\mathbb{F}_b}(\Lambda_5) = \dim_{\mathbb{F}_b}(\Lambda_3) + \dim_{\mathbb{F}_b}(\Lambda_4) \ge d_{s+1,2} - d_{s+1,1} + 1 + sd_0e\dot{m} - r$$

with $r=(g+1)(e_0+s)$, $e=e_1e_2...e_s$ and $\dot{m}=[m\epsilon]$.

Let $\dot{m}_1 = d_0e\dot{m}$, $\epsilon = \eta_1(2sd_0e)^{-1}$, $\dddot{m}_i = 0$, $1\le i\le s$, and $\dddot{m}_{s+1} = d_{s+1,1}+g$, $d_{s+1,1} = t+(s-1)d_0[m\epsilon]e$, $d_{s+1,2} = t-1+sd_0[m\epsilon]e = d_{s+1,1}+\dot{m}_1-1$ (see (4.115)), $\tilde{d}_{i,2} = d_0[m\epsilon]e/e_i - 1 = d_{i,2}+g = \dot{m}_1/e_i - 1$ $(i\in[1,s])$,

$$\dot{\theta}_{n,j_1e_s+j_2}^{(i)} := \mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(\beta_{j_2}^{(i)} f_{n,j_1}^{(i)}) \qquad \text{and} \qquad \dot{\theta}_{n,j+1}^{(s+1)} := f_{n,j}^{(s+1)} = \bar{a}_{j-g}(n) \ \ (\text{see } (4.120))$$

for $0\le j_1\le \tilde{d}_{i,2}$, $1\le j_2\le e_i$, $1\le i\le s, 2g\le j$, and let

$$\Lambda_6 = \Big\{ \Big(\Big(\dot{\theta}_{\dddot{m}_i+d_0e\dot{j}_i+\ddot{j}_i}^{(i)}\Big)_{0\le \dot{j}_i<\dot{m}, 1\le \ddot{j}_i\le d_0e, 1\le i\le s+1} \Bigm| n\in[0,b^m) \Big\}.$$

It is easy to verify that $\Lambda_6=\Lambda_5$ and $\dim_{\mathbb{F}_b}(\Lambda_6) = (s+1)\dot{m}_1 - \dot{r}$ with $0\le \dot{r}\le r=(g+1)(e_0+s)$.

Let $m\ge |2g-2+2(t+g-2)(\eta_1^{-1}-1)| + 2t + 2/\epsilon$. Applying Lemma 2, with $\dot{s}=s+1$, we get that there exists $B_i\subset\{0,...,\dot{m}-1\}$, $1\le i\le s+1$ such that

$$\Lambda_7 = \mathbb{F}_b^{(s+1)\dot{m}_1 - d_0eB}, \qquad \text{where} \quad B = \#B_1+...+\#B_{s+1} \le (g+1)(e_0+s),$$

and

$$\Lambda_7 = \Big\{ \Big(\dot{\theta}_{\dddot{m}_i+d_0e\dot{j}_i+\ddot{j}_i}^{(i)} \Bigm| \dot{j}_i\in\bar{B}_i,\ \ddot{j}_i\in[1,d_0e],\ i\in[1,s+1] \Big) \Bigm| n\in[0,b^m) \Big\}$$

with $\bar{B}_i = \{0,...,\dot{m}-1\}\setminus B_i$. Hence

$$\Big\{ \Big(f_{n,\dddot{m}_i+\dot{j}_id_0e/e_i+\ddot{j}_i-1}^{(i)} \Big| \dot{j}_i\in\bar{B}_i, \ddot{j}_i\in[1,\frac{d_0e}{e_i}], i\in[1,s+1] \Big) \Big| n\in[0,b^m) \Big\} = \prod_{i=1}^{s} F_{P_i}^{\chi_i} \mathbb{F}_b^{\chi_{s+1}}$$

with $e_{s+1}=1$, $\chi_i = d_0e(\dot{m}-\#B_i)/e_i$, $1\le i\le s+1$.
Taking into account that $\sigma_{P_i}: F_{P_i}\to Z_{b^{e_i}}$ is a bijection (see (3.21)), we obtain

$$\Big\{ \Big(\sigma_{P_i}(f_{n,\dddot{m}_i+\dot{j}_id_0e/e_i+\ddot{j}_i-1}^{(i)}) \Bigm| \dot{j}_i\in\bar{B}_i, \ddot{j}_i\in[1,\frac{d_0e}{e_i}], i\in[1,s] \Big),$$

$$\big(a_{\dddot{m}_{s+1}+\dot{j}_{s+1}d_0e+\ddot{j}_{s+1}-1-g}(n) \big| \dot{j}_{s+1}\in\bar{B}_{s+1}, \ddot{j}_{s+1}\in[1,d_0e] \big) \Big| n\in[0,b^m) \Big\} = Z_b^{(s+1)\dot{m}_1-d_0eB}.$$

Let $\tilde{B}_i = \bar{B}_i$, $1 \leq i \leq s$, and let $\tilde{B}_{s+1} = \{\dot{m} - j - 1 | j \in \bar{B}_{s+1}\}$. From (4.103), we derive

$$\Big\{ \Big(x^{(i)}_{n,\ddot{m}_i + \dot{j}_i d_0 e + \ddot{j}_i - 1} \;\Big|\; \dot{j}_i \in \tilde{B}_i, \ddot{j}_i \in [1, d_0 e], i \in [1, s+1] \Big) \Big| n \in [0, b^m) \Big\} = Z_b^{(s+1)\dot{m}_1 - d_0 e B},$$

where $x_n^{(s+1)} = \sum_{j=1}^m x_{n,j}^{(s+1)} b^{-j} := n/b^m$, and $x_{n,j}^{(s+1)} = a_{m-j-1}(n)$ $(1 \leq j \leq m)$, $\ddot{m}_i = \dddot{m}_i = 0$ for $1 \leq i \leq s$ and $\ddot{m}_{s+1} = m - t - s\dot{m}_1 = m - 1 - (\dddot{m}_{s+1} + \dot{m}_1 - 1 - g)$.

By Lemma 17 and Theorem L, we obtain that $(\mathbf{x}_n)_{n \geq 0}$ is a $d-$admissible (t, s)-sequence with $\mathbf{x}_n = (x_n^{(1)}, ..., x_n^{(s)})$, $d = g + e_0$ and $t = g + e_0 - s$.
Now applying Corollary 1 with $\dot{s} = s + 1$, $\tilde{r} = 0$, $\tilde{m} = m$ and $\hat{e} = e = e_1 ... e_{s+1}$, we derive

$$\min_{0 \leq Q < b^m} \min_{\mathbf{w} \in E_m^s} b^m D^*((\mathbf{x}_n \oplus \mathbf{w}, n \oplus Q/b^m)_{0 \leq n < b^m}) \geq 2^{-2} b^{-d} K_{d,t,s+1}^{-s} \eta_1^s m^s,$$

with $m \geq 2^{2s+3} b^{d+t+s+1} (d+t)^{s+1} s^{2s} e(g+1)(e_0+s)\eta_1^{-s}$, and $\eta_1 = (1 + \deg((t_{s+1})_\infty))^{-1}$. Using Lemma B, we get the first assertion in Theorem 4.

Consider the second assertion in Theorem 4.
By (3.23)-(3.25), we get that the net $(\mathbf{x}_n)_{0 \leq n < b^m}$ is constructed similarly to the construction of the Niederreiter-Özbudak net (see (4.61)-(4.69) and (3.15)). The difference is that in the construction of Section 3.3 the map $\sigma_i : F_{P_i} \to \mathbb{F}_b^{e_i}$ is linear, while in the construction of Section 3.4 this map may be nonlinear.

It is easy to verify that this does not affect the proof of bound (3.31) and Theorem 4 follows . □

4.5. **Niederreiter-Xing sequence. Sketch of the proof of Theorem 5.** First we will prove that

$$\dot{\mathcal{C}}_m = \mathcal{M}_m^\perp(P_1, ..., P_s; G_m) \quad \text{for} \quad m \geq g + 1. \tag{4.125}$$

By (2.26) and (3.34), we get

$$\dot{\mathcal{C}}_m = \Big\{ \Big(\sum_{r=0}^{m-1} \dot{c}_{j,r}^{(i)} \bar{a}_r(n) \Big)_{0 \leq j \leq m-1, 1 \leq i \leq s} \;\Big|\; 0 \leq n < b^m \Big\}.$$

Using (4.58) with $\tilde{G} = (g-1)P_{s+1}$, we derive $G_m^\perp = L_m$, where $L_m = \mathcal{L}((m - g + 1)P_{s+1} + W)$. From (3.33), we have

$$\{f^\perp \mid f^\perp \in L_m\} = \{\dot{f}_n := \sum_{r=0}^{m-1} a_r(n) \dot{v}_r \mid n \in [0, b^m)\}.$$

Applying (3.34), we obtain

$$\dot{f}_n \tau_i = \sum_{j=0}^\infty \dot{f}_{n,j}^{(i)} t_i^j, \quad \text{where} \quad \dot{f}_{n,j}^{(i)} = \sum_{r=0}^{m-1} \dot{c}_{j,r}^{(i)} \bar{a}_r(n) \in \mathbb{F}_b, \ i \in [1, s], \ j \geq 0.$$

Therefore

$$\dot{\mathcal{C}}_m = \{(\dot{f}^{(i)}_{n,j})_{0\leq j\leq m-1, 1\leq i\leq s} \mid 0 \leq n < b^m\}. \tag{4.126}$$

We use notations (4.59)-(4.69) with the following modifications. In (4.61) we take the field $\mathbb{F}_b$ instead of F_{P_i} , and in (4.62) we consider the map $\vartheta_i^{\perp}$ as the identical map ($1 \leq i \leq s$). By (4.63), we have $\dot{\theta}^{\perp}_{i,j}(f_n) = \dot{f}^{(i)}_{n,j-1}$ for $1 \leq j \leq m$, and $\dot{\theta}^{\perp}_i(\dot{f}_n) = (\dot{f}^{(i)}_{n,0}, ..., \dot{f}^{(i)}_{n,m-1})$, $1 \leq i \leq s$. According to (4.69) and (4.126) we get

$$\Xi_m = \dot{\Xi}_m = \{\dot{\theta}^{\perp}(f^{\perp}) | f^{\perp} \in \mathcal{L}(G_m^{\perp})\} = \{\dot{\theta}^{\perp}(\dot{f}_n) | n \in [0, b^m)\}$$
$$= \{(\dot{\theta}^{\perp}_1(\dot{f}_n), ..., \dot{\theta}^{\perp}_s(\dot{f}_n)) | n \in [0, b^m)\} = \{(\dot{f}^{(i)}_{n,j})_{0\leq j\leq m-1, 1\leq i\leq s} \mid 0 \leq n < b^m\} = \dot{\mathcal{C}}_m.$$

Now applying (3.13), (3.32) and Lemma 12, we obtain (4.125). By [DiPi, ref. 8.9], we have

$$\delta_m(\mathcal{M}_m) = \delta_m(\mathcal{M}_m(P_1, ..., P_s; G_m)) \geq m - g + 1 \quad \text{for} \quad m \geq g + 1.$$

Taking into account Proposition C, we get that $\mathbf{x}_n(\dot{C})_{n\geq 0}$ is a digital $(\mathbf{T}, s)$ sequence with $T(m) = g$ for $m \geq g + 1$.

Now the $d-$admissible property follow from Lemma 16. In order to complete the proof of Theorem 5, we use Theorem 3 and Theorem 4. □

4.6. **General $d-$admissible (t,s)-sequences. Proof of Theorem 6.** First we will prove Lemma 20. We need the following notations:

Let $\tilde{C}^{(1)}, ..., \tilde{C}^{(\dot{s})}$ are $m \times m$ generating matrices of a digital $(t, m, \dot{s})$-net $(\tilde{\mathbf{x}}_n)_{n=0}^{b^m-1}$ in base b, $\tilde{x}^{(\dot{s})}_n \neq \tilde{x}^{(\dot{s})}_k$ for $n \neq k$, $\tilde{C}^{(i)} = (\tilde{c}^{(i)}_{r,j})_{1\leq r,j\leq m}$, $\tilde{\mathbf{c}}^{(i)}_j = (\tilde{c}^{(i)}_{1,j}, ..., \tilde{c}^{(i)}_{m,j}) \in \mathbb{F}_b^m$, $i \in [1, \dot{s}]$, $\tilde{\mathbf{c}}_j = (\tilde{\mathbf{c}}^{(1)}_j, ..., \tilde{\mathbf{c}}^{(\dot{s})}_j) \in \mathbb{F}_b^{m\dot{s}}$ ($1 \leq j \leq m$). Let $\phi : Z_b \mapsto \mathbb{F}_b$ be a bijection with $\phi(0) = \bar{0}$, and let $n = \sum_{j=1}^m a_j(n) b^{j-1}$, $\mathbf{n} = (\bar{a}_1(n), ..., \bar{a}_m(n)) \in \mathbb{F}_b^m$, $\bar{a}_j(n) = \phi(a_j(n))$, $\tilde{\mathbf{y}}_n = (\tilde{\mathbf{y}}^{(1)}_n, ..., \tilde{\mathbf{y}}^{(\dot{s})}_n) \in \mathbb{F}_b^{m\dot{s}}$, $\tilde{\mathbf{y}}^{(i)}_n = (\tilde{y}^{(i)}_{n,1}, ..., \tilde{y}^{(i)}_{n,m}) \in \mathbb{F}_b^m$,

$$\tilde{\mathbf{x}}_n = (\tilde{x}^{(1)}_n, ..., \tilde{x}^{(\dot{s})}_n), \quad \tilde{x}^{(i)}_n = \sum_{j=1}^m \phi^{-1}(\tilde{y}^{(i)}_{n,j})/b^j \quad \text{for} \quad 1 \leq i \leq \dot{s}, \tag{4.127}$$

$$\tilde{\mathbf{y}}^{(i)}_n = \mathbf{n}(\tilde{\mathbf{c}}^{(i)}_1, ..., \tilde{\mathbf{c}}^{(i)}_m)^{\top} := \sum_{j=1}^m \bar{a}_j(n)\tilde{\mathbf{c}}^{(i)}_j = \mathbf{n}\tilde{C}^{(i)\top} \quad \text{for} \quad 1 \leq i \leq \dot{s}. \tag{4.128}$$

Hence

$$\tilde{\mathbf{y}}_n = \sum_{j=1}^m \bar{a}_j(n)\tilde{\mathbf{c}}_j, \quad \text{for} \quad 0 \leq n < b^m.$$

We put

$$\tilde{\Phi}_m = \{\tilde{\mathbf{x}}_n | n \in [0, b^m)\}, \ \tilde{\Psi}_m = \{\tilde{\mathbf{y}}_n | n \in [0, b^m)\}, \ \tilde{Y}_m = \{\tilde{\mathbf{y}}^{(\dot{s})}_n | n \in [0, b^m)\}.$$

We see that $\tilde{\Psi}_m$ is a vector space over $\mathbb{F}_b$, with $\dim(\tilde{\Psi}_m) \le m$. Taking into account that $\tilde{x}_n^{(\dot{s})} \neq \tilde{x}_k^{(\dot{s})}$ for $n \neq k$, we obtain $\dim(\tilde{\Psi}_m) = m$, $\tilde{\mathfrak{c}}_1, ..., \tilde{\mathfrak{c}}_m$ is the basis of $\tilde{\Psi}_m$ and $\tilde{Y}_m = \mathbb{F}_b^m$.
Let $d \ge 1$, $d_0 = d + t$, $m \ge 4d_0(s+1)$, $\dot{m} = [(m-t)/(2d_0(\dot{s}-1))]$,

$$d_1^{(\dot{s})} = m - t + 1 - (\dot{s}-1)d_0\dot{m} \quad \text{and} \quad d_2^{(\dot{s})} = m - t - (\dot{s}-2)d_0\dot{m}. \tag{4.129}$$

Bearing in mind that $\tilde{\Phi}_m$ is a $(t, m, \dot{s})$ net, we get that for each $j \in [1, (\dot{s}-1)d_0\dot{m}]$ with $j = (j_1 - 1)(\dot{s}-1) + j_2$, $j_1 \in [1, d_0\dot{m}]$ and $j_2 \in [1, \dot{s}-1]$ there exists $n(j) \in [0, b^m)$ such that

$$\tilde{x}^{(\dot{s})}_{n(j),r_1} = \delta_{(j_1-1)(\dot{s}-1)+j_2,r_1} \quad \text{and} \quad \tilde{x}^{(i)}_{n(j),r_2} = \delta_{i,j_2}\delta_{j_1,r_2} \tag{4.130}$$

for all $r_1 \in [1, (\dot{s}-1)d_0\dot{m}]$, $r_2 \in [1, d_0\dot{m}]$, $i \in [1, \dot{s}-1]$.

Taking into account that $Y_m = \mathbb{F}_b^m$, we derive that there exists $n(j) \in [0, b^m)$ with

$$\tilde{y}^{(\dot{s})}_{n(j),r} = \delta_{j,r} \quad \text{for} \quad (\dot{s}-1)d_0\dot{m} + 1 \le j \le m, \quad 1 \le r \le m. \tag{4.131}$$

We take a basis $\dot{\mathfrak{f}}_1, ..., \dot{\mathfrak{f}}_m$ of $\tilde{\Psi}_m$ in the following way:

Let $\dot{\mathfrak{f}}_j = (\dot{\mathfrak{f}}_j^{(1)}, ..., \dot{\mathfrak{f}}_j^{(\dot{s})}) \in \mathbb{F}_b^{m\dot{s}}$ with $\dot{\mathfrak{f}}_j^{(i)} = (\dot{\mathfrak{f}}_{1,j}^{(i)}, ..., \dot{\mathfrak{f}}_{m,j}^{(i)}) \in \mathbb{F}_b^m$, $i \in [1, \dot{s}]$, $j \in [1, m]$. For $j \in [1, m]$, we put $\dot{\mathfrak{f}}_j := \tilde{\mathbf{y}}_{n(j)}$. We have from (4.130) and (4.131) that

$$\dot{\mathfrak{f}}^{(\dot{s})}_{(j_1-1)(\dot{s}-1)+j_2,r_1} = \delta_{(j_1-1)(\dot{s}-1)+j_2,r_1} \quad \text{and} \quad \dot{\mathfrak{f}}^{(i)}_{(j_1-1)(\dot{s}-1)+j_2,r_2} = \delta_{i,j_2}\delta_{j_1,r_2}$$

for $r_1 \in [1, (\dot{s}-1)d_0\dot{m}]$, $r_2 \in [1, d_0\dot{m}]$, $i \in [1, \dot{s}-1]$, $j_1 \in [1, d_0\dot{m}]$, $j_2 \in [1, \dot{s}-1]$ and

$$\dot{\mathfrak{f}}^{(\dot{s})}_{j,r} = \delta_{j,r} \quad \text{for} \quad (\dot{s}-1)d_0\dot{m} + 1 \le j \le m, \ 1 \le r \le m. \tag{4.132}$$

It is easy to see that the vectors $\dot{\mathfrak{f}}_1, ..., \dot{\mathfrak{f}}_m \in \tilde{\Psi}_m$ are linearly independent over $\mathbb{F}_b$. Thus $\dot{\mathfrak{f}}_1, ..., \dot{\mathfrak{f}}_m$ is a basis of $\tilde{\Psi}_m$.
Let

$$\dot{\mathbf{y}}_n^{(i)} = (\dot{y}^{(i)}_{n,1}, ..., \dot{y}^{(i)}_{n,m}) := \mathbf{n}(\dot{\mathfrak{f}}_1^{(i)}, ..., \dot{\mathfrak{f}}_m^{(i)}) = \sum_{j=1}^{m} \bar{a}_j(n)\dot{\mathfrak{f}}_j^{(i)} = \mathbf{n}\dot{\mathcal{F}}^{(i)\,\top}, \tag{4.133}$$

where $\dot{\mathcal{F}}^{(i)} = (\dot{\mathfrak{f}}^{(i)}_{r,j})_{1\le r,j\le m}$ for $1 \le i \le \dot{s}$. Hence

$$\dot{\mathbf{y}}_n := (\dot{\mathbf{y}}_n^{(1)}, ..., \dot{\mathbf{y}}_n^{(\dot{s})}) = \sum_{j=1}^{m} \bar{a}_j(n)\dot{\mathfrak{f}}_j \quad \text{for} \quad 0 \le n < b^m.$$

We put

$$\dot{\Psi}_m = \{\dot{\mathbf{y}}_n \mid 0 \le n < b^m\}.$$

It is easy to see that $\dot{\Psi}_m = \tilde{\Psi}_m$.

For $\ddot{\mathfrak{f}}_j = (\ddot{\mathfrak{f}}_j^{(1)}, ..., \ddot{\mathfrak{f}}_j^{(\dot{s})})$ with $\ddot{\mathfrak{f}}_j^{(i)} = (\ddot{\mathfrak{f}}_{1,j}^{(i)}, ..., \ddot{\mathfrak{f}}_{m,j}^{(i)})$, we define

$$\ddot{\mathfrak{f}}_j = \dot{\mathfrak{f}}_j \ \text{ for } \ j \in [(\dot{s}-1)d_0\dot{m}+1, m] \ \text{ and } \ \ddot{\mathfrak{f}}_j^{(i)} = \dot{\mathfrak{f}}_j^{(i)} \text{ for } i \in [1, \dot{s}-1],\ j \in [1,m],$$

$$\ddot{\mathfrak{f}}_{j,r}^{(\dot{s})} = \bar{0} \quad \text{for} \quad j \in [1, (\dot{s}-1)d_0\dot{m}],\ r \in [d_1^{(\dot{s})}, d_2^{(\dot{s})}], \quad \text{and} \quad \ddot{\mathfrak{f}}_{j,r}^{(\dot{s})} = \dot{\mathfrak{f}}_{j,r}^{(\dot{s})} \tag{4.134}$$

for $j \in [1, (\dot{s}-1)d_0\dot{m}]$ and $r \in [1,m] \setminus [d_1^{(\dot{s})}, d_2^{(\dot{s})}]$. Let

$$\ddot{\mathbf{y}}_n^{(i)} = (\ddot{y}_{n,1}^{(i)}, ..., \ddot{y}_{n,m}^{(i)}) := \mathbf{n}(\ddot{\mathfrak{f}}_1^{(i)}, ..., \ddot{\mathfrak{f}}_m^{(i)}) = \sum_{j=1}^{m} \bar{a}_j(n)\ddot{\mathfrak{f}}_j^{(i)} = \mathbf{n}\ddot{\mathcal{F}}^{(i)\,\top}, \tag{4.135}$$

where $\ddot{\mathcal{F}}^{(i)} = (\ddot{\mathfrak{f}}_{r,j}^{(i)})_{1 \le r,j \le m}$ for $1 \le i \le \dot{s}$. Hence

$$\ddot{\mathbf{y}}_n := (\ddot{\mathbf{y}}_n^{(1)}, ..., \ddot{\mathbf{y}}_n^{(\dot{s})}) = \sum_{j=1}^{m} \bar{a}_j(n)\ddot{\mathfrak{f}}_j \quad \text{for} \quad 0 \le n < b^m. \tag{4.136}$$

We put

$$\ddot{\Psi}_m = \{\ddot{\mathbf{y}}_n \mid 0 \le n < b^m\} \qquad \text{and} \qquad \ddot{Y}_m = \{\ddot{\mathbf{y}}_n^{(\dot{s})} | n \in [0, b^m)\}. \tag{4.137}$$

Now let $\dot{\mathbf{x}}_n = (\dot{x}_n^{(1)}, ..., \dot{x}_n^{(\dot{s})})$ and $\ddot{\mathbf{x}}_n = (\ddot{x}_n^{(1)}, ..., \ddot{x}_n^{(\dot{s})})$, where

$$\dot{x}_n^{(i)} = \sum_{j=1}^{m} \phi^{-1}(\dot{y}_{n,j}^{(i)})/b^j, \quad \text{and} \quad \ddot{x}_n^{(i)} = \sum_{j=1}^{m} \phi^{-1}(\ddot{y}_{n,j}^{(i)})/b^j$$

for $1 \le i \le \dot{s}$. We have

$$\tilde{\Phi}_m = \{\tilde{\mathbf{x}}_n \mid 0 \le n < b^m\} = \{\dot{\mathbf{x}}_n \mid 0 \le n < b^m\} \quad \text{and} \quad \ddot{Y}_m = \mathbb{F}_b^m. \tag{4.138}$$

Bearing in mind that $\dot{\mathfrak{f}}_1, ..., \dot{\mathfrak{f}}_m$ and $\tilde{\mathfrak{c}}_1, ..., \tilde{\mathfrak{c}}_m$ are two basis of the vector space $\tilde{\Psi}_m$, we get that there exists a nonsingular matrix $B = (b_{j,r})_{1 \le j,r \le m}$ with $b_{j,r} \in \mathbb{F}_b$ such that $(\dot{\mathfrak{f}}_1, ..., \dot{\mathfrak{f}}_m)^\top = B(\tilde{\mathfrak{c}}_1, ..., \tilde{\mathfrak{c}}_m)^\top$. Hence

$$\dot{\mathfrak{f}}_k = \sum_{r=1}^{m} b_{k,r}\tilde{\mathfrak{c}}_r, \quad \text{and} \quad \dot{\mathfrak{f}}_{k,j}^{(i)} = \sum_{r=1}^{m} b_{k,r}\tilde{c}_{r,j}^{(i)},$$

for $1 \le k, j \le m, 1 \le i \le \dot{s}$. Therefore

$$(\dot{\mathfrak{f}}_1^{(i)}, ..., \dot{\mathfrak{f}}_m^{(i)})^\top = B(\tilde{\mathfrak{c}}_1^{(i)}, ..., \tilde{\mathfrak{c}}_m^{(i)})^\top \text{ and } \tilde{C}^{(i)} = \dot{\mathcal{F}}^{(i)}B^{-1\,\top} \text{ for } i \in [1, \dot{s}]. \tag{4.139}$$

Let $n' \in [0, b^m)$, $\mathbf{n}' = (\bar{a}_1(n'), ..., \bar{a}_m(n'))$, and let $\mathbf{n}' = \mathbf{n}B^{-1}$.
Using (4.128) and (4.133), we get

$$\dot{\mathbf{y}}_{n'}^{(i)} = \mathbf{n}'\dot{\mathcal{F}}^{(i)\top} = \mathbf{n}'(\dot{\mathfrak{f}}_1^{(i)}, ..., \dot{\mathfrak{f}}_m^{(i)})^\top = \mathbf{n}B^{-1}B(\tilde{\mathfrak{c}}_1^{(i)}, ..., \tilde{\mathfrak{c}}_m^{(i)})^\top$$

$$= \mathbf{n}(\tilde{\mathfrak{c}}_1^{(i)}, ..., \tilde{\mathfrak{c}}_m^{(i)})^\top = \mathbf{n}\tilde{C}^{(i)\top} = \tilde{\mathbf{y}}_n^{(i)}, \quad \text{for} \quad 1 \le i \le \dot{s} \quad \text{and} \quad 0 \le n < b^m.$$

Let $\breve{C}^{(i)} = (\breve{c}^{(i)}_{r,j})_{1\le r,j\le m} := \ddot{\mathcal{F}}^{(i)} B^{-1\,\top}$, $1 \le i \le \dot{s}$, $\breve{\mathbf{c}}^{(i)}_j = (\breve{c}^{(i)}_{1,j},...,\breve{c}^{(i)}_{m,j})$, $1 \le i \le \dot{s}$, $1 \le j \le m$ and let $\breve{\mathbf{y}}_n := \ddot{\mathbf{y}}_{n'}$, $\breve{\mathbf{x}}_n := \ddot{\mathbf{x}}_{n'}$ for $\mathbf{n}' = \mathbf{n}B^{-1}$. We have

$$\breve{\mathbf{y}}^{(i)}_n = \ddot{\mathbf{y}}^{(i)}_{n'} = \mathbf{n}'\ddot{\mathcal{F}}^{(i)\top} = \mathbf{n}B^{-1}\ddot{\mathcal{F}}^{(i)\top} = \mathbf{n}\breve{C}^{(i)\top} \text{ for } 1 \le i \le \dot{s},\ 0 \le n < b^m. \tag{4.140}$$

Hence, $\breve{C}^{(1)},...,\breve{C}^{(\dot{s})}$ are generating matrices of the net $(\breve{\mathbf{x}}_n)_{0\le n<b^m}$. According to (4.134) and (4.139), we obtain $\ddot{\mathcal{F}}^{(i)} = \dot{\mathcal{F}}^{(i)}$,

$$\breve{C}^{(i)} = \tilde{C}^{(i)} \quad \text{for} \quad 1 \le i \le \dot{s}-1, \quad \text{and} \quad \breve{C}^{(\dot{s})} - \tilde{C}^{(\dot{s})} = (\ddot{\mathcal{F}}^{(\dot{s})} - \dot{\mathcal{F}}^{(\dot{s})})B^{-1\,\top}. \tag{4.141}$$

Let $(B^{-1})^\top = (\hat{b}_{r,j})_{1\le r,j\le m}$, $\Delta c_{r,j} = \breve{c}^{(\dot{s})}_{r,j} - \tilde{c}^{(\dot{s})}_{r,j}$ and $\Delta \mathfrak{f}_{r,j} = \ddot{\mathfrak{f}}^{(\dot{s})}_{r,j} - \dot{\mathfrak{f}}^{(\dot{s})}_{r,j}$ for $1 \le r,j \le m$. Applying (4.133), (4.135) and (4.141), we derive

$$\Delta c_{r,j} = \sum_{l=1}^{m} \Delta \mathfrak{f}_{r,l}\hat{b}_{l,j} \quad \text{for} \quad 1 \le r,j \le m. \tag{4.142}$$

From (4.134) and (4.139), we get

$$\Delta c_{r,j} = \breve{c}^{(\dot{s})}_{r,j} - \tilde{c}^{(\dot{s})}_{r,j} = 0 \text{ for } r \in [(\dot{s}-1)d_0\dot{m}+1, m],\ 1 \le j \le m. \tag{4.143}$$

By (4.139) and (4.132), we have

$$\tilde{c}^{(\dot{s})}_{r,j} = \sum_{l=1}^{m} \dot{\mathfrak{f}}^{(\dot{s})}_{r,l}\hat{b}_{l,j} = \hat{b}_{r,j} \quad \text{for} \quad r \in [(\dot{s}-1)d_0\dot{m}]+1, m] \text{ and } 1 \le j \le m. \tag{4.144}$$

Using (4.129), we obtain $d^{(\dot{s})}_1 > (\dot{s}-1)d_0\dot{m}$. By (4.134), (4.142) and (4.144), we get

$$\Delta c_{r,j} = \sum_{l=d^{(\dot{s})}_1}^{d^{(\dot{s})}_2} \Delta \mathfrak{f}_{r,l}\tilde{c}_{l,j} \quad \text{for} \quad r \in [1, (\dot{s}-1)d_0\dot{m}] \quad \text{and} \quad 1 \le j \le m. \tag{4.145}$$

Lemma 20. *With notations as above. Let $\dot{s} \ge 3$, $(\tilde{\mathbf{x}}_n)_{0\le n<b^m}$ be a digital $(t,m,\dot{s})$-net in base b, $\tilde{x}^{\dot{s}}_n \ne \tilde{x}^{\dot{s}}_k$ for $n \ne k$. Then $(\breve{\mathbf{x}}_n)_{0\le n<b^m}$ is a digital $(t,m,\dot{s})$-net in base b with $\breve{x}^{\dot{s}}_n \ne \breve{x}^{\dot{s}}_k$ for $n \ne k$,*

$$\left\|\breve{\mathbf{x}}^{(\dot{s})}_n\right\|_b = \left\|\tilde{\mathbf{x}}^{(\dot{s})}_n\right\|_b \quad \text{for} \quad 0 < n < b^m \tag{4.146}$$

and

$$\Lambda = \mathbb{F}_b^{\dot{s}d_0\dot{m}}, \quad \text{for} \quad m \ge 2d_0\dot{s},\ \dot{m} = [(m-t)/(2d_0(\dot{s}-1))], \tag{4.147}$$

where

$$\Lambda = \{(\breve{y}^{(1)}_{n,d^{(1)}_1},...,\breve{y}^{(1)}_{n,d^{(1)}_2},...,\breve{y}^{(\dot{s})}_{n,d^{(\dot{s})}_1},...,\breve{y}^{(\dot{s})}_{n,d^{(\dot{s})}_2}) \mid n \in [0,b^m)\}$$

with $d_1^{(i)} = 1$, $d_2^{(i)} = d_0\dot{m}$ *for* $1 \leq i < \dot{s}$, $d_1^{(\dot{s})} = m - t + 1 - (\dot{s}-1)d_0\dot{m}$ *and* $d_2^{(\dot{s})} = m - t - (\dot{s}-2)d_0\dot{m}$.

Proof. By (4.140), we have $\check{\mathbf{y}}_n = \ddot{\mathbf{y}}_{n'}$, $\check{\mathbf{x}}_n = \ddot{\mathbf{x}}_{n'}$ and $\tilde{\mathbf{y}}_n = \dot{\mathbf{y}}_{n'}$, $\tilde{\mathbf{x}}_n = \dot{\mathbf{x}}_{n'}$ for $\mathbf{n}' = \mathbf{n}B^{-1}$. Hence, in order to prove the lemma, it is sufficient to take $\ddot{\mathbf{x}}_n$ instead of and $\check{\mathbf{x}}_n$ and $\dot{\mathbf{x}}_n$ instead of $\tilde{\mathbf{x}}_n$. Applying (4.137) and (4.138), we derive that $\ddot{x}_n^{\dot{s}} \neq \ddot{x}_k^{\dot{s}}$ for $n \neq k$.

Suppose that $a_j(n) = 0$ for $1 \leq j \leq (\dot{s}-1)d_0\dot{m}$. By (4.134) and (4.136), we get $\|\ddot{x}_n^{\dot{s}}\|_b = \|\dot{x}_n^{\dot{s}}\|_b$.

Let $a_j(n) = 0$ for $1 \leq j < j_0 \leq (\dot{s}-1)d_0\dot{m}$ and let $a_{j_0}(n) \neq 0$. From (4.134) and (4.136), we have $\left\|\ddot{x}_n^{(\dot{s})}\right\|_b = \left\|\dot{x}_n^{(\dot{s})}\right\|_b = b^{-j_0}$. Hence $\left\|\ddot{\mathbf{x}}_n^{(\dot{s})}\right\|_b = \left\|\dot{\mathbf{x}}_n^{(\dot{s})}\right\|_b$ for all $n \in [1, b^m)$ and (4.146) follows.

Let $\mathbf{d} = (d_1, ..., d_{\dot{s}})$, $d_i \geq 0$ $(i = 1, ..., \dot{s})$, $\ddot{\mathbf{v}}_{\mathbf{d}} = (\ddot{v}_1^{(1)}, ..., \ddot{v}_{d_1}^{(1)}, ..., \ddot{v}_1^{(\dot{s})}, ..., \ddot{v}_{d_{\dot{s}}}^{(\dot{s})}) \in \mathbb{F}_b^{\dot{d}}$, with $\dot{d} = d_1 + ... + d_{\dot{s}}$, and let

$$\ddot{U}_{\ddot{\mathbf{v}}_{\mathbf{d}}} = \{0 \leq n < b^m \mid \ddot{y}_{n,j}^{(i)} = v_j^{(i)},\ 1 \leq j \leq d_i,\ 1 \leq i \leq \dot{s}\}. \tag{4.148}$$

In order to prove that $(\ddot{\mathbf{x}}_n)_{0 \leq n < b^m}$ is a $(t, m, \dot{s})$ net, it is sufficient to verify that $\#\ddot{U}_{\ddot{\mathbf{v}}_{\mathbf{d}}} = b^{m-\dot{d}}$ for all $\ddot{\mathbf{v}}_{\mathbf{d}} \in \mathbb{F}_b^{\dot{d}}$ and all $\mathbf{d}$ with $\dot{d} \leq m - t$. By (4.133), (4.134) and (4.135), we get

$$\dot{\mathbf{y}}_n^{(i)} = \sum_{j=1}^{m} \bar{a}_j(n)\dot{\mathfrak{f}}_j^{(i)} \quad \text{and} \quad \ddot{\mathbf{y}}_n^{(i)} = \sum_{j=1}^{m} \bar{a}_j(n)\ddot{\mathfrak{f}}_j^{(i)}, \quad \text{with} \quad \ddot{\mathfrak{f}}_j^{(i)} = \dot{\mathfrak{f}}_j^{(i)} \tag{4.149}$$

for $1 \leq i \leq \dot{s}-1$, $1 \leq j \leq m$ and $i = \dot{s}$, $(\dot{s}-1)d_0\dot{m} + 1 \leq j \leq m$, $0 \leq n < b^m$. Hence

$$\dot{\mathbf{y}}_n^{(i)} - \ddot{\mathbf{y}}_n^{(i)} = 0 \ \text{ for } \ 1 \leq i \leq \dot{s}-1,\ \ \dot{\mathbf{y}}_n^{(\dot{s})} - \ddot{\mathbf{y}}_n^{(\dot{s})} = \sum_{r=1}^{(\dot{s}-1)d_0\dot{m}} \bar{a}_r(n)(\dot{\mathfrak{f}}_r^{(\dot{s})} - \ddot{\mathfrak{f}}_r^{(\dot{s})}) \tag{4.150}$$

and $\dot{\mathbf{y}}_{n,j}^{(\dot{s})} - \ddot{\mathbf{y}}_{n,j}^{(\dot{s})} = 0$ for $j \in [1, (\dot{s}-1)d_0\dot{m}]$, $0 \leq n < b^m$. Let

$$\dot{v}_j^{(i)} := \ddot{v}_j^{(i)} \text{ for } j \in [1, d_i], i \in [1, \dot{s}-1] \text{ and } \dot{v}_j^{(\dot{s})} := \ddot{v}_j^{(\dot{s})} \text{ for } j \in [1, \min(d_{\dot{s}}, (\dot{s}-1)d_0\dot{m})].$$

For $d_{\dot{s}} > (\dot{s}-1)d_0\dot{m}$ and $j \in [(\dot{s}-1)d_0\dot{m}+1, d_{\dot{s}}]$, we define

$$\dot{v}_j^{(\dot{s})} = \ddot{v}_j^{(\dot{s})} + \sum_{r=1}^{(\dot{s}-1)d_0\dot{m}} \ddot{v}_r^{(\dot{s})}(\dot{\mathfrak{f}}_{r,j}^{(\dot{s})} - \ddot{\mathfrak{f}}_{r,j}^{(\dot{s})}).$$

By (4.132) and (4.149), we get

$$\dot{y}_{n,j}^{(\dot{s})} = \dot{v}_j^{(\dot{s})} \Longleftrightarrow \bar{a}_j(n) = \dot{v}_j^{(\dot{s})} = \ddot{v}_j^{(\dot{s})}, \quad \text{for} \quad j \in [1, \min(d_{\dot{s}}, (\dot{s}-1)d_0\dot{m})],\ n \in [0, b^m).$$

Using (4.150), we obtain for $n \in [0, b^m)$ that

$$\ddot{\mathbf{y}}_{n,j}^{(i)} = \ddot{v}_j^{(i)} \Longleftrightarrow \dot{\mathbf{y}}_{n,j}^{(i)} = \dot{v}_j^{(i)} \quad \text{for} \quad 1 \le j \le d_i,\ 1 \le i \le \dot{s}. \tag{4.151}$$

Let

$$\dot{U}_{\dot{\mathbf{v}}_{\mathbf{d}}} = \{0 \le n < b^m \mid \dot{y}_{n,j}^{(i)} = \dot{v}_j^{(i)},\ 1 \le j \le d_i,\ 1 \le i \le \dot{s}\}$$

with $\dot{\mathbf{v}}_{\mathbf{d}} = (\dot{v}_1^{(1)}, ..., \dot{v}_{d_1}^{(1)}, ..., \dot{v}_1^{(\dot{s})}, ..., \dot{v}_{d_{\dot{s}}}^{(\dot{s})})$.

Taking into account that $(\dot{\mathbf{x}}_n)_{0 \le n < b^m}$ is a $(t, m, \dot{s})$-net in base b, we get from (4.148) and (4.151) that $\#\ddot{U}_{\ddot{\mathbf{v}}_{\mathbf{d}}} = \#\dot{U}_{\dot{\mathbf{v}}_{\mathbf{d}}} = b^{m-\dot{d}}$.

Now consider the statement (4.147). Let $\ddot{\mathbf{v}} = (\ddot{v}^{(1)}_{d_1^{(1)}}, ..., \ddot{v}^{(1)}_{d_2^{(2)}}, ..., \ddot{v}^{(\dot{s})}_{d_1^{(\dot{s})}}, ..., \ddot{v}^{(\dot{s})}_{d_2^{(\dot{s})}}) \in \mathbb{F}_b^{\dot{d}}$, with $\dot{d} = d_2^{(1)} + ... + d_2^{(\dot{s}-1)} + d_2^{(\dot{s})} - d_1^{(\dot{s})} + 1$. It is easy to see that to obtain (4.147), it is sufficient to verify that $\ddot{U}'_{\ddot{\mathbf{v}}} \neq \emptyset$ for all $\ddot{\mathbf{v}} \in \mathbb{F}_b^{\dot{d}}$. where

$$\ddot{U}'_{\ddot{\mathbf{v}}} = \{0 \le n < b^m \mid \ddot{y}_j^{(i)} = \ddot{v}_j^{(i)},\ d_1^{(i)} \le j \le d_2^{(i)},\ 1 \le i \le \dot{s}\}.$$

According to (4.135) and (4.136), $\ddot{U}'_{\ddot{\mathbf{v}}} \neq \emptyset$ if there exists $n \in [0, b^m)$ such that

$$\sum_{r=1}^{m} \bar{a}_r(n) \ddot{\mathfrak{f}}_{j,r}^{(i)} = \ddot{v}_j^{(i)} \quad \text{for all} \quad d_1^{(i)} \le j \le d_2^{(i)} \ \text{and}\ 1 \le i \le \dot{s}. \tag{4.152}$$

By (4.132) and (4.134), we have that (4.152) is true only if $\bar{a}_j(n) = \ddot{v}_j^{(\dot{s})}$

for $d_1^{(\dot{s})} \le j \le d_2^{(\dot{s})}$. Let $n_0 = \sum_{j=d_1^{(\dot{s})}}^{d_2^{(\dot{s})}} \phi^{-1}(\ddot{v}_j^{(\dot{s})}) b^{j-1}$ and let

$$n = n_0 + \sum_{i=1}^{\dot{s}-1} \sum_{j=d_1^{(i)}}^{d_2^{(i)}} \phi(\ddot{v}_j^{(i)} - \ddot{y}_{n_0,j}^{(i)}) b^{(i-1)d_0\dot{m}+j-1}.$$

Therefore $\bar{a}_j(n) = \ddot{v}_j^{(\dot{s})}$ for $j \in [d_1^{(\dot{s})}, d_2^{(\dot{s})}]$ and $\bar{a}_{(i-1)d_0\dot{m}+j}(n) = \ddot{v}_j^{(i)}$ for $j \in [d_1^{(i)}, d_2^{(i)}]$, $i \in [1, \dot{s}-1]$. Using (4.132) and (4.134), we get that (4.152) is true and $\ddot{U}'_{\ddot{\mathbf{v}}} \neq \emptyset$ for all $\ddot{\mathbf{v}} \in \mathbb{F}_b^{\dot{d}}$. Hence (4.147) is proved, and Lemma 20 follows. □

End of the proof of Theorem 6. Let $C^{(1)}, ..., C^{(s)} \in \mathbb{F}_b^{\infty \times \infty}$ be the generating matrices of a digital (t, s)-sequence $(\mathbf{x}_n)_{n \ge 0}$. For any $m \in \mathbb{N}$ we denote the $m \times m$ left-upper sub-matrix of $C^{(i)}$ by $[C^{(i)}]_m$.

Let $m_k = s^2 d_0 (2^{2k+2} - 1)$, $k = 0, 1, ...$,

$$x_n^{(i,k)} = \sum_{j=1}^{m_k} \phi^{-1}(y_{n,j}^{(i,k)})/b^j, \quad \mathbf{y}_n^{(i,k)} = \mathbf{n}[C^{(i)\,\top}]_{m_k} \tag{4.153}$$

and $\mathbf{y}_n^{(i,k)} = (y_{n,1}^{(i,k)}, ..., y_{n,m_k}^{(i,k)})$ for $n \in [0, b^{m_k})$, $i \in [1, s]$.

For $x = \sum_{j\geq 1} x_j p_i^{-j}$, where $x_i \in Z_b = \{0,...,b-1\}$, we define the truncation

$$[x]_m = \sum_{1\leq j\leq m} x_j b^{-j} \quad \text{with} \quad m \geq 1.$$

If $x = (x^{(1)},...,x^{(s)}) \in [0,1)^s$, then the truncation $[\mathbf{x}]_m$ is defined coordinatewise, that is, $[\mathbf{x}]_m = ([x^{(1)}]_m,...,[x^{(s)}]_m)$.

By (2.14) - (2.16), we have

$$[\mathbf{x}_n]_{m_k} = \mathbf{x}_n^{(k)} := (x_n^{(1,k)},...,x_n^{(s,k)}) \quad \text{for} \quad n \in [0, b^{m_k}). \tag{4.154}$$

Let $\hat{C}^{(s+1,0)} = (\hat{c}_{i,j}^{(s+1,0)})_{1\leq i,j\leq m_0}$ with $\hat{c}_{i,j}^{(s+1,0)} = \delta_{i,m_0-j+1}$, $i,j = 1,...,m_0$. We will use (4.127) - (4.141) to construct a sequence of matrices $\hat{C}^{(s+1,k)} \in \mathbb{F}_b^{m_k\times m_k}$ $(k = 1,2,...)$, satisfying the following induction assumption:

For given sequence of matrices $\hat{C}^{(s+1,0)},...,\hat{C}^{(s+1,k-1)}$ there exists a matrix $\hat{C}^{(s+1,k)} = (\hat{c}_{i,j}^{(s+1,k)})_{1\leq i,j\leq m_k}$ such that

$$\hat{c}_{m_k-i+1,j}^{(s+1,k)} = \hat{c}_{m_{k-1}-i+1,j}^{(s+1,k-1)} \ \text{ for } \ i,j \in [1,m_{k-1}] \quad \text{and} \quad \hat{c}_{m_k-i+1,j}^{(s+1,k)} = 0 \tag{4.155}$$

for $i \in [m_{k-1}+1, m_k]$, $j \in [1, m_{k-1}]$, $(x_n^{(1,k)},...,x_n^{(s,k)},\hat{x}_n^{(s+1,k)})_{0\leq n<b^{m_k}}$ is a $(t, m_k, s+1)$-net in base b with

$$\hat{x}_n^{(s+1,k)} \neq \hat{x}_l^{(s+1,k)} \ \text{ for } n \neq l \text{ and } \left\|\hat{x}_n^{(s+1,k)}\right\|_b = \|n\|_b\, b^{-m_k} \text{ for } 0 \leq n < b^{m_k}, \tag{4.156}$$

where

$$\hat{x}_n^{(s+1,k)} = \sum_{j=1}^{m_k} \phi^{-1}(y_{n,j}^{(s+1,k)})/b^j, \quad \mathbf{y}_n^{(s+1,k)} = \mathbf{n}\hat{C}^{(s+1,m_k)\,\top} \tag{4.157}$$

and $\mathbf{y}_n^{(s+1,k)} = (y_{n,1}^{(s+1,k)},...,y_{n,m_k}^{(s+1,k)})$ for $n \in [0, b^{m_k})$.

Let $k = 1$. We take $\hat{c}_{i,j}^{(s+1,1)} = \delta_{i,m_1-j+1}$ for $i,j = 1,...,m_1$.
Now assume we known $\hat{C}^{(s+1,k)}$ and we want to construct $\hat{C}^{(s+1,k+1)}$. We first construct $\tilde{C}^{(s+1,k+1)} = (\tilde{c}_{i,j}^{(s+1,k+1)})_{1\leq i,j\leq m_{k+1}}$ as following

$$\tilde{c}_{m_{k+1}-i+1,j}^{(s+1,k+1)} = \hat{c}_{m_k-i+1,j}^{(s+1,k)} \quad \text{for} \quad i,j \in [1,m_k], \qquad \tilde{c}_{i,j}^{(s+1,k+1)} = \delta_{i,m_{k+1}-j+1} \tag{4.158}$$

$$\text{for} \quad i \in [1, m_{k+1}-m_k],\ j \in [1, m_{k+1}] \quad \text{and} \quad \tilde{c}_{i,j}^{(s+1,k+1)} = \bar{0}$$

for $(i,j) \in [1, m_{k+1}-m_k] \times [1, m_k]$ and $(i,j) \in [m_{k+1}-m_k+1, m_{k+1}] \times [m_k + 1, m_{k+1}]$.

Lemma 21. *With notations as above,* $(x_n^{(1,k+1)},...,x_n^{(s,k+1)},\tilde{x}_n^{(s+1,k+1)})_{0\le n<b^{m_{k+1}}}$ *is a* $(t,m_{k+1},s+1)$*-net in base* b *with* $\tilde{x}_n^{(s+1,k+1)} \ne \tilde{x}_l^{(s+1,k+1)}$ *for* $n \ne l$*, and*

$$\left\| \tilde{x}_n^{(s+1,k+1)} \right\|_b = \|n\|_b \, b^{-m_{k+1}} \quad \text{for} \quad 0<n<b^{m_{k+1}}. \tag{4.159}$$

Proof. Let $\mathbf{d}=(d_1,...,d_{s+1})$, $\mathbf{v_d}=(v_1^{(1)},...,v_{d_1}^{(1)},...,v_1^{(s+1)},...,v_{d_{s+1}}^{(s+1)}) \in \mathbb{F}_b^{\dot{d}}$ with $\dot{d}=d_1+...+d_{s+1}$,

$$\tilde{U}_{\mathbf{v_d}} = \{0\le n<b^{m_{k+1}} \mid y_{n,j}^{(i,k)} = v_j^{(i)}, \quad 1\le j\le d_i,\ 1\le i\le s$$
$$\text{and} \quad \tilde{y}_{n,j}^{(s+1,k+1)} = v_j^{(s+1)}, \quad 1\le j\le d_{s+1}\}. \tag{4.160}$$

In order to prove that $(x_n^{(1,k+1)},...,x_n^{(s,k+1)},\tilde{x}_n^{(s+1,k+1)})_{0\le n<b^{m_{k+1}}}$ is a $(t,m_{k+1},s+1)$-net, it is sufficient to verify that $\#\tilde{U}_{\mathbf{v_d}} = b^{m_{k+1}-\dot{d}}$ for all $\mathbf{v_d}\in\mathbb{F}_b^{\dot{d}}$ and all $\mathbf{d}$ with $\dot{d}\le m_{k+1}-t$.

Suppose that $d_{s+1}\le m_{k+1}-m_k$.
Let $n\in[0,b^{m_{k+1}})$, $n_0\equiv n \pmod{b^{m_{k+1}-d_{s+1}}}$, $n_0\in[0,b^{m_{k+1}-d_{s+1}})$ and let $n_1 = n-n_0$. It is easy to see that

$$\tilde{y}_{n,j}^{(s+1,k+1)} = \tilde{y}_{n_0,j}^{(s+1,k+1)} + \tilde{y}_{n_1,j}^{(s+1,k+1)}.$$

Let $j\in[1,m_{k+1}-m_k]$. By (4.158), we get

$$\tilde{y}_{n,j}^{(s+1,k+1)} = \sum_{r=1}^{m_{k+1}} \bar{a}_r(n)\tilde{c}_{j,r}^{(s+1,k+1)} = \sum_{r=1}^{m_{k+1}-m_k} \bar{a}_r(n)\delta_{j,m_{k+1}+1-r} = \bar{a}_{m_{k+1}+1-j}(n). \tag{4.161}$$

Let $\ddot{n} = \sum_{j=1}^{d_{s+1}} \phi(v_j^{(s+1)}) b^{m_{k+1}-j}$. By (4.160), we get $n\in\tilde{U}_{\mathbf{v_d}} \Leftrightarrow n_1=\ddot{n}$ and $n_0\in\tilde{U}'_{\mathbf{v_d}}$, where

$$\tilde{U}'_{\mathbf{v_d}} = \{0\le \dot{n}<b^{m_{k+1}-d_{s+1}} \mid y_{\dot{n},j}^{(i,k+1)} = v_j^{(i)} - y_{\ddot{n},j}^{(i,k+1)},\ 1\le j\in[1,d_i], i\in[1,s]\}.$$

Bearing in mind (4.157), (4.158), (4.160) and that $(\mathbf{x}(n))_{0\le n<b^{m_{k+1}-d_{s+1}}}$ is a $(t,m_{k+1}-d_{s+1},s)$-net in base b , we obtain $\#\tilde{U}_{\mathbf{v_d}} = \#\tilde{U}'_{\mathbf{v_d}} = b^{m_{k+1}-\dot{d}}$.

Now let $d_{s+1}>m_{k+1}-m_k$. Let $n\in[0,b^{m_{k+1}})$, $n_0\equiv n \pmod{b^{m_k}}$, $n_0\in[0,b^{m_k})$ and let $n_1=n-n_0$. We have

$$\tilde{y}_{n,j}^{(s+1,k+1)} = \tilde{y}_{n_0,j}^{(s+1,k+1)} + \tilde{y}_{n_1,j}^{(s+1,k+1)}.$$

Let $\ddot{n}=\sum_{j=1}^{m_{k+1}-m_k}\phi(v_j^{(s+1)})b^{m_{k+1}-j}$. By (4.160) and (4.161), we get

$$n\in\tilde{U}_{\mathbf{v_d}}\Leftrightarrow n_1=\ddot{n}\ \text{ and }\ n_0\in\{0\le\dot{n}<b^{m_k}\mid y_{\dot{n},j}^{(i,k+1)}=v_j^{(i)}-y_{\ddot{n},j}^{(i,k+1)},\ 1\le j\le d_i,$$
$$1\le i\le s\ \text{ and }\ \ y_{\dot{n},j}^{(s+1,k+1)}=v_j^{(s+1)}-y_{\ddot{n},j}^{(s+1,k+1)},\quad m_{k+1}-m_k+1\le j\le d_{s+1}\}.$$

Let $j\in[m_{k+1}-m_k+1,m_{k+1}]$ and let $j_0=m_{k+1}+1-j\in[1,m_k]$.
By (4.158), we derive

$$\tilde{y}_{\dot{n},j}^{(s+1,k+1)}=\tilde{y}_{\dot{n},m_{k+1}+1-j_0}^{(s+1,k+1)}=\sum_{r=1}^{m_{k+1}}\bar{a}_r(\dot{n})\tilde{c}_{m_{k+1}+1-j_0,r}^{(s+1,k+1)}=\sum_{r=1}^{m_k}\bar{a}_r(\dot{n})\tilde{c}_{m_{k+1}+1-j_0,r}^{(s+1,k+1)}$$

$$(4.162)\qquad =\sum_{r=1}^{m_k}\bar{a}_r(\dot{n})\tilde{c}_{m_k+1-j_0,r}^{(s+1,k)}=\tilde{y}_{\dot{n},m_k+1-j_0}^{(s+1,k)}\quad\text{for all}\quad\dot{n}\in[0,b^{m_k}).$$

We have that $y_{\dot{n},j}^{(i,k+1)}=y_{\dot{n},j}^{(i,k)}$ $(i=1,....,s)$ and $y_{\dot{n},j}^{(s+1,k+1)}=y_{\dot{n},m_k+1-j_0}^{(s+1,k)}$ for $\dot{n}\in[0,b^{m_k})$. Hence

$$n\in\tilde{U}_{\mathbf{v_d}}\Leftrightarrow n_1=\ddot{n}\text{ and }n_0\in\tilde{U}_{\mathbf{v_d}}^{'}=\Big\{0\le\dot{n}<b^{m_k}\mid y_{\dot{n},j}^{(i,k)}=v_j^{(i)}-y_{\ddot{n},j}^{(i,k+1)},j\in[1,d_i],$$
$$i\in[1,s],\ \text{ and }\ \ y_{\dot{n},j-m_{k+1}+m_k}^{(s+1,k)}=v_{j-m_{k+1}+m_k}^{(s+1)}-y_{\ddot{n},j}^{(s+1,k+1)},\ \ j\in(m_{k+1}-m_k,d_{s+1}]\Big\}.$$

Taking into account that $(x_n^{(1,k)},...,x_n^{(s,k)},\tilde{x}_n^{(s+1,k)}))_{0\le n<b^{m_k}}$ is a $(t,m_k,s+1)$-net in base b, we obtain $\#\tilde{U}_{\mathbf{v_d}}=\#\tilde{U}_{\mathbf{v_d}}^{'}=b^{m_k-(\dot{d}-m_{k+1}+m_k)}=b^{m_{k+1}-\dot{d}}$. Therefore $(x_n^{(1,k+1)},...,x_n^{(s,k+1)},\tilde{x}_n^{(s+1,k+1)})_{0\le n<b^{m_{k+1}}}$ is a $(t,m_{k+1},s+1)$-net in base b.

From (4.158), (4.161), (4.162) and the induction assumption, we get that $\tilde{x}_n^{(s+1,k+1)}\neq\tilde{x}_l^{(s+1,k+1)}$ for $n\neq l$.

Consider the assertion (4.159). Let $n\in[0,b^{m_{k+1}})$ and let

$$(4.163)\qquad \left\|\tilde{x}_n^{(s+1,k+1)}\right\|_b=b^{-j_1}.$$

Hence $\tilde{y}_{n,j}^{(s+1,k+1)}=0$ for $1\le j\le j_1-1$ and $\tilde{y}_{n,j_1}^{(s+1,k+1)}\neq 0$ (see (1.4)).

Let $j_1\in[1,m_{k+1}-m_k]$. By (4.161), we get $\bar{a}_{m_{k+1}+1-j}(n)=0$ for $1\le j\le j_1-1$ and $\bar{a}_{m_{k+1}+1-j_1}(n)\neq 0$. Therefore $\|n\|_b=\left\|\sum_{i=1}^{m_{k+1}}a_i(n)b^{i-1}\right\|_b=b^{m_{k+1}-j_1}$.

Now let $j_1\in[m_{k+1}-m_k+1,m_{k+1}]$. From (4.161), we obtain $\bar{a}_{m_{k+1}+1-j}(n)=0$ for $1\le j\le m_{k+1}-m_k$. Hence $n\in[0,b^{m_k})$. Using (4.158) and (4.161), we have $\tilde{y}_{n,j}^{(s+1,k)}=\tilde{y}_{n,j-m_{k+1}+m_k}^{(s+1,k)}$ for $m_{k+1}-m_k+1\le j\le j_1$. Therefore $\tilde{y}_{n,j}^{(s+1,k)}=0$ for $1\le j\le j_1-m_{k+1}+m_k-1$ and $\tilde{y}_{n,j_1-m_{k+1}+m_k}^{(s+1,k)}\neq 0$. Using the induction assumption (4.156), we get $b^{-j_1+m_{k+1}-m_k}=\left\|\tilde{x}_n^{(s+1,k)}\right\|_b=\|n\|_b\,b^{-m_k}$.

By (4.163), we obtain $\left\|\tilde{x}_n^{(s+1,k+1)}\right\|_b = \|n\|_b b^{-m_{k+1}}$. Thus assertion (4.159) is proved and Lemma 21 follows. $\square$

Now we apply (4.127) - (4.141) with $\dot{s} = s+1$, $m = m_{k+1}$, $\tilde{C}^{(i)} := [C^{(i)}]_{m_{k+1}}$ $(i = 1, ..., s)$ and $\tilde{C}^{(s+1)} := \tilde{C}^{(s+1,k+1)}$ to construct matrices $\check{C}^{(i)}$ $(i = 1, ..., s+1)$.

From (4.141), we have

$$\check{C}^{(i)} = \tilde{C}^{(i)} = [C^{(i)}]_{m_{k+1}} \quad \text{for} \quad i = 1, ..., s. \tag{4.164}$$

Let $\hat{C}^{(s+1,k+1)} := \check{C}^{(s+1)}$. According to (4.143) and (4.158), we get

$$\hat{c}_{r,j}^{(s+1,k+1)} - \tilde{c}_{r,j}^{(s+1,k+1)} = 0 \quad \text{for} \quad r \in [sd_0\dot{m}_{k+1}+1, m_{k+1}] \text{ and } 1 \leq j \leq m_{k+1}. \tag{4.165}$$

By (4.129) and (4.145), we obtain for $r \in [1, sd_0\dot{m}_{k+1}]$ and $1 \leq j \leq m_{k+1}$

$$\hat{c}_{r,j}^{(s+1,k+1)} - \tilde{c}_{r,j}^{(s+1,k+1)} = \sum_{l=d_1^{(s+1,k+1)}}^{d_2^{(s+1,k+1)}} \Delta\mathfrak{f}_{r,l}^{(s+1,k+1)} \tilde{c}_{l,j}^{(s+1,k+1)}, \tag{4.166}$$

where $d_1^{(s+1,k+1)} = m_{k+1} - t + 1 - sd_0\dot{m}_{k+1}$, $d_2^{(s+1,k+1)} = m_{k+1} - t - (s-1)d_0\dot{m}_{k+1}$, $m_{k+1} = s^2 d_0(2^{2k+4}-1)$, $d_0 = d+t$ and $\dot{m}_{k+1} = [(m_{k+1}-t)/(2sd_0)]$.

We have $d_1^{(s+1,k+1)} > (s-1)d_0\dot{m}_{k+1}$, $\dot{m}_{k+1} = 2^{2k+3}-1$ for $k = 0, 1, ...$ and

$$m_{k+1} - d_2^{(s+1,k+1)} \geq (s-1)d_0\dot{m}_{k+1} \geq 2^{-1}s^2 d_0(2^{2k+3}-1) > m_k.$$

By (4.158), we obtain $\tilde{c}_{r,j}^{(s+1,k+1)} = 0$ for $r \leq d_2^{(s+1,k+1)} < m_{k+1} - m_k$ and $1 \leq j \leq m_k$.

From (4.166), we derive

$$\hat{c}_{r,j}^{(s+1,k+1)} - \tilde{c}_{r,j}^{(s+1,k+1)} = 0 \quad \text{for} \quad r \in [1, sd_0\dot{m}_{k+1}] \quad \text{and} \quad 1 \leq j \leq m_k. \tag{4.167}$$

Bearing in mind that

$$m_{k+1} - sd_0\dot{m}_{k+1} = s^2 d_0(2^{2k+4}-1) - s^2 d_0(2^{2k+3}-1) = s^2 d_0 2^{2k+3} > m_k,$$

we get from (4.165) and (4.158)

$$\hat{c}_{m_{k+1}-i+1,j}^{(s+1,k+1)} = \tilde{c}_{m_{k+1}-i+1,j}^{(s+1,k+1)} = \hat{c}_{m_k-i+1,j}^{(s+1,k)} \quad \text{for} \quad 1 \leq i,j \leq m_k. \tag{4.168}$$

Applying (4.158), (4.165) and (4.167), we have

$$\hat{c}_{i,j}^{(s+1,k+1)} = \tilde{c}_{i,j}^{(s+1,k+1)} = 0, \text{ for } 1 \leq i \leq m_{k+1} - m_k,\ 1 \leq j \leq m_k$$

Now using (4.168), we obtain (4.155).

We see that (4.156) follows from (4.159) and (4.146). Consider the net $(\hat{\mathbf{x}}_n^{(k+1)})_{n=0}^{b^{m_{k+1}}-1}$ with $\hat{\mathbf{x}}_n^{(k+1)} = (x_n^{(1,k+1)}, ..., x_n^{(s,k+1)}, \hat{x}_n^{(s+1,k+1)}) := \check{\mathbf{x}}_n = (\check{x}_n^{(1)}, ..., \check{x}_n^{(s+1)})$. Let

$$\Lambda_{k+1} = \left\{ \left((y_{n,1}^{(i,k+1)}, ..., y_{n,d^{(i,k+1)}}^{(i,k+1)})_{1 \leq i \leq s}, \hat{y}_{n,d_1^{(s+1,k+1)}}^{(s+1,k+1)}, ..., \hat{y}_{n,d_2^{(s+1,k+1)}}^{(s+1,k+1)} \right) \;\middle|\; n \in [0, b^{m_{k+1}}) \right\}$$

with $d^{(i,k+1)} = d_0 \dot{m}_{k+1}$ for $1 \le i \le s$. Using (4.129), (4.164) and Lemma 20, we obtain

$$\Lambda_{k+1} = \mathbb{F}_b^{(s+1)d_0\dot{m}_{k+1}}, \quad \text{for} \quad \dot{m}_{k+1} = [(m_{k+1} - t)/(2sd_0)] = s(2^{k+1} - 1), \tag{4.169}$$

and $(\hat{\mathbf{x}}_n^{(k+1)})_{0 \le n < b^{m_{k+1}}}$ is a $(t, m_{k+1}, s+1)$-net in base b. Thus we have that $\hat{C}^{(s+1,k+1)}$ satisfy the induction assumption.

Let $C^{(s+1,k+1)} = (c_{i,j}^{(s+1,k+1)})_{1 \le i,j \le m_{k+1}}$ where $c_{i,j}^{(s+1,k+1)} := \hat{c}_{m_{k+1}-i+1,j}^{(s+1,k+1)}$ for $1 \le i, j \le m_{k+1}$. By (4.155), we get

$$[C^{(s+1,k+1)}]_{m_k} = C^{(s+1,k)} \text{ and } c_{i,j}^{(s+1,k+1)} = 0, \ i \in (m_k, m_{k+1}], \ j \in [1, m_k]. \tag{4.170}$$

Now let $C^{(s+1)} = (c_{i,j}^{(s+1)})_{i,j \ge 1} = \lim_{k \to \infty} C^{(s+1,k)}$ i.e. $[C^{(s+1)}]_{m_k} := C^{(s+1,k)}$, $k = 1, 2, \ldots$. We define

$$h_k(n) := h_{k,1}(n) + \ldots + h_{k,m_k}(n) b^{m_k - 1} := \hat{x}_n^{(s+1,k)} b^{m_k} \quad \text{for} \quad 0 \le n < b^{m_k}. \tag{4.171}$$

From (4.157), we have

$$\begin{aligned} \phi(h_{k,i}(n)) &= \phi(\hat{x}_{n,m_k-i+1}^{(s+1,k)}) = \hat{y}_{n,m_k-i+1}^{(s+1,k)} = \textstyle\sum_{j=1}^{m_k} \bar{a}_j(n) \hat{c}_{m_k-i+1,j}^{(s+1,k)} \\ &= \textstyle\sum_{j=1}^{m_k} \bar{a}_j(n) c_{m_k-i+1,j}^{(s+1,k)} \quad \text{for} \quad 0 \le n < b^{m_k}. \end{aligned} \tag{4.172}$$

Applying (4.170), we obtain for $n \in [0, b^{m_k})$ that

$$h_{k,i}(n) = 0 \text{ for } i > m_k \text{ and } h_k(n) = h_{k-1}(n) \in [0, b^{m_{k-1}}) \text{ for } n \in [0, b^{m_{k-1}}). \tag{4.173}$$

For $n \in [1, b^{m_k})$, we get from (4.172) and (4.156) that

$$\|h_k(n)\|_b = \|n\|_b \,. \tag{4.174}$$

Let $l \ne n \in [0, b^{m_k})$. Using (4.156), we have $(\hat{y}_{l,1}^{(s+1,k)}, \ldots, \hat{y}_{l,m_k}^{(s+1,k)}) \ne (\hat{y}_{n,1}^{(s+1,k)}, \ldots, \hat{y}_{n,m_k}^{(s+1,k)})$. Hence $(h_{k,1}(l), \ldots, h_{k,m_k}(l)) \ne (h_{k,1}(n), \ldots, h_{k,m_k}(n))$ and $h_k(l) \ne h_k(n)$.

Therefore h_k is a bijection from $[0, b^{m_k})$ to $[0, b^{m_k})$. We define $h_k^{-1}(n)$ such that $h_k(h_k^{-1}(n)) = n$ for all $n \in [0, b^{m_k})$.
Let $n \in [0, b^{m_k})$ and $l = h_k^{-1}(n)$, then $l \in [0, b^{m_k})$ and $h_{k+1}(l) = h_k(l) = n$. Thus

$$h_{k+1}^{-1}(n) = h_k^{-1}(n) = l \quad \text{for} \quad n \in [0, b^{m_k}). \tag{4.175}$$

Let $h(n) = \lim_{k \to \infty} h_k(n)$, and $h^{-1}(n) = \lim_{k \to \infty} h_k^{-1}(n)$.
Let $n \in [0, b^{m_k})$ and let $l = h_k^{-1}(n)$. By (4.173) and (4.175), we get

$$h(n) = h_k(n) = l, \quad h^{-1}(l) = h_k^{-1}(l) = n, \quad \text{and} \quad h^{-1}(h(n)) = n.$$

Consider the $d-$admissible property of the sequence $(\mathbf{x}_{h^{-1}(n)})_{n \ge 0}$. It is sufficient to take $k = 0$ in (1.4).

Let $n \in [0, b^{m_k})$. By (4.174), we have $\|h(n)\|_b = \|h_k(n)\|_b = \|n\|_b$. Taking into account Definition 5 and that $(\mathbf{x}_n)_{n\geq 0}$ is a $d-$admissible sequence, we obtain

$$\|n\|_b \left\|\mathbf{x}_{h^{-1}(n)}\right\|_b = \|h(l)\|_b \|\mathbf{x}_l\|_b = \|l\|_b \|\mathbf{x}_l\|_b \geq b^{-d}, \quad \text{with} \quad l = h^{-1}(n). \tag{4.176}$$

Hence $(\mathbf{x}_{h^{-1}(n)})_{n\geq 0}$ is a $d-$admissible sequence.

By the induction assumption, $([\mathbf{x}_n]_{m_k}, h_k(n)/b^{m_k})_{0\leq n<b^{m_k}}$ is a $(t, m_k, s+1)$-net in base b for $k \geq 1$. Hence $(\mathbf{x}_n, h(n)/b^{m_k})_{0\leq n<b^{m_k}}$ and $(\mathbf{x}_{h^{-1}(n)}, n/b^{m_k})_{0\leq n<b^{m_k}}$ are also $(t, m_k, s+1)$-nets in base b for $k \geq 1$. By Lemma 1, $(\mathbf{x}_{h^{-1}(n)})_{n\geq 0}$ is a (t, s)-sequence in base b.

Let $N \in [b^{m_k}, b^{m_{k+1}})$. Applying Lemma B, we get

$$\begin{aligned}
\sigma &:= 1 + \min_{0\leq Q<b^{m_k}, \mathbf{w}\in E^s_{m_k}} \max_{1\leq M\leq N} MD^*((\mathbf{x}_{h^{-1}(n\ominus Q)} \oplus \mathbf{w})_{0\leq n<M}) \\
&\geq 1 + \min_{0\leq Q<b^{m_k}, \mathbf{w}\in E^s_{m_k}} \max_{1\leq M\leq b^{m_k}} MD^*((\mathbf{x}_{h^{-1}(n\ominus Q)} \oplus \mathbf{w})_{0\leq n<M}) \\
&\geq \min_{0\leq Q<b^{m_k}, \mathbf{w}\in E^s_{m_k}} b^{m_k} D^*((\mathbf{x}_{h^{-1}(n\ominus Q)} \oplus \mathbf{w}, n/b^{m_k})_{0\leq n<b^{m_k}}) \\
&\geq \min_{0\leq Q<b^{m_k}, \mathbf{w}\in E^s_{m_k}} b^{m_k} D^*((\mathbf{x}_l \oplus \mathbf{w}, h(l) \oplus Q/b^{m_k})_{0\leq l<b^{m_k}})
\end{aligned}$$

where $l = h^{-1}(n \ominus Q)$ and $n = h(l) \oplus Q$. Bearing in mind that $h(n) = h_k(n)$ for $0 \leq n < b^{m_k}$, and that $\hat{x}^{(s+1,k)}_n = h_k(n)/b^{m_k}$ for $0 \leq n < b^{m_k}$, we get

$$\sigma \geq \min_{0\leq Q<b^{m_k}, \mathbf{w}\in E^s_{m_k}} b^{m_k} D^*((\mathbf{x}_n \oplus \mathbf{w}, \hat{x}^{(s+1,k)}_n \oplus (Q/b^{m_k}))_{0\leq n<b^{m_k}}). \tag{4.177}$$

By (4.176) and (1.4), we obtain that $(\mathbf{x}_n, h(n)/b^{m_k})_{0\leq n<b^{m_k}}$ is a d-admissible net.

Applying (4.154) and the induction assumption, we get that $(\mathbf{x}_n, h(n)/b^{m_k})_{0\leq n<b^{m_k}}$ is a $(t, m_k, s+1)$ net in base b. Let

$$\Lambda'_k = \Big\{ \Big((y^{(i)}_{n,1}, ..., y^{(i)}_{n,d^{(i,k)}})_{1\leq i\leq s}, \ \hat{y}^{(s+1,k)}_{n,d^{(s+1,k)}_1}, ..., \hat{y}^{(s+1,k)}_{n,d^{(s+1,k)}_2} \Big) \ \Big| \ n \in [0, b^{m_k}) \Big\}.$$

Using (4.153), (4.154) and (4.171) , we obtain $y^{(i)}_{n,j} = y^{(i,k)}_{n,j}$ for $1 \leq j \leq m_k$, $1 \leq i \leq s$, and $h(n)/b^{m_k} = \hat{x}^{(s+1,k)}_n$. By (4.169), we have

$$\Lambda'_k = \Lambda_k = \mathbb{F}_b^{(s+1)d_0\dot{m}}, \quad \text{for} \quad \dot{m} = [(m_k - t)/(2sd_0)] = d^{(s+1,k)}_2 - d^{(s+1,k)}_1 + 1.$$

Now we apply Corollary 2 with $\dot{s} = s + 1$, $\epsilon = (2sd_0)^{-1}$, $\eta = \hat{e} = 1$, $\tilde{r} = t$, $m = m_k$, $\tilde{m} = m - t$, $\ddot{m}_{s+1} = d^{(s+1,k)}_1 - 1$, $B_i = \emptyset$ for $i \in [1, s+1]$, and $B = 0$. Taking into account (4.177), we get the assertion in Theorem 6. □

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