ON SOME IDENTITIES AND GENERATING FUNCTIONS FOR PELL-LUCAS NUMBERS

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Abstract. Generating functions for Pell and Pell-Lucas numbers are obtained. Applications are given for some results recently obtained by Mansour [12]; by using an alternative approach that considers the action of operator $\delta^{k}_{e_1 e_2}$ to the series $\sum_{j=0}^{\infty} a_j (e_1 z)^j$.

1. Introduction and the Main Result

Let us specify a second-order linear recurrence sequence $(U_n(a,b;p,q))_{n\geq 0}$, or briefly $(U_n)_{n\geq 0}$ by

$$U_{n+2} = pU_{n+1} + qU_n, ,$$

with $U_0 = a, U_1 = b$ and $n > 0$ (see [6, 11]) for example. This sequence was introduced in 1965 by Horadam [6, 7], which were generalized innumerable sequences, mostly depending on p, q, a and b (see [8]). Some examples of such sequences are the Pell number $(P_n)_{n\geq 0}$ and Pell-Lucas number sequences $(Q_n)_{n\geq 0}$; when one has $p = 2, q = b = 1, a = 0$ and $p = b = a = 2, q = 1$ respectively. In this paper we give some new generating functions for Pell and Pell-Lucas numbers.

In this contribution, we shall define a new useful operator denoted by $\delta^{k}_{e_1 e_2}$ for which we can formulate, extend, and prove new results based on our previous ones [2, 3, 4]. In order to determine generating functions for Pell and Pell-Lucas numbers and Chebychev polynomials of second kind, we combine between our indicated past techniques and these presented polishing approaches.

In order to render the work self-contained, we give the necessary preliminary tools; we recall some definitions and results.

Definition 1.1. [1] *Let A and B be any two alphabets, then we give $S_j(A-B)$ by the following form:*

$$\frac{\prod_{b\epsilon B}(1-bz)}{\prod_{a\epsilon A}(1-az)} = \sum_{j=0}^{\infty} S_j(A-B) z^j,$$

with the condition $S_j(A-B) = 0$ for $j < 0$.

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Corollary 1.2. *Taking $A = 0$ in* (1.1) *gives*

$$\prod_{b\in B}(1-zb) = \sum_{j=0}^{\infty} S_j(-B)z^j.$$

Definition 1.3. [3] *Given a function g on $\mathbb{R}^n$, the divided difference operator is defined as follows*

$$\partial_{x_i x_{i+1}}(g) = \frac{g(x_1,\cdots,x_i,x_{i+1},\cdots x_n) - g(x_1,\cdots x_{i-1},x_{i+1},x_i,x_{i+2}\cdots x_n)}{x_i - x_{i+1}}.$$

Definition 1.4. [4] *Given an alphabet $E = \{e_1, e_2\}$, the symmetrizing operator $\delta^k_{e_1e_2}$ is defined by*

$$\delta^k_{e_1e_2}(f) = \frac{e_1^k f(e_1) - e_2^k f(e_2)}{e_1 - e_2} \text{ for all } j \in \mathbb{N}. \tag{1.1}$$

If $f(e_1) = e_1$, the operator (1.1) gives us

$$\delta^k_{e_1e_2}(f) = S_k(e_1+e_2) = \partial_{e_1e_2}(e_1^{k+1}).$$

Proposition 1.5. [5] *Let $E = \{e_1, e_2\}$, we define the operator $\delta^k_{e_1e_2}$ as follows:*

$$\delta^k_{e_1e_2}f\ (e_1) = S_{k-1}(e_1+e_2)f\ (e_1) + e_2^k\partial_{e_1e_2}f\ (e_1), \text{ for all } k \in \mathbb{N}.$$

Proposition 1.6. [10] *The relations*

$$\begin{aligned} 1)\ P_{-j} &= (-1)^{j+1}P_j, \\ 2)\ Q_{-j} &= (-1)^j Q_j \end{aligned}$$

hold for all $j \geq 0$.

In this paper, we shall combine all these above tools in a unified way. Then it seems that all our results can be treated as special cases of the following theorem.

Theorem 1.7. *Given an alphabet $E = \{e_1, e_2\}$, two sequences $\sum_{j=0}^{+\infty} a_j z^j$, $\sum_{j=0}^{+\infty} b_j z^j$ such that $\left(\sum_{j=0}^{+\infty} a_j z^j\right)\left(\sum_{j=0}^{+\infty} b_j z^j\right) = 1$, then*

$$\sum_{j=0}^{\infty} a_j\ \partial_{e_1e_2}(e_1^{k+j})z^j = \frac{\sum\limits_{j=0}^{k-1} b_j(e_1e_2)^j\partial_{e_1e_2}(e_1^{k-j})z^j - (e_1e_2z)^k\sum\limits_{j=0}^{\infty} b_{j+k+1}\partial_{e_1e_2}(e_1^{j+1})z^{j+1}}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)}. \tag{1.2}$$

The paper is organized as follows. In Section 2 we give the proof our main Theorem. In Section 3 we give some applications for this Theorem.

2. **Proof of main result**

In this section, we present a proof of Theorem 1.6.

Let $\sum\limits_{j=0}^{\infty} a_j z^j$ and $\sum\limits_{j=0}^{\infty} b_j z^j$ be two sequences such that $\left(\sum_{j=0}^{+\infty} a_j z^j\right)\left(\sum_{j=0}^{+\infty} b_j z^j\right) = 1$.

In the first instance, since $f(e_1) = \sum\limits_{j=0}^{\infty} a_j e_1^j z^j$, we have

$$\begin{aligned} \delta_{e_1e_2}^k f\ (e_1) &= \delta_{e_1e_2}^k \left(\sum_{j=0}^{\infty} a_j e_1^j z^j\right) \\ &= \sum_{j=0}^{\infty} a_j\ \partial_{e_1e_2}(e_1^{k+j}) z^j, \end{aligned}$$

which is the left-hand side of (1.2). On the other hand, since

$$f(e_1) = \frac{1}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j},$$

we have

$$\begin{aligned} \partial_{e_1e_2} f(e_1) &= \frac{1}{e_1 - e_2}\left(\frac{1}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j} - \frac{1}{\sum\limits_{j=0}^{\infty} b_j e_2^j z^j}\right) \\ &= \frac{1}{e_1 - e_2}\left(\frac{\sum\limits_{j=0}^{\infty} b_j e_2^j z^j - \sum\limits_{j=0}^{\infty} b_j e_1^j z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)}\right) \\ &= \frac{\sum\limits_{j=0}^{\infty} b_j \frac{e_2^j - e_1^j}{e_1 - e_2} z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)} \\ &= -\frac{\sum\limits_{j=0}^{\infty} b_j S_j(e_1 + e_2) z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)} \end{aligned}$$

By Proposition 1.5, it follows that

$$\begin{aligned}
\delta_{e_1e_2}^{k} f\ (e_1) &= S_{k-1}(e_1+e_2) f\ (e_1) + e_2^k \partial_{e_1e_2} f\ (e_1) \\
&= \frac{S_{k-1}(e_1+e_2)}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j} - \frac{\sum\limits_{j=0}^{\infty} b_j S_{j-1}(e_1+e_2) z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)} \\
&= \frac{\sum\limits_{j=0}^{\infty} b_j \left[e_2^j S_{k-1}(e_1+e_2) - e_2^k S_{j-1}(e_1+e_2)\right] z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)},
\end{aligned}$$

which can also be cast in the form

$$\begin{aligned}
\delta_{e_1e_2}^{k} f\ (e_1) = {} & \frac{\sum\limits_{j=0}^{k-1} b_j \left[e_2^j S_{k-1}(e_1+e_2) - e_2^k S_{j-1}(e_1+e_2)\right] z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)} \\
& + \frac{\sum\limits_{j=k+1}^{\infty} b_j \left[e_2^j S_{k-1}(e_1+e_2) - e_2^k S_{j-1}(e_1+e_2)\right] z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)} \\
= {} & \frac{\sum\limits_{j=0}^{k-1} b_j (e_1e_2)^j \partial_{e_1e_2}(e_1^{k-j}) z^j - (e_1e_2z)^k \sum\limits_{j=0}^{\infty} b_{j+k+1} \partial_{e_1e_2}(e_1^{j+1}) z^{j+1}}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)}.
\end{aligned}$$

This completes the proof.

3. **On the Generating Functions of Some Numbers and Polynomials**

In this part, we derive several new generating functions of some known numbers and polynomials. Indeed, we consider our previous Theorem in order to derive Pell, Pell-Lucas numbers and Chebychev polynomials of the second kind.

Lemma 3.1. *Given an alphabet $E = \{e_1, e_2\}$, we have*

$$\sum_{j=0}^{\infty} \partial_{e_1e_2}(e_1^{j+1}) z^j = \frac{1}{1-(e_1+e_2)z+e_1e_2z^2}. \tag{3.1}$$

Proof. Let $\sum_{j\geq 0}^{\infty} e_1^j z^j$ and $(1-e_1z)$ be two sequences such that $\sum_{j\geq 0}^{\infty} e_1^j z^j = \frac{1}{1-e_1z}$, the left-hand side of the formula (3.1) can be written as:

$$\begin{aligned}
\delta_{e_1e_2}\sum_{j=0}^{\infty} e_1^j z^j &= \frac{e_1\sum_{j=0}^{\infty} e_1^j z^j - e_2\sum_{j=0}^{\infty} e_1^j z^j}{e_1-e_2} \\
&= \sum_{j=0}^{\infty} \frac{e_1^{j+1} - e_2{}^{j+1}}{e_1-e_2} z^j \\
&= \sum_{j=0}^{\infty} \partial_{e_1e_2}(e_1^{j+1}) z^j.
\end{aligned}$$

while the right-hand side can be expressed as

$$\begin{aligned}
\delta_{e_1e_2}\left(\frac{1}{1-e_1z}\right) &= \frac{e_1\frac{1}{1-e_1z} - e_2\frac{1}{1-e_2z}}{e_1-e_2} \\
&= \\
&= \frac{e_1(1-e_2z)-e_2(1-e_1z)}{(e_1-e_2)(1-e_1z)(1-e_2z)} \\
&= \frac{e_1-e_2}{(e_1-e_2)(1-(e_1+e_2)z+e_1e_2z^2)} \\
&= \frac{1}{1-(e_1+e_2)z+e_1e_2z^2}.
\end{aligned}$$

This completes the proof. □

Lemma 3.2. *Given an alphabet $E = \{e_1, e_2\}$, we have*

$$\sum_{j=0}^{\infty} \partial_{e_1e_2}(e_1^{j+2}) z^j = \frac{e_1+e_2-e_1e_2z}{1-(e_1+e_2)z+e_1e_2z^2}. \tag{3.2}$$

Proof. Let $\sum_{j\geq 0}^{\infty} e_1^j z^j$ and $(1-e_1z)$ be two sequences such that $\sum_{j\geq 0}^{\infty} e_1^j z^j = \frac{1}{1-e_1z}$, the left-hand side of the formula (3.2), can be written as

$$\begin{aligned}
\delta_{e_1e_2}^2\sum_{j=0}^{\infty} e_1^j z^j &= \frac{e_1^2\sum_{j=0}^{\infty} e_1^j z^j - e_2^2\sum_{j=0}^{\infty} e_1^j z^j}{e_1-e_2} \\
&= \sum_{j=0}^{\infty} \frac{e_1^{j+2} - e_2{}^{j+2}}{e_1-e_2} z^j \\
&= \sum_{j=0}^{\infty} \partial_{e_1e_2}(e_1^{j+2}) z^j.
\end{aligned}$$

while the right-hand side can be expressed as

$$
\begin{aligned}
\delta_{e_1e_2}^2\left(\frac{1}{1-e_1z}\right) &= \frac{e_1^2\frac{1}{1-e_1z}-e_2^2\frac{1}{1-e_2z}}{e_1-e_2} \\
&= \\
&= \frac{e_1^2(1-e_2z)-e_2^2(1-e_1z)}{(e_1-e_2)(1-e_1z)(1-e_2z)} \\
&= \frac{(e_1-e_2)(e_1+e_2-e_1e_2z)}{(e_1-e_2)(1-(e_1+e_2)z+e_1e_2z^2)} \\
&= \frac{e_1+e_2-e_1e_2z}{1-(e_1+e_2)z+e_1e_2z^2}
\end{aligned}
$$

This completes the proof. □

This case consists of two related parts. **Firstly**, by replacing e_2 by $(-e_2)$ in (3.1) and (3.2), we obtain

$$\sum_{j=0}^{\infty} \partial_{e_1[-e_2]}(e_1^{j+1})z^j = \frac{1}{1-(e_1-e_2)z-e_1e_2z^2}, \tag{3.3}$$

$$\sum_{j=0}^{\infty} \partial_{e_1[-e_2]}(e_1^{j+2})z^j = \frac{e_1-e_2+e_1e_2z}{1-(e_1-e_2)z-e_1e_2z^2}. \tag{3.4}$$

Choosing e_1 and e_2 such that $\begin{cases} e_1e_2 = 1 \\ e_1-e_2 = 2 \end{cases}$ and substituting in (3.3) and (3.4), we obtain

$$\sum_{j=0}^{\infty} \partial_{e_1[-e_2]}(e_1^{j+1})z^j = \frac{1}{1-2z-z^2}, \tag{3.5}$$

$$\sum_{j=0}^{\infty} \partial_{e_1[-e_2]}(e_1^{j+2})z^j = \frac{2+z}{1-2z-z^2}. \tag{3.6}$$

Multiplying the equation (3.5) by (-2) and added to (3.6), we have

$$\sum_{j=0}^{\infty} P_jz^j = \frac{z}{1-2z-z^2} \text{ with } P_j = \partial_{e_1[-e_2]}(e_1^{j+2}) - 2\partial_{e_1[-e_2]}(e_1^{j+1}), \tag{3.7}$$

where (3.7) establishes the generating function for Pell numbers [12].

Multiplying (3.5) by (6) and adding to (3.6) multiplied by (-2), gives the following generating function

$$\sum_{j=0}^{\infty} Q_jz^j = \frac{2-2z}{1-2z-z^2} \text{ with } Q_j = 6\partial_{e_1[-e_2]}(e_1^{j+1}) - 2\partial_{e_1[-e_2]}(e_1^{j+2}), \tag{3.8}$$

Replacing z by $(-z)$ in (3.7) and (3.8), we have the following theorems.

Theorem 3.3. *For $j \in \mathbb{N}$, the generating function of the Pell numbers is given by*

$$\sum_{j=0}^{\infty} P_{-j} z^j = \frac{z}{1+2z-z^2}.$$

Proof. The ordinary generating function associated is defined by

$$G(P_j, z) = \sum_{j=0}^{\infty} P_j z^j,$$

using the initial conditions, we have

$$\begin{aligned}\sum_{j=0}^{\infty} P_j z^j &= P_0 + P_1 z + \sum_{j=2}^{\infty} P_j z^j \\ &= z + \sum_{j=2}^{\infty} 2P_{j-1} z^j + \sum_{j=2}^{\infty} P_{j-2} z^j.\end{aligned}$$

If we consider the case when $n = j-2$ and $p = j-1$, the right-hand side expand as

$$\begin{aligned}&= z + 2z \sum_{j=1}^{\infty} P_j z^j + z^2 \sum_{j=2}^{\infty} P_{j-2} z^{j-2} \\ &= z + 2z \sum_{n=0}^{\infty} P_n z^n + z^2 \sum_{p=0}^{\infty} P_p z^p,\end{aligned}$$

which is equivalent to

$$\begin{aligned}(1-2z-z^2) \sum_{j=0}^{\infty} P_j z^j &= z \\ \sum_{j=0}^{\infty} P_j z^j &= \frac{z}{1-2z-z^2}.\end{aligned}$$

Replacing z by $(-z)$, we have

$$\sum_{j=0}^{\infty} (-1)^j P_j z^j = \frac{-z}{1+2z-z^2},$$

therefore

$$\sum_{j=0}^{\infty} (-1)^{j+1} P_j z^j = \frac{z}{1+2z-z^2},$$

$$\sum_{j=0}^{\infty} P_{-j} z^j = \frac{z}{1+2z-z^2}.$$

This completes the proof. □

Theorem 3.4. *For $j \in \mathbb{N}$, the generating function of the Pell-Lucas numbers is given by*

$$\sum_{j=0}^{\infty} Q_{-j} z^j = \frac{2+2z}{1+2z-z^2}.$$

Proof. The ordinary generating function associated is defined by

$$G(Q_j, z) = \sum_{j=0}^{\infty} Q_j z^j.$$

Following the same procedure and using the initial conditions, we can write

$$\begin{aligned}
\sum_{j=0}^{\infty} Q_j z^j &= Q_0 + Q_1 z + \sum_{j=2}^{\infty} Q_j z^j \\
&= 2 + 2z + \sum_{j=2}^{\infty} 2Q_{j-1} z^j + \sum_{j=2}^{\infty} Q_{j-2} z^j,
\end{aligned}$$

If we take $n = j-2$ and $p = j-1$. The right-hand side can be expanded as

$$\begin{aligned}
&= 2 + 2z + 2z \sum_{j=1}^{\infty} Q_j z^j + z^2 \sum_{j=2}^{\infty} Q_{j-2} z^{j-2} \\
&= 2 + 2z + 2z \left(\sum_{n=0}^{\infty} Q_n z^n - Q_0 \right) + z^2 \sum_{p=0}^{\infty} Q_p z^p,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
(1 - 2z - z^2) \sum_{j=0}^{\infty} Q_j z^j &= 2 - 2z \\
\sum_{j=0}^{\infty} Q_j z^j &= \frac{2-2z}{1-2z-z^2}.
\end{aligned}$$

Replacing z by $(-z)$, we have

$$\sum_{j=0}^{\infty} (-1)^j Q_j z^j = \frac{2+2z}{1+2z-z^2},$$

therefore

$$\sum_{j=0}^{\infty} Q_{-j} z^j = \frac{2+2z}{1+2z-z^2}.$$

This completes the proof. □

Lastly, replacing e_1 by $2e_1$ and e_2 by $(-2e_2)$ in (3.3) and (3.4), and under the condition $4e_1e_2 = -1$, we obtain, for $x = e_1 - e_2$,

$$\sum_{j=0}^{\infty} U_j(x)z^j = \frac{1}{1-2xz+z^2}, \text{ such that } U_j(x) = \partial_{2e_1[-2e_2]}(e_1^{j+1}).$$

$$\sum_{j=0}^{+\infty} \partial_{2e_1[-2e_2]}(e_1^{j+2})z^j = \frac{2x-z}{1-2xz+z^2}, \tag{3.9}$$

which represents a generating function for Chebychev polynomials of second kind,[2] such that $\partial_{2e_1[-2e_2]}(e_1^{j+2}) = (2x-z)U_j(e_1-e_2)$.

Moreover, we deduce from (3.9), the following generating function:

$$\sum_{j=0}^{+\infty} \left[\partial_{2e_1[-2e_2]}(e_1^{j+2}) - x\partial_{2e_1[-2e_2]}(e_1^{j+1})\right] z^j = \frac{2x-(x+1)z}{1-2xz+z^2}.$$

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