WALLS IN BARGRAPHS

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Abstract. Bargraphs are lattice paths in $\mathbb{N}_0^2$ with three allowed types of steps; up $(0,1)$, down $(0,-1)$ and horizontal $(1,0)$. They start at the origin with an up step and terminate immediately upon return to the x-axis. A wall of size r is a maximal sequence of r adjacent up steps. In this paper we develop the generating function for the total number of walls of fixed size $r \geq 1$. We then derive asymptotic estimates for the mean number of such walls.

1. Introduction

Bargraphs are lattice paths in $\mathbb{N}_0^2$, starting at the origin and ending upon first return to the x-axis. The allowed steps are the up step, $u = (0,1)$, the down step, $d = (0,-1)$ and the horizontal step, $h = (1,0)$. The first step has to be an up step and the horizontal steps must all lie above the x-axis. An up step cannot follow a down step and vice versa. It is clear that the number of down steps must equal the number of up steps. Related lattice paths such as Dyck paths and Motzkin paths have been studied extensively (see [8, 15]) whereas until recently bargraphs, which are fundamental combinatorial structures, have not attracted the same amount of interest. The present authors have studied height, levels, area and peaks of bargraphs in [1, 2, 3, 4].

Previously, Bousquet-Mélou and Rechnitzer in [5] enumerated (by their site-perimeter) the simplest family of polyominoes that are not fully convex, namely bargraphs. The generating function is shown to be a q-series into which an algebraic series has been substituted. Geraschenko in [12] also studied bargraphs, which were named skylines. Wall polyominoes were investigated by Feretić, in [10]. Bargraphs models arise frequently in statistical physics, see for example [6, 9, 16, 18, 21, 22, 23]. In addition, bargraphs are commonly used in probability theory to represent frequency diagrams and are also related to compositions of integers [17].

In this paper we investigate walls, which are maximal sequences of one or more adjacent up steps. The study of walls in bargraphs is related to the modelling of tethered polymers under pulling forces, see [19, 20]. These pulling forces have vertical

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and horizontal components and tend to be resisted by what is known as the stiffness of the polymers. The polymers undergo phase changes, called the stretched (adsorption) phase, where the polymer is stretched vertically. The stretched phase occurs only when the vertical force is not zero. The degree of stretching is a function of the size of the walls and fundamentally determines where a phase change occurs. We will fix the height of the walls to be r, and determine the number of walls of height r for each $r \geq 1$. A wall can be thought of as a maximal ascent; work on ascents in compositions and partitions can be found in [7, 13, 14]. We find the generating functions $F(x, y, w)$ where x counts the horizontal steps, y the up vertical steps and w the number of walls.

In this paper, we use a return to the first level decomposition consisting of three cases shown in Figure 1 below. This is a simplification of the so-called wasp-waist decomposition which had five cases and was previously used in [1, 2, 3, 4, 5]

Note that part α of case 1 and 3 may be empty; other parts are not.

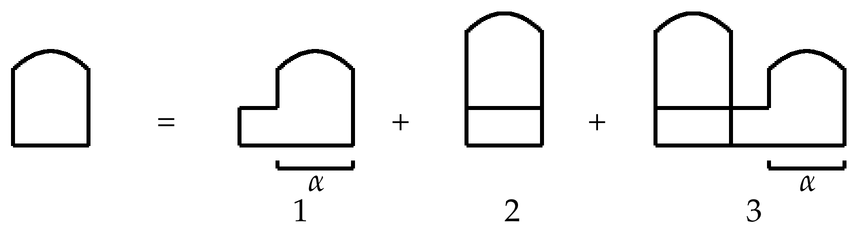

Figure 1. Return to the first level decomposition of bargraphs

This decomposition is different to that of the previous papers [1, 2, 3, 4, 5] which all used a five case decomposition.

2. Definitions and background

A *wall* in a bargraph is a subword consisting of a maximal number of adjacent up steps. A *wall of size r* is a wall consisting of precisely r up steps (and therefore neither preceded nor followed by another adjacent up step).

We define $F_r(x, y, w)$ to be the generating function that counts bargraphs where the number of walls of size r are counted by the variable w. The variables x and y count the horizontal and vertical semi-perimeters respectively. We will also use the generating function $F_{r,s}(x, y, w)$ for bargraphs with walls of size r and where the first column is of height s. The generating function for all bargraphs can be found in [5] amongst others. It is given by

$$B(x, y) = \frac{1 - x - y - xy - \sqrt{(1 - x - y - xy)^2 - 4x^2 y}}{2x}. \tag{2.1}$$

If we substitute $z = y = x$ we obtain the generating function for the semi-perimeter counted by z, often called the isotropic generating function

$$B(z,z) = \frac{1 - 2z - z^2 - \sqrt{1 - 4z + 2z^2 + z^4}}{2z}. \tag{2.2}$$

To find the asymptotics for $B(z,z)$, we must first compute the dominant singularity ρ which is the positive root of $D := 1 - 4z + 2z^2 + z^4 = 0$. We find

$$\rho = \frac{1}{3}\left(-1 - \frac{4 \times 2^{2/3}}{(13 + 3\sqrt{33}\,)^{1/3}} + \left(2(13 + 3\sqrt{33}\,)\right)^{1/3}\right) = 0.295598\cdots. \tag{2.3}$$

Then $B(z,z) \sim C_\rho - \sqrt{\frac{1-\rho-\rho^3}{\rho}}\left(1 - \frac{z}{\rho}\right)^{1/2}$ where $C_\rho = 0.543689\cdots$.

Singularity analysis is a method for finding the asymptotics of the coefficients of a generating function by studying the behaviour of the function near its dominant singularities. By singularity analysis (see [11]) we have

$$[z^n]B(z,z) \sim \frac{1}{2}\sqrt{\frac{1-\rho-\rho^3}{\pi \rho n^3}}\,\rho^{-n}. \tag{2.4}$$

3. Walls of size one

In this section, we find the generating function $F_1 := F_1(x,y,w)$ for bargraphs where walls of size one are counted by the variable w. We first obtain $F_{1,1} := F_{1,1}(x,y,w)$ which counts bargraphs where the first column is of height 1. To find $F_{1,1}$ we use the decomposition illustrated below, where the 1's and 2's specify the height of its indicated column:

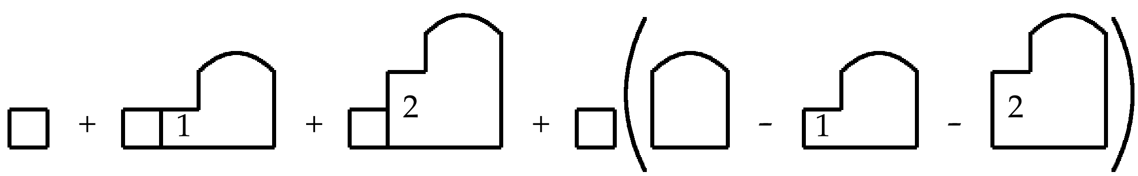

Figure 2. Decomposition for bargraphs with first column of height 1

This yields

$$F_{1,1} = xyw + xF_{1,1} + xw^2F_{1,2} + xw(F_1 - F_{1,1} - F_{1,2}), \tag{3.1}$$

where $F_{1,2}$ is the generating function for bargraphs whose first column is of height 2.

The generating function $F_{1,2}$ is obtained with the use of the following decomposition

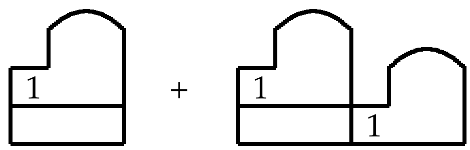

FIGURE 3. Decomposition for bargraphs with first column of height 2

Thus

$$F_{1,2} = \frac{yF_{1,1}}{w} + \frac{F_{1,1}^2}{w^2}. \tag{3.2}$$

For the raised case 2 from Figure 1, we shall denote the generating function by F_1^R, where the decomposition is illustrated below

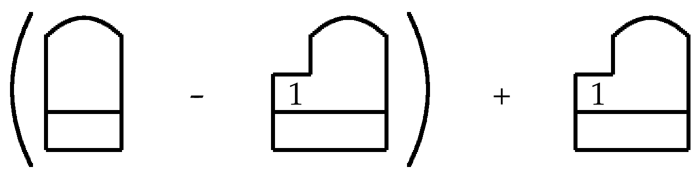

FIGURE 4. Decomposition for raised bargraphs

This yields the formula for F_1^R as

$$F_1^R = (F_1 - F_{1,1})y + \frac{F_{1,1}y}{w}. \tag{3.3}$$

Thus by the general decomposition in Figure 1

$$F_1 = F_{1,1} + F_1^R + \frac{F_1^R F_{1,1}}{yw}. \tag{3.4}$$

Our aim is to solve for F_1. Substituting equations (3.1), (3.2) and (3.3) into (3.4) we obtain the following cubic equation in F_1

$$F_1^3x^2 + F_1^2x(-2+2x+y+wxy) + F_1(1-2x+x^2-y+xy-wxy+2wx^2y-xy^2+wxy^2)$$
$$= xy(w - wx + y - wy). \tag{3.5}$$

In order to find the total number of walls of size one, we differentiate (3.5) with respect to w and then put $w = 1$. For simplicity, we use $F_1(1)$ for $F_1(x,y,1)$ and $F_1'(1)$ for $\frac{\partial F_1(x,y,w)}{\partial w}\big|_{w=1}$. Thus, we have

$$y^2(1+F_1(1)) + (-2+x-4F_1(1)+4xF_1(1)+3xF_1^2(1))F_1'(1)$$
$$+ y\Big(-1 + F_1(1)(-1+2F_1'(1)) + x(1+F_1(1))(1+F_1(1)+2F_1'(1))\Big) = \frac{-1+y}{x}F_1'(1).$$

However, we know that $F_1(1)$ is the generating function for all bargraphs (see [1, 2]). So

$$F_1(1) = \frac{1}{2x}\left(1 - x - y - xy - \sqrt{(1-x-y-xy)^2 - 4x^2y}\right). \tag{3.6}$$

The solution for $F_1'(1)$ is

$$F_1'(1) = \frac{xy - 2x^2y - xy^2 + (x - x^2 - 4x^2y - xy^2)F_1(1) + (x - 2x^2 - xy - 2x^2y)F_1^2(1) - x^2F_1^3(1)}{1 - 2x + x^2 - y + 2x^2y + (-4x + 4x^2 + 2xy + 2x^2y)F_1(1) + 3x^2F_1^2(1)}. \tag{3.7}$$

Substituting the expression for $F_1(1)$ from (3.6) into the derivative (3.7), we get

$$F_1'(1) = \frac{xy(1-y)\left[1 - x(1-y) + y + \sqrt{X}\right]}{1 + x^2(1-y)^2 - 2y + y^2 + (1+y)\sqrt{X} - x(1-y)(2 + 2y + \sqrt{X})} \tag{3.8}$$

where $X = (1-y)(1 + x^2(1-y) - y - 2x(1+y))$.

The series expansion for this derivative up to the term in x^5 and y^5 with the pertinent term in bold illustrated in Figure 5 is

$$xy + x^2(y + 2y^2 + 2y^3 + 2y^4 + 2y^5) + x^3(y + \mathbf{6y^2} + 12y^3 + 18y^4 + 24y^5) + x^4(y + 12y^2 + 42y^3 + 92y^4 + 162y^5) + x^5(y + 20y^2 + 110y^3 + 340y^4 + 780y^5) + \cdots.$$

We illustrate below, in bold, the six walls of size one in the bargraphs represented by x^3y^2. Note that the first 3 bargraphs in Figure 5 contain no wall of size 1.

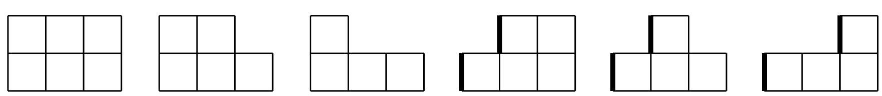

FIGURE 5. The 6 walls of size 1 corresponding to x^3y^2

3.1. **Asymptotics.** We are now going to consider an asymptotic expression for the total number of walls of size 1 as the semi-perimeter of the bargraph tends to infinity. For this, we use the isotropic generating function where y is replaced by x. After this substitution, x measures the total semi-perimeter in the bargraph.

$$\left.\frac{\partial F_1(x,x)}{\partial w}\right|_{w=1} = \frac{(1-x)x^2(1 + x^2 + \sqrt{1 - 4x + 2x^2 + x^4})}{1 - 4x + x^4 + \sqrt{1 - 4x + 2x^2 + x^4} + x^2(2 + \sqrt{1 - 4x + 2x^2 + x^4})}. \tag{3.9}$$

We use the dominant singularity ρ as found in (2.3), as before it is the positive root of $D = 1 - 4x + 2x^2 + x^4$.

As $x \to \rho$ we find that the expression for $\frac{\partial F_1(x,x)}{\partial w}\Big|_{w=1}$ is asymptotic to

$$\frac{\partial F_1(x,x)}{\partial w}\Big|_{w=1} \sim \frac{\rho^2 - \rho^3}{2\sqrt{\rho(1-\rho-\rho^3}(1-\frac{x}{\rho})^{1/2}}.$$

By singularity analysis, see [11], we find that

$$[x^n]\frac{\partial F_1(x,x)}{\partial w}\Big|_{w=1} \sim \frac{\rho^2 - \rho^3}{2\sqrt{\pi\rho n}\sqrt{1-\rho-\rho^3}}\,\rho^{-n} \quad \text{as } n \to \infty.$$

In order to find the average number of walls of size 1, we divide by the number of bargraphs of size n which was found in (2.4). Thus, we have the following result

Theorem 1. *The average number of walls of size one is asymptotic to* $\dfrac{\rho^2 - \rho^3}{1-\rho-\rho^3}\, n$ *as* $n \to \infty$. *Numerically this is* $C\,n$ *where* $C = 0.0907039$.

4. Walls of size r

In the previous section, we considered walls of size 1, in this section we consider walls of size r where $r > 1$. Again, we shall use $F_{r,s}$ as the generating function for bargraphs with walls of size r starting with a first column of height s.

First, we consider $F_{r,1}$ for $r > 1$. The decomposition for $F_{r,1}$ is given below

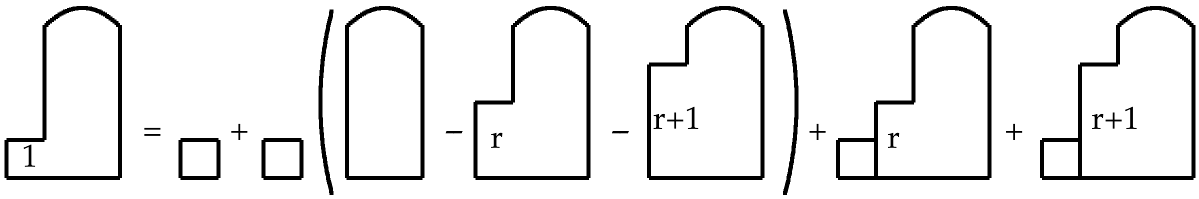

FIGURE 6. Decomposition for $F_{r,1}$

Thus

$$F_{r,1} = xy + x(F_r - F_{r,r} - F_{r,r+1}) + \frac{x}{w}F_{r,r} + xwF_{r,r+1}, \tag{4.1}$$

where the decomposition for bargraphs with generating function $F_{r,r}$ is

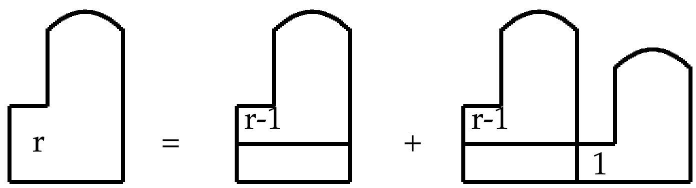

FIGURE 7. Decomposition for $F_{r,r}$

so that

$$F_{r,r} = ywF_{r,r-1} + wF_{r,r-1}F_{r,1} = wF_{r,r-1}(y + F_{r,1}). \tag{4.2}$$

Next, we consider the generating function $F_{r,i}$, where $1 < i < r$. The decomposition is the same as in Figure 7 except that all r's are replaced by i's. Hence

$$F_{r,i} = yF_{r,i-1} + F_{r,i-1}F_{r,1} = F_{r,i-1}(y + F_{r,1}). \tag{4.3}$$

Iterating (4.3) yields

$$F_{r,i} = (y + F_{r,1})^{i-1}F_{r,1}. \tag{4.4}$$

Similarly we compute $F_{r,r}$ as

$$F_{r,r} = w(y + F_{r,1})F_{r,r-1} = w(y + F_{r,1})^{r-1}F_{r,1}. \tag{4.5}$$

To find $F_{r,r+1}$, we use a similar decomposition to that shown in Figure 7 except that every r is replaced by $r + 1$.

Thus we obtain

$$F_{r,r+1} = \frac{yF_{r,r}}{w} + \frac{F_{r,r}F_{r,1}}{w} = (y + F_{r,1})^r F_{r,1}. \tag{4.6}$$

Again, using the decomposition from Figure 1

$$F_r = F_{r,1} + F_r^R + \frac{F_r^R F_{r,1}}{y}. \tag{4.7}$$

The raised case 2 from Figure 1 has generating function F_r^R with decomposition:

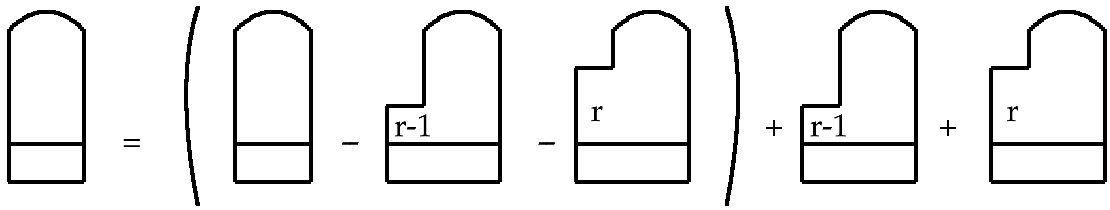

FIGURE 8. Decomposition for F_r^R

Therefore,

$$F_r^R = y(F_r - F_{r,r-1} - F_{r,r}) + ywF_{r,r-1} + \frac{y}{w}F_{r,r}. \tag{4.8}$$

Using (4.4) this simplifies to

$$F_r^R = \left(F_r - (y+F_{r,1})^{r-2}F_{r,1} - F_{r,r}\right) y + yw(y+F_{r,1})^{r-2}F_{r,1} + \frac{y}{w}F_{r,r}. \tag{4.9}$$

We eliminate F_r^R, $F_{r,r}$ and $F_{r,r+1}$ from (4.1), (4.5), (4.6), (4.7) and (4.9) to obtain two equations in $F_{r,1}$ and F_r. These are

$$F_{r,1} = xy + x(F_r - w(y+F_{r,1})^{r-1}F_{r,1} - (y+F_{r,1})^r F_{r,1}) + x(y+F_{r,1})^{r-1}F_{r,1} + xw(y+F_{r,1})^r F_{r,1} \tag{4.10}$$

and

$$\begin{aligned} &F_r \\ &= \Big[(F_r - (y+F_{r,1})^{r-2}F_{r,1} - w(y+F_{r,1})^{r-1}F_{r,1})y + yw(y+F_{r,1})^{r-2}F_{r,1} + y(y+F_{r,1})^{r-1}F_{r,1}\Big] \\ &\left(\frac{F_{r,1}}{y}+1\right) + F_{r,1}. \end{aligned} \tag{4.11}$$

Since, we are interested in the total number of walls of size $r > 1$, we differentiate equations (4.10) and (4.11) as before. For the derivative of equation (4.10) we have

$$\begin{aligned} &F'_{r,1}(1) \\ &= \frac{x\left[F_{r,1}^2(1)(y+F_{r,1}(1))^r + yF'_r(1) + F_{r,1}(1)(-(y+F_{r,1}(1))^r + y(y+F_{r,1}(1))^r + F'_r(1))\right]}{y+F_{r,1}(1)} \end{aligned} \tag{4.12}$$

The derivative of equation (4.11) yields

$$\begin{aligned} F'_r(1) = \frac{1}{y+F_{r,1}(1)} &\left[F_{r,1}(1)((y+F_{r,1}(1))^r - y((y+F_{r,1}(1))^r - 2F'_r(1)) + (1+F_r(1))F'_{r,1}(1))\right. \\ &\left. + F_{r,1}^2(1)(-(y+F_{r,1}(1))^r + F'_{r,1}(1)) + y(yF'_r(1) + (1+F_r(1))F'_{r,1}(1))\right]. \end{aligned} \tag{4.13}$$

We solve the two simultaneous equations (4.12) and (4.13) to finally obtain

$$F_r'(1) = \frac{(1-x-xF_{r,1}(1))F_{r,1}(1)(1-y-F_{r,1}(1))(y+F_{r,1}(1))^{r-1}}{1-x-y-xF_{r,1}(1)-F_{r,1}(1)}.$$

However, we know from (2.1)

$$F_r(1) = B(x,y) = \frac{1-x-y-xy-\sqrt{(1-x-y-xy)^2-4x^2y}}{2x}.$$

Thus, the derivative becomes

$$F_r'(1) = \frac{2^{1-r}xy(-1+y)(1+x(-1+y)+y-\sqrt{X})^r}{\sqrt{X}(-1+x-y-xy+\sqrt{X})} \tag{4.14}$$

where $X = (1-y)(1+x^2(1-y)-y-2x(1+y))$ as in the previous section.
For example where $r=2$ we obtain

$$F_2'(1) = \frac{xy(-1+y)(-1+x-y-xy+\sqrt{(-1+y)(-1+x^2(-1+y)+y+2x(1+y)})}{2\sqrt{(-1+y)(-1+x^2(-1+y)+y+2x(1+y)}},$$

with series expansion (up to the term in x^5 and y^5)

$$xy^2 + x^2(2y^2+2y^3+2y^4+2y^5) + x^3(\mathbf{3y^2}+9y^3+15y^4+21y^5)$$
$$+\, x^4(4y^2+24y^3+64y^4+124y^5) + x^5(5y^2+50y^3+200y^4+525y^5) + \cdots .$$

As before, we illustrate the term $3x^3y^2$ with the three walls of size two shown in bold below.

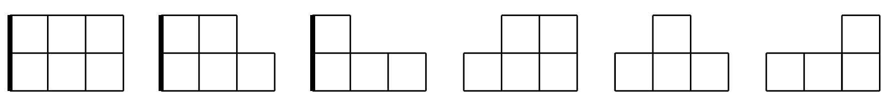

FIGURE 9. The 3 walls of size 2 corresponding to x^3y^2

4.1. **Asymptotics.** We now find an asymptotic expression for the total number of walls of size r as the semi-perimeter of the bargraph tends to infinity. As before, we use the isotropic generating function which is the expression for $F_r'(1)$ in (4.14) with $y=x$

$$F_r'(1)\big|_{y=x} = \frac{2^{1-r}(1-x)x^2\left(1+x^2-\sqrt{1-4x+2x^2+x^4}\right)^{r-1}}{\sqrt{1-4x+2x^2+x^4}}.$$

The dominant singularity is still ρ, the positive root of $D = 1-4x+2x^2+x^4$.

Thus as $x \to \rho$, we have

$$F_r'(1)\big|_{y=x} \sim \frac{2^{-r}(1-\rho)\rho^2(1+\rho^2)^{r-1}}{\sqrt{\rho(1-\rho-\rho^3)}\sqrt{1-x/\rho}}.$$

By singularity analysis, we obtain the following estimate for the coefficients

$$[x^n]F_r'(1)\big|_{y=x} \sim \frac{2^{-r}(1-\rho)\rho^2(1+\rho^2)^{r-1}}{\sqrt{n\pi\rho(1-\rho-\rho^3)}}\,\rho^{-n} \quad \text{as } n \to \infty.$$

Thus, after dividing by the asymptotic number of bargraphs in (2.4), we have the following result

Theorem 2. *The average number of walls of size r is asymptotic to $\frac{2^{1-r}(1-\rho)\rho^2(1+\rho^2)^{r-1}}{1-\rho-\rho^3} n$ as $n \to \infty$.*

Numerically this average is $C_r n$ where for example $C_2 = 0.04931$, and $C_3 = 0.02681$.

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