ZETA SERIES GENERATING FUNCTION TRANSFORMATIONS RELATED TO POLYLOGARITHM FUNCTIONS AND THE k-ORDER HARMONIC NUMBERS

MAXIE D. SCHMIDT

Abstract. We define a new class of generating function transformations related to polylogarithm functions, Dirichlet series, and Euler sums. These transformations are given by an infinite sum over the j^{th} derivatives of a sequence generating function and sets of generalized coefficients satisfying a non-triangular recurrence relation in two variables. The generalized transformation coefficients share a number of analogous properties with the Stirling numbers of the second kind and the known harmonic number expansions of the unsigned Stirling numbers of the first kind.

We prove a number of properties of the generalized coefficients which lead to new recurrence relations and summation identities for the k-order harmonic number sequences. Other applications of the generating function transformations we define in the article include new series expansions for the polylogarithm function, the alternating zeta function, and the Fourier series for the periodic Bernoulli polynomials. We conclude the article with a discussion of several specific new "almost" linear recurrence relations between the integer-order harmonic numbers and the generalized transformation coefficients, which provide new applications to studying the limiting behavior of the zeta function constants, $\zeta(k)$, at integers $k \geq 2$.

1. Introduction

The *Stirling numbers of the second kind*, $\left\{ {n \atop k} \right\}$, are defined for $n, k \geq 0$ by the triangular recurrence relation [6, §6.1][1]

$$\left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\} + [n = k = 0]_{\delta}. \tag{1}$$

It is also known, or at least straightforward to prove by induction, that for any sequence, $\langle g_n \rangle$, whose formal ordinary power series (OGF) is denoted by $G(z)$, and natural numbers $m \geq 1$, we have a generating function transformation of the form [6, *cf.* §7.4][2]

$$\sum_{n \geq 0} n^m g_n z^n = \sum_{j=0}^{m} \left\{ {m \atop j} \right\} z^j G^{(j)}(z). \tag{2}$$

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[1] The notation for *Iverson's convention*, $[n = k]_{\delta} = \delta_{n,k}$, is consistent with its usage in [6].

[2] Variants of (2) can be found in [9, §26.8(v)]. A special case of the identity for $f_n \equiv 1$ appears in [6, eq. (7.46); §7.4]. Other related expansions for converting between powers of the differential operator D and the operator $\vartheta := zD$ are known as sums involving the Stirling numbers of the first and second kinds [6, Ex. 6.13; *cf.* §6.5]. The particular identity that $[z^n]\left((zD)^k F(z)\right) = n^k f_n$ is stated in [13, §2.2].

We seek to study the properties of a related set of coefficients that provide the corresponding negative-order generating function transformations of the form

$$\sum_{n\geq 1} \frac{g_n}{n^k} z^n = \sum_{j\geq 1} \left\{ {k+2 \atop j} \right\}_* z^j G^{(j)}(z), \tag{3}$$

for integers $k > 0$. We readily see that the generalized coefficients, $\left\{ {k \atop j} \right\}_*$, are defined by a two-index, non-triangular recurrence relation of the form

$$\begin{aligned} \left\{ {k \atop j} \right\}_* &= -\frac{1}{j} \left\{ {k \atop j-1} \right\}_* + \frac{1}{j} \left\{ {k-1 \atop j} \right\}_* + [k = j = 1]_\delta \\ &= \sum_{1\leq m\leq j} \binom{j}{m} \frac{(-1)^{j-m}}{j! m^{k-2}}, \end{aligned} \tag{4}$$

which provides a number of new properties, identities, and sequence applications involving these numbers.

We likewise obtain a number of new, interesting relations between the *r-order harmonic numbers*, $H_n^{(r)} = \sum_{k=1}^n k^{-r}$ and $H_n^{(r)}(t) = \sum_{k=1}^n t^k/k^r$, by their corresponding ordinary generating functions, $\mathrm{Li}_r(z)/(1-z)$ and $\mathrm{Li}_r(tz)/(1-z)$, through our study of the generalized transformation coefficients, $\left\{ {k \atop j} \right\}_*$, in (4). Most of the series expansions for special functions we define through (3) are new, and moreover, provide rational partial series approximations to the infinite series in z. Section 2.1 provides the details to a combinatorial proof of the zeta series transformations defined by (3) and (4).

Examples. The Dirichlet-generating-function-like series defined formally by (3) can be approximated up to any finite order $u \geq 1$ by the terms of typically rational truncated Taylor series. We cite a few notable examples of these truncated ordinary generating functions in the following equations where $k \in \mathbb{N}$, $a, b, r, t \in \mathbb{R}$, and $\omega_a = \exp(2\pi\imath/a)$ denotes the *primitive a^{th} root of unity* which has a distinct notation from the formal series variable w indexing the sums in the first six formulas below:

$$\sum_{1\leq n\leq u} \frac{z^n}{n^k} = [w^u] \left(\sum_{j=1}^{u} \left\{ {k+2 \atop j} \right\}_* \frac{(wz)^j j!}{(1-wz)^{j+1}(1-w)} \right) \tag{5a}$$

$$\sum_{1\leq n\leq u} \frac{z^n}{n^k n!} = [w^u] \left(\sum_{j=1}^{u} \left\{ {k+2 \atop j} \right\}_* \frac{(wz)^j e^{wz}}{(1-w)} \right) \tag{5b}$$

$$\sum_{1\leq n\leq u} H_n^{(k)} z^n = [w^u] \left(\sum_{j=1}^{u} \left\{ {k+2 \atop j} \right\}_* \frac{(wz)^j j!}{(1-wz)^{j+2}(1-w)} \right) \tag{5c}$$

$$\sum_{1\leq n\leq u} \left(\sum_{m=1}^{n} \frac{t^m}{m^k} \right) z^n = [w^u] \left(\sum_{j=1}^{u} \left\{ {k+2 \atop j} \right\}_* \frac{(wtz)^j j!}{(1-wtz)^{j+1}(1-wz)(1-w)} \right) \tag{5d}$$

$$\text{(5e)}\quad \sum_{1\leq n\leq u}\left(\sum_{m=1}^{n}\frac{r^m}{m^k m!}\right)z^n = [w^u]\left(\sum_{j=1}^{u}\left\{\begin{matrix}k+2\\ j\end{matrix}\right\}_* \frac{(wrz)^j e^{wrz}}{(1-wz)(1-w)}\right)$$

$$\text{(5f)}\quad \sum_{1\leq n\leq u}\frac{H_n^{(k)}}{n!}z^n = [w^u]\left(\sum_{j=1}^{u}\left\{\begin{matrix}k+2\\ j\end{matrix}\right\}_* \frac{(wz)^j}{(j+1)}\frac{e^{wz}(j+1+wz)}{(1-w)}\right)$$

(5g)

$$\sum_{1\leq n\leq u}\frac{z^n}{(an+b)^s}+\frac{[b>0]_\delta}{b^s} = [t^{au+b}]\left(\sum_{m=0}^{a-1}\sum_{j=1}^{au+b}\left\{\begin{matrix}s+2\\ j\end{matrix}\right\}_* \frac{\omega_a^{-mb}z^{b/a}\left(\left(tz^{1/a}\right)^j j!\right.}{a\left(\left(1-\omega_a^m tz^{1/a}\right)^{j+1}(1-t)\right.}\right).$$

These expansions follow easily as consequences of a few generating function operations and transformation results. First, for any fixed scalar t, the j^{th} derivative of the geometric series satisfies

$$\frac{d^{(j)}}{dz^{(j)}}\left[\frac{1}{(1-tz)}\right] = \frac{t^j j!}{(1-tz)^{j+1}},$$

which implies the finite and infinite series variants of (5a) in the formulas listed above. We also have an integral transform that converts the ordinary generating function, $F(z)$, of any sequence into its corresponding exponential generating function, $\widehat{F}(z)$, according to [6, p. 566]

$$\widehat{F}(z) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} F\left(ze^{-\imath\vartheta}\right)e^{e^{\imath\vartheta}}d\vartheta.$$

This integral transformation together with a well-known expansion of generating functions for the binomial coefficients shows that

$$\text{(6)}\quad \frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{\left(wze^{-\imath\vartheta}\right)^j}{\left(1-wze^{-\imath\vartheta}\right)^{j+2}}e^{e^{\imath\vartheta}}d\vartheta = \frac{(wz)^j e^{wz}}{(j+1)!}\left(j+1+wz\right),$$

which implies the second to last expansion in (5f). Lastly, there is a known "*series multisection*" generating function transformation over arithmetic progressions of a sequence for integers $a>1, b\geq 0$ of the form [8, §1.2.9]

$$\sum_{n\geq 0} f_{an+b}z^{an+b} = \sum_{0\leq m<a}\frac{\omega_a^{-mr}}{a}F\left(\omega_a^m z\right)$$

Since the geometric series ordinary generating function, and its j^{th} derivatives, are always rational, we may also give similar statements about the partial sums of the *Euler sum* generating functions of the forms studied in [4, 2, 12].

Comparison to Known Series. For comparison, we summarize a pair of known series identities for the *polylogarithm function*, $\mathrm{Li}_s(z)$, and the *modified Hurwitz zeta function*,

$\Phi(z,s,\alpha,\beta) = \sum_{n\geq 1} z^n/(\alpha n+\beta)^s = \alpha^{-s} z\Phi(z,s,\beta/\alpha+1)$, as follows [9, §25.12(ii), 25.14] [7, eq. (6); Thm. 2.1; §2]:

$$\operatorname{Li}_s(z) = \sum_{k\geq 0}\left(-\frac{z}{1-z}\right)^{k+1}\sum_{0\leq m\leq k}\binom{k}{m}\frac{(-1)^{m+1}}{(m+1)^s} \tag{7}$$
$$\Phi(z,s,\alpha,\beta) = \sum_{k\geq 0}\left(-\frac{z}{1-z}\right)^{k+1}\sum_{0\leq m\leq k}\binom{k}{m}\frac{(-1)^{m+1}}{(\alpha m+\alpha+\beta)^s}.$$

The new generalized coefficients, $\left\{ {k \atop j} \right\}_*$, defining the transformations in (3) satisfy several key properties and generating functions analogous to those of the Stirling numbers of the second kind, and are closely-related to the known harmonic number expansions of the unsigned triangle of the *Stirling numbers of the first kind* [1, 10]. We explore several initial properties and relations of these coefficients in the next section.

2. Initial Properties, Ordinary Generating Functions, and Relations to the Stirling Numbers

2.1. **Proof of the Transformation Identity in (3).**

We first prove that the recurrence relation in (4) holds for the generalized transformation coefficients in (3), which is then used to extrapolate new results providing summation and harmonic number identities for these sequences.

Proof of (4). The proof proceeds by an inductive argument similar to the proof that can be given from (1) for the generating function transformations involving the Stirling numbers of the second kind cited in the introduction. We first observe that

$$\sum_{n\geq 1}\frac{g_n}{n^k}z^n = \sum_{n\geq 1}\frac{(n\cdot g_n)}{n^{k+1}}z^n$$

for all $k\in\mathbb{N}$. Since the ordinary generating function for the sequence, $\langle n g_n\rangle$, is given by $zG'(z)$, and the j^{th} derivative of $zG'(z)$ is $jG^{(j)}(z)+zG^{(j+1)}(z)$, we may write that

$$\begin{aligned}\sum_{j\geq 1}\left\{ {k+2 \atop j} \right\}_* z^j G^{(j)}(z) &= \sum_{j\geq 1}\left\{ {k+3 \atop j} \right\}_* z^j\left(jG^{(j)}(z)+zG^{(j+1)}(z)\right)\\ &= \sum_{j\geq 2} z^j\left(j\left\{ {k+3 \atop j} \right\}_* + \left\{ {k+3 \atop j-1} \right\}_*\right) + [k=j=1]_\delta.\end{aligned}$$

We then conclude that the non-triangular recurrence relation in (4) defines the series transformation coefficients in (3). □

2.2. **Exact Expansions of the Transformation Coefficients.**

The recurrence relation in (4) leads us to compute the first few terms of these sequences given in Table 1 and Table

k \ j	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0
2	0	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{24}$	$\frac{1}{120}$	$-\frac{1}{720}$	$\frac{1}{5040}$	$-\frac{1}{40320}$
3	0	1	$-\frac{3}{4}$	$\frac{11}{36}$	$-\frac{25}{288}$	$\frac{137}{7200}$	$-\frac{49}{14400}$	$\frac{121}{235200}$	$-\frac{761}{11289600}$
4	0	1	$-\frac{7}{8}$	$\frac{85}{216}$	$-\frac{415}{3456}$	$\frac{12019}{432000}$	$-\frac{13489}{2592000}$	$\frac{726301}{889056000}$	$-\frac{3144919}{28449792000}$
5	0	1	$-\frac{15}{16}$	$\frac{575}{1296}$	$-\frac{5845}{41472}$	$\frac{874853}{25920000}$	$-\frac{336581}{51840000}$	$\frac{129973303}{124467840000}$	$-\frac{1149858589}{7965941760000}$
6	0	1	$-\frac{31}{32}$	$\frac{3661}{7776}$	$-\frac{76111}{497664}$	$\frac{58067611}{1555200000}$	$-\frac{68165041}{9331200000}$	$\frac{187059457981}{156829478400000}$	$-\frac{3355156783231}{20074173235200000}$

TABLE 1. A Table of the Generalized Coefficients $\left\{ {k \atop j} \right\}_*$

k \ j	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0
2	0	1	1	1	1	1	1	1	1
3	0	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$	$\frac{363}{140}$	$\frac{761}{280}$
4	0	1	$\frac{7}{4}$	$\frac{85}{36}$	$\frac{415}{144}$	$\frac{12019}{3600}$	$\frac{13489}{3600}$	$\frac{726301}{176400}$	$\frac{3144919}{705600}$
5	0	1	$\frac{15}{8}$	$\frac{575}{216}$	$\frac{5845}{1728}$	$\frac{874853}{216000}$	$\frac{336581}{72000}$	$\frac{129973303}{24696000}$	$\frac{1149858589}{197568000}$
6	0	1	$\frac{31}{16}$	$\frac{3661}{1296}$	$\frac{76111}{20736}$	$\frac{58067611}{12960000}$	$\frac{68165041}{12960000}$	$\frac{187059457981}{31116960000}$	$\frac{3355156783231}{497871360000}$

TABLE 2. A Table of the Scaled Coefficients $\left\{ {k \atop j} \right\}_* \times (-1)^{j-1} \cdot j!$

2. We are also able to compute the next explicit formulas for variable k and fixed small special cases of $j \geq 1$ as follows:

$$\begin{aligned}
\left\{ {k \atop 1} \right\}_* &= \quad [k \geq 1]_\delta \\
\left\{ {k \atop 2} \right\}_* &= -\left(1 - 2^{1-k}\right) [k \geq 2]_\delta \\
\left\{ {k \atop 3} \right\}_* &= \quad \frac{1}{2}\left(1 - 2 \cdot 2^{1-k} + 3^{1-k}\right) [k \geq 2]_\delta \\
\left\{ {k \atop 4} \right\}_* &= -\frac{1}{6}\left(1 - 3 \cdot 2^{1-k} + 3 \cdot 3^{1-k} - 4^{1-k}\right) [k \geq 2]_\delta
\end{aligned} \tag{8}$$

$$\left\{ {k \atop 5} \right\}_* = \frac{1}{24}\left(1 - 4\cdot 2^{1-k} + 6\cdot 3^{1-k} - 4\cdot 4^{1-k} + 5^{1-k}\right)[k \geq 2]_\delta .$$

The inductive proof of the full explicit summation formula expanded in (10) we obtain from the special cases above is left as an exercise to the reader. We compare this formula to the analogous identity for the Stirling numbers of the second kind as follows [9, §26.8]:

$$\left\{ {k \atop j} \right\} = \sum_{m=1}^{j} \binom{j}{m} \frac{(-1)^{j-m} m^k}{j!} \tag{9}$$

$$\left\{ {k+2 \atop j} \right\}_* = \sum_{m=1}^{j} \binom{j}{m} \frac{(-1)^{j-m}}{j!\, m^k} \quad . \tag{10}$$

For further comparison, observe that the forms of both (9) and (10) lead to the following similar pair of ordinary generating functions in z with respect to the upper index $k > 0$ and fixed $j \in \mathbb{Z}^+$ [9, §26.8(ii)]:

$$\begin{aligned} \sum_{k=0}^{\infty} \left\{ {k \atop j} \right\} z^k &= \frac{z^j}{(1-z)(1-2z)\cdots(1-jz)} \\ \sum_{k=0}^{\infty} \left\{ {k \atop j} \right\}_* z^k &= \left(\frac{(-1)^{j+1} z^2}{(1-z)(2-z)\cdots(j-z)}\right)[j \geq 2]_\delta + \left(\frac{z}{(1-z)}\right)[j = 1]_\delta . \end{aligned} \tag{11}$$

We also compare the generalized coefficient formula in (10) and its generating function representation in (11) to the *Nörlund-Rice integral* of a meromorphic function f over a suitable contour given by [5]

$$\sum_{1 \leq m \leq j} \binom{j}{m} (-1)^{j-m} f(m) = \frac{j!}{2\pi\imath} \oint \frac{f(z)}{z(z-1)(z-2)\cdots(z-j)} dz.$$

Corollary 2.1 (Harmonic Number Formulas)**.** *For $j \in \mathbb{N}$, the following formulas provide expansions of the coefficients from* (4) *and* (10) *at the fixed cases of $k \in [2, 6] \subseteq \mathbb{N}$:*

$$\begin{aligned} \left\{ {2 \atop j} \right\}_* &= \frac{(-1)^{j-1}}{j!} \\ \left\{ {3 \atop j} \right\}_* &= \frac{(-1)^{j-1}}{j!} H_j \\ \left\{ {4 \atop j} \right\}_* &= \frac{(-1)^{j-1}}{2j!}\left(H_j^2 + H_j^{(2)}\right) \\ \left\{ {5 \atop j} \right\}_* &= \frac{(-1)^{j-1}}{6j!}\left(H_j^3 + 3H_j H_j^{(2)} + 2H_j^{(3)}\right) \\ \left\{ {6 \atop j} \right\}_* &= \frac{(-1)^{j-1}}{24j!}\left(H_j^4 + 6H_j^2 H_j^{(2)} + 3\left(H_j^{(2)}\right)^2 + 8H_j H_j^{(3)} + 6H_j^{(4)}\right). \end{aligned} \tag{12}$$

Proof. Both the *Mathematica* software suite and the package `Sigma`[3] are able to obtain these formulas for small special cases [11]. Larger special cases of $k \geq 7$ are easiest to compute by first generating a recurrence corresponding to the sum in (10), and then solving the resulting non-linear recurrence relation with the `Sigma` package routines.

A more general heuristic harmonic-number-based recurrence formula that generates these expansions for all $k \geq 2$ is suggested along the lines of the analogous formulas for the Stirling numbers of the first kind in the references as [1, §2]

$$\left\{ {k+2 \atop j} \right\}_* = \sum_{0 \leq m < k} \frac{H_j^{(m+1)}}{k} \left\{ {k+1-m \atop j} \right\}_* + \left(\frac{(-1)^{j-1}}{j!} \right) [k=0]_\delta . \tag{13}$$

A short proof of the identity in (13) for all $k \geq 1$ is given through the exponential of a generating function for the r-order harmonic numbers and properties of the *Bell polynomials*, $Y_n(x_1, x_2, \ldots, x_n)$, in [3]. □

k	$t_0^{(k)}(j)$	$t_1^{(k)}(j)$
2	0	2
3	0	$2H_j$
4	$H_j^{(2)}$	H_j^2
5	$H_j H_j^{(2)}$	$\frac{1}{3}\left(H_j^3 + 2H_j^{(3)}\right)$
6	$\frac{1}{2}\left(H_j^2 H_j^{(2)} + H_j^{(4)}\right)$	$\frac{1}{12}\left(H_j^4 + 3\left(H_j^{(2)}\right)^2 + 8H_j H_j^{(3)}\right)$
7	$\frac{1}{6}\left(H_j^3 H_j^{(2)} + 2H_j^{(2)} H_j^{(3)} + 3H_j H_j^{(4)}\right)$	$\frac{1}{60}\left(H_j^5 + 15H_j\left(H_j^{(2)}\right)^2 + 20H_j^2 H_j^{(3)} + 24H_j^{(5)}\right)$

TABLE 3. The Harmonic Number Remainder Terms in (14) and (15)

Example 2.2 (Comparison of the Formulas)**.** The similarities between the harmonic number expansions of the Stirling numbers of the first kind and the related expansions of (10) given in (12) suggest another interpretation for the generalized coefficient representations. More precisely, for $k \in \mathbb{N}$, let the respective functions $t_0^{(k)}(j)$ and $t_1^{(k)}(j)$ denote the remainder terms in the forms of the coefficients in (10) defined by the following pair of equations:

$$\begin{aligned} t_0^{(k+2)}(j) &= \left\{ {k+2 \atop j} \right\}_* \cdot (-1)^{j-1} \cdot j! - \left[{j+1 \atop k+1} \right] \frac{1}{j!} \\ t_1^{(k+2)}(j) &= \left\{ {k+2 \atop j} \right\}_* \cdot (-1)^{j-1} \cdot j! + \left[{j+1 \atop k+1} \right] \frac{1}{j!}. \end{aligned} \tag{14}$$

[3] https://www.risc.jku.at/research/combinat/software/Sigma/

The harmonic number formulas for these generalized coefficient sums are recovered from the remainder terms in (14) and the Stirling number expansions as

$$\begin{aligned}\left\{ {k+2 \atop j} \right\}_* &= \frac{(-1)^{j-1}}{j!} \left(\left[{j+1 \atop k+1} \right] \frac{1}{j!} + t_0^{(k+2)}(j) \right) \\ \left\{ {k+2 \atop j} \right\}_* &= \frac{(-1)^{j}}{j!} \left(\left[{j+1 \atop k+1} \right] \frac{1}{j!} - t_1^{(k+2)}(j) \right).\end{aligned} \tag{15}$$

The heuristic method identified in the proof of Corollary 2.1 allows for the form of both functions to be computed for the next several special case formulas extending the expansions cited in the corollary. For comparison, a table of the first several of these remainder functions is provided in Table 3 for $2 \leq k \leq 7$.

3. Recurrence Relations and Other Identities for Harmonic Number Sequences

3.1. **Finite Sum Expansions of the Transformation Coefficients.**

Proposition 3.1 (Integer-Order Harmonic Number Identities). *For natural numbers $k \geq 1$, the generalized coefficients in* (4) *satisfy the following identities:*

$$\begin{aligned}\left\{ {k+2 \atop j} \right\}_* &= (j+1) \sum_{i=0}^{j-1} \frac{(-1)^{j-1-i} H_{i+1}^{(k)}}{(j-1-i)!(i+2)!} \\ \left\{ {k+2 \atop j} \right\}_* &= (j+1) \sum_{i=0}^{j-1} \frac{(-1)^{j-1-i}}{(j-1-i)!} \left(\frac{H_{i+2}^{(k)}}{(i+2)!} - \frac{1}{(i+2)!(i+2)^k} \right).\end{aligned} \tag{16}$$

Proof. Let the coefficient terms, $c_j(i)$, be defined as in the next equation.

$$c_j(i) := \frac{(-1)^{j-i}}{i!(j-i)!} \tag{17}$$

It follows from (10) that

$$\begin{aligned}\left\{ {k+2 \atop j} \right\}_* &= \sum_{i=0}^{j-1} \left(c_j(j-i) - c_j(j+1-i) \right) H_{j-i}^{(k)} \\ &= \sum_{i=0}^{j-1} \left(\frac{(-1)^i}{i!(j-i)!} + \frac{(-1)^i i}{i!(j+1-i)!} \right) H_{j-i}^{(k)} \\ &= (j+1) \sum_{i=0}^{j-1} \frac{(-1)^i H_{j-i}^{(k)}}{i!(j+1-i)!} \\ &= (j+1) \sum_{i=0}^{j-1} \frac{(-1)^{j-1-i} H_{i+1}^{(k)}}{(j-1-i)!(i+2)!}\end{aligned}$$

$$= (j+1) \sum_{i=0}^{j-1} \frac{(-1)^{j-1-i}}{(j-1-i)!(i+2)!} \left(H_{i+2}^{(k)} - \frac{1}{(i+2)^k} \right). \qquad \square$$

Proposition 3.2 (Formulas Involving Real-Order Harmonic Numbers)**.** *For $k \in \mathbb{Z}^+$ and real $r \geq 0$, the generalized coefficients in* (4) *satisfy the following identities:*

$$\left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_* = \sum_{i=0}^{j-1} \frac{(-1)^{j-1-i} H_{i+1}^{(k-r)}}{(j-1-i)!(i+1)!} \left(\frac{1}{(i+1)^r} + \frac{(j-1-i)}{(i+2)^{r+1}} \right)$$

$$\left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_* = \sum_{i=0}^{j-1} \frac{(-1)^{j-1-i} H_{i+1}^{(k-r)}}{(j-1-i)!(i+1)!} \left(\frac{1}{(i+1)^r} - \frac{1}{(i+2)^r} + \frac{(j+1)}{(i+2)^{r+1}} \right).$$

Proof. Let the coefficient terms, $c_j(i)$, be defined as in (17). It follows from (10) that

$$\left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_* = \sum_{i=0}^{j-1} \left(\frac{c_j(j-i)}{(j-i)^r} - \frac{c_j(j+1-i)}{(j+1-i)^r} \right) H_{j-i}^{(k-r)}$$

$$= \sum_{i=0}^{j-1} \frac{(-1)^i H_{j-i}^{(k-r)}}{i!(j-i)!} \left(\frac{1}{(j-i)^r} + \frac{i}{(j+1-i)^{r+1}} \right).$$

The identities in the proposition follow similarly by interchanging the summation indices in the last equation. $\square$

3.2. **Exponential Harmonic Number Sums.** The *Mathematica* `Sigma` package [11] is able to obtain the formulas given in Corollary 2.1 by a straightforward procedure. The package is also able to verify the related results that

$$\sum_{i=1}^{j} \frac{H_i}{i!} \frac{(-1)^{j-i}}{(j-i)!} = \frac{(-1)^{j-1}}{j\, j!} \tag{18}$$

$$\sum_{i=1}^{j} \frac{H_i^{(2)}}{i!} \frac{(-1)^{j-i}}{(j-i)!} = \frac{(-1)^{j-1}}{j\, j!} H_j$$

$$\sum_{i=1}^{j} \frac{H_i^{(3)}}{i!} \frac{(-1)^{j-i}}{(j-i)!} = \frac{(-1)^{j-1}}{2j\, j!} \left(H_j^2 + H_j^{(2)} \right)$$

$$\sum_{i=1}^{j} \frac{H_i^{(4)}}{i!} \frac{(-1)^{j-i}}{(j-i)!} = \frac{(-1)^{j-1}}{6j\, j!} \left(H_j^3 + 3H_j H_j^{(2)} + 2H_j^{(3)} \right),$$

and then that

$$\frac{H_j}{j!} = \sum_{i=1}^{j} \frac{(-1)^{i-1}}{i\, i!(j-i)!} \tag{19}$$

$$\frac{H_j^{(2)}}{j!} = \sum_{i=1}^{j} \frac{(-1)^{i-1}}{i\, i!(j-i)!} H_i$$

$$\begin{aligned}
\frac{H_j^{(3)}}{j!} &= \sum_{i=1}^{j} \frac{(-1)^{i-1}}{2i\ i!(j-i)!}\left(H_i^2 + H_i^{(2)}\right) \\
\frac{H_j^{(4)}}{j!} &= \sum_{i=1}^{j} \frac{(-1)^{i-1}}{6i\ i!(j-i)!}\left(H_i^3 + 3H_iH_i^{(2)} + 2H_i^{(3)}\right),
\end{aligned}$$

by considering the generating functions over each side of the equations in (18). An alternate, direct approach using the capabilities of `Sigma` for the special cases of the sums in (19) is also used to obtain the closed-forms of these sums.

Proposition 3.3 provides a generalization of the coefficient sums given by these special cases. The particular expansions of the previous identities and their generalized forms in the proposition immediately imply relations between the exponential generating functions of the r-order harmonic numbers and of the generalized coefficients in (4).

Proposition 3.3. *For $k \in \mathbb{N}$ and $j \in \mathbb{Z}^+$, the generalized coefficients and exponential harmonic numbers are related through the following sums:*

$$\left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_* \cdot \frac{1}{j} = \sum_{m=0}^{j} \frac{H_m^{(k+1)}}{m!} \frac{(-1)^{j-m}}{(j-m)!} \tag{20}$$

$$\frac{H_j^{(k+1)}}{j!} = \sum_{i=1}^{j} \left\{ \begin{matrix} k+2 \\ i \end{matrix} \right\}_* \cdot \frac{1}{i(j-i)!}. \tag{21}$$

Proof of Equation (6). We first notice that the integral representation for the reciprocal gamma function cited in the introduction is restated in the following form for all integers $n \geq 0$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\imath nt} e^{e^{\imath t}} dt = \frac{1}{n!}.$$

Then by expanding the left-hand-side of (6) as a generating function for the binomial coefficients, $\binom{n+j+1}{j+1}$, we see that

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{\left(wze^{-\imath\vartheta}\right)^j}{\left(1-wze^{-\imath\vartheta}\right)^{j+2}} e^{e^{\imath\vartheta}} d\vartheta &= \frac{1}{2\pi}\int_{-\pi}^{+\pi}\left(\sum_{n\geq 0} \frac{(j+1+n)!}{(j+1)!\cdot n!}\left(wze^{-\imath t}\right)^{n+j}\right) e^{e^{\imath t}} dt \\
&= \frac{(wz)^j}{(j+1)!} \times \sum_{n\geq 0} \frac{(j+1+n)}{n!}(wz)^n \\
&= \frac{(wz)^j}{(j+1)!} \times \left((j+1)e^{wz} + \sum_{n\geq 1}\frac{(wz)^n}{(n-1)!}\right) \\
&= \frac{(wz)^j e^{wz}}{(j+1)!}(j+1+wz).
\end{aligned}$$

□

Proof. We recall the series for the exponential harmonic numbers cited in (5f) of the introduction in the limiting case where $u \to \infty$ given by

$$\frac{H_n^{(k+1)}}{n!} = [z^n]\left(\sum_{j\geq 1} \left\{ {k+3 \atop j} \right\}_* z^j e^z \left(1 + \frac{z}{j+1}\right)\right).$$

The formula stated in (5f) follows easily from (5a) and the proof of (6) given above. Next, we see that the generating function, $\widehat{H}_k(z)$, of the *k-order exponential harmonic numbers*, $H_n^{(k)}/n!$, is given by

$$\begin{aligned}
\widehat{H}_{k+1}(z)e^{-z} &= \sum_{j\geq 1} \left\{ {k+3 \atop j} \right\}_* z^j \left(1 + \frac{z}{j+1}\right) \\
&= \sum_{j\geq 1} \left\{ {k+3 \atop j} \right\}_* z^j + \sum_{j\geq 2} \left\{ {k+3 \atop j-1} \right\}_* \frac{z^j}{j} \\
&= \sum_{j\geq 1} \left\{ {k+2 \atop j} \right\}_* \frac{z^j}{j},
\end{aligned}$$

by (4), which implies the second identity in (21). The proof of either (20) or (21) implies the result in the other equation by a formal power series, or generating function convolution, argument for establishing the forms of (19) from the first set of results in (18) (and vice versa). □

Remark 3.4 (Functional Equations Resulting from the Binomial Transform)**.** Notice that the results in (19) imply new forms of functional equations between the *polylogarithm functions*, $\mathrm{Li}_s(z)/(1-z)$, when $s = 2, 3$. For example, by integrating the generating function for the first-order harmonic numbers and applying the binomial transform, the second identity in the previous equations leads to the known functional equation for the *dilogarithm function*, $\mathrm{Li}_2(z)$, providing that [9, §25.12(i)] [14]

$$\mathrm{Li}_2(z) = -\frac{1}{2}\operatorname{Log}(1-z)^2 - \mathrm{Li}_2\left(-\frac{z}{1-z}\right).$$

Similarly, the third identity in (19) implies a new functional equation between products of the natural logarithm, the dilogarithm function, and the *trilogarithm function*, $\mathrm{Li}_3(z)$, in the following form (*cf. Landen's formula* for the trilogarithm):

$$\begin{aligned}
\mathrm{Li}_3(z) = -\frac{1}{6}\operatorname{Log}(1-z)^3 + \frac{1}{2}\operatorname{Log}(1-z)^2 \operatorname{Log}\left(-\frac{z}{1-z}\right) - \operatorname{Log}(1-z)\,\mathrm{Li}_2\left(\frac{1}{1-z}\right) \\
- \operatorname{Log}(1-z)\,\mathrm{Li}_2\left(-\frac{z}{1-z}\right) - \mathrm{Li}_3\left(\frac{1}{1-z}\right) - \mathrm{Li}_3\left(-\frac{z}{1-z}\right) - \zeta(3).
\end{aligned}$$

Remark 3.5 (Exponential Generating Functions for Harmonic Numbers)**.** The *first-order harmonic numbers*, $H_n \equiv H_n^{(1)}$, have an explicit closed-form exponential generating function in z given by [1, *cf.* §5]

$$\widehat{H}_1(z) := \sum_{n=0}^{\infty} H_n \frac{z^n}{n!} = (-e^z) \sum_{k=1}^{\infty} \frac{(-z)^k}{k!\, k} \equiv e^z \left(\gamma + \Gamma(0,z) + \operatorname{Log}(z)\right). \tag{22}$$

where γ denotes *Euler's gamma constant* [9, §5.2(ii)] and $\Gamma(a,z)$ denotes the *incomplete gamma function* [9, §8]. No apparent simple analogs to the closed-form function on the right-hand-side of (22) are known for the exponential harmonic number generating functions, $\widehat{H}_k(z)$, when $k \geq 2$.

However, we are able to easily relate these exponential generating functions to the generating functions of the sequence, $\left\{ {k \atop j} \right\}_*$, by applying Proposition 3.3. In particular, if we define the ordinary generating function of the sequences, $\left\{ {k \atop j} \right\}_*$, over $j \geq 1$ for fixed k by $\widetilde{S}_{k,*}(z)$, the proposition immediately implies that (see Section 4.2)

$$\widehat{H}_{k+1}(z) = e^z \int_0^z \frac{\widetilde{S}_{k+2,*}(t)}{t} dt.$$

We compare these integral formulas to the somewhat simpler formal series expansion for the exponential harmonic numbers, $\widehat{H}_n^{(r)} = H_n^{(r)}/n!$, in the example from (5f) of the introduction in the form of

$$\widehat{H}_r(z) = \sum_{n \geq 1} \widehat{H}_n^{(r)} z^n = \sum_{j \geq 1} \left\{ {k+2 \atop j} \right\}_* \frac{z^j e^z}{(j+1)} \left(j+1+z\right).$$

Other relations between the exponential harmonic numbers, the generalized coefficients, $\left\{ {k+2 \atop j} \right\}_*$, and the sequences, $M_{k+1}^{(d)}(z)$, are considered below in Section 4.2.

3.3. New Recurrences and Expansions of the k-Order Harmonic Numbers in Powers of n.

Remark 3.6 (Formulas for Integral Powers of n)**.** For positive $n \in \mathbb{N}$, the following finite sums define the forms of the integral powers of n, given by n^k and n^{-k} for $k \in \mathbb{Z}^+$, respectively in terms of sums over the Stirling numbers of the second kind and the generalized transformation coefficients from (4):

$$n^k = \sum_{j=1}^{k} \left\{ {k \atop j} \right\} \frac{n!}{(n-j)!} \tag{23}$$

$$\frac{1}{n^k} = \sum_{j=1}^{n} \left\{ {k+2 \atop j} \right\}_* \frac{n!}{(n-j)!}. \tag{24}$$

A formula related to (23) cited in the references [9, eq. (26.8.34); §26.8(v)] is re-stated as follows for scalar-valued $x \neq 0, 1$:

$$\sum_{j=0}^{n} j^k x^j = \sum_{j=0}^{k} \left\{ {k \atop j} \right\} x^j \frac{d^{(j)}}{dx^{(j)}} \left[\frac{1-x^{n+1}}{1-x} \right].$$

For fixed $k \in \mathbb{N}$ and $n \geq 0$, these partial sums can also be expressed in closed-form through the *Bernoulli numbers*, B_n, defined as in [9, §] [6, §6.5] by

$$S_k(n) := \sum_{j=0}^{n-1} j^k = \sum_{m=0}^{k} \binom{k+1}{m} \frac{B_m n^{k+1-m}}{(k+1)}.$$

For $k \in \mathbb{Z}^+$ and $n \in \mathbb{N}$, the integer-order harmonic number sequences, $H_n^{(k)}$, can then be defined recursively in terms of the generalized coefficient forms as follows when $n \geq 1$ and where $H_0^{(k)} \equiv 0$ for all $k \in \mathbb{Z}^+$:

$$\begin{aligned} H_n^{(k)} &= H_{n-1}^{(k)} + \frac{1}{n^k} [n \geq 1]_\delta \\ &= H_{n-1}^{(k)} + \sum_{j=1}^{n} \left\{ {k+2 \atop j} \right\}_* \frac{n!}{(n-j)!}. \end{aligned} \tag{25}$$

We are now concerned with applying the results in the previous propositions to (24) in order to obtain new recurrence relations and sums for the r-order harmonic number sequences.

Corollary 3.7 (Recurrences for the k-Order Harmonic Numbers). *Suppose that $k \in \mathbb{Z}^+$ and let $r \in [0, k) \subseteq \mathbb{R}$. The k-order harmonic numbers satisfy each of the following recurrence relations:*

$$H_n^{(k)} = H_{n-1}^{(k)} + \sum_{1 \leq i \leq j \leq n} \binom{n}{j} \left\{ {k+1 \atop i} \right\}_* (-1)^{j-i} (i-1)!$$

$$H_n^{(k)} = H_{n-1}^{(k)} + \sum_{1 \leq m \leq i \leq j \leq n} \binom{n}{j} \binom{i}{m} (-1)^{j+m} H_m^{(k)}$$

$$H_n^{(k)} = H_{n-1}^{(k)} + \sum_{0 \leq i < j \leq n} \binom{n}{j} \binom{j}{i+1} (-1)^{j-1-i} H_{i+1}^{(k-r)} \left(\frac{1}{(i+1)^r} - \frac{1}{(i+2)^r} + \frac{(j+1)}{(i+2)^{r+1}} \right).$$

Proof. The first recurrence relation results by applying (25) to the identity

$$\left\{ {k+2 \atop j} \right\}_* = \frac{1}{j!} \sum_{1 \leq i \leq j} \left\{ {k+1 \atop i} \right\}_* (-1)^{j-i} (i-1)!.$$

The second and third identities are similarly obtained from (25) respectively combined with Proposition 3.3 and Proposition 3.2. ☐

Proposition 3.8 (Formulas for the k-Order Harmonic Numbers in Powers of n)**.** *Let $k \in \mathbb{N}$ and $r \in [0,1) \subseteq \mathbb{R}$. For each $n \in \mathbb{N}$, the k-order harmonic numbers are given respectively through the next finite sums involving positive integer powers of $n+1$.*

$$H_n^{(k)} = \sum_{0 \leq j \leq n} \left(\sum_{0 \leq m \leq j+1} \begin{bmatrix} j+1 \\ m \end{bmatrix} \begin{Bmatrix} k+2 \\ j \end{Bmatrix}_* \frac{(-1)^{j+1-m}(n+1)^m}{(j+1)} \right) \tag{26}$$

Proof. For $n, j \in \mathbb{N}$ we have the next expansion of the binomial coefficients as a finite sum over powers of $n+1$ given in (i).

$$\begin{aligned} \binom{n+1}{j+1} &= \frac{(n+1)!}{(j+1)! \cdot (n+1-(j+1))!} \\ &= \sum_{m=0}^{j+1} \begin{bmatrix} j+1 \\ m \end{bmatrix} \frac{(-1)^{j+1-m}(n+1)^m}{(j+1)!}. \end{aligned} \tag{i}$$

The result in (26) then follows from the identity

$$H_n^{(k)} = \sum_{0 \leq j \leq n} \binom{n+1}{j+1} \begin{Bmatrix} k+2 \\ j \end{Bmatrix}_* j!. \qquad \square$$

4. Applications

4.1. **New Series for Polylogarithm-Related Functions and Special Values.** For the geometric series special case of the transform in (3) with $g_n \equiv 1$ for all n, the next transformation stated in (27) is employed to expand the polylogarithm function, $\mathrm{Li}_s(z) = \sum_{n \geq 1} z^n / n^s$, in terms of only the j^{th} derivatives, $G^{(j)}(z) := j!/(1-z)^{j+1}$, as a series analog to (7) given by

$$\mathrm{Li}_s(z) = \sum_{j=1}^{\infty} \begin{Bmatrix} s+2 \\ j \end{Bmatrix}_* \frac{z^j j!}{(1-z)^{j+1}}. \tag{27}$$

Corollary 4.1 (Polylogarithm Functions)**.** *The polylogarithm functions, $\mathrm{Li}_k(z)$, for $k \in [1,4] \subseteq \mathbb{N}$ are expanded as the following special case series:*

$$\begin{aligned} \mathrm{Li}_1(z) &= \sum_{j=1}^{\infty} (-1)^{j-1} H_j \frac{z^j}{(1-z)^{j+1}} \\ \mathrm{Li}_2(z) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2} \left(H_j^2 + H_j^{(2)} \right) \frac{z^j}{(1-z)^{j+1}} \\ \mathrm{Li}_3(z) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{6} \left(H_j^3 + 3H_j H_j^{(2)} + 2H_j^{(3)} \right) \frac{z^j}{(1-z)^{j+1}} \\ \mathrm{Li}_4(z) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{24} \left(H_j^4 + 6H_j^2 H_j^{(2)} + 3\left(H_j^{(2)} \right)^2 + 8H_j H_j^{(3)} + 6H_j^{(4)} \right) \frac{z^j}{(1-z)^{j+1}}. \end{aligned}$$

Proof. These series follow from the coefficient identities given in (12) of Corollary 2.1 applied to the transformed series of the polylogarithm function in (27). □

Example 4.2 (The Alternating Zeta Function)**.** Let $s \in \mathbb{Z}^+$ and consider the following forms of the *alternating zeta function*, $\zeta^*(s) = \mathrm{Li}_s(-1)$, defined as in the references [4, §7] [4].

$$\zeta^*(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \left(1 - 2^{1-s}\right) \zeta(s) \cdot [s > 1]_\delta + \mathrm{Log}(2) \cdot [s = 1]_\delta \tag{28}$$

Since $\zeta^*(s) \equiv -\mathrm{Li}_s(-1)$, the transformed series for the polylogarithm function in (27) leads to series expansions given by

$$\zeta^*(s) = \sum_{j=1}^{\infty} \left\{ {s+2 \atop j} \right\}_* \frac{(-1)^{j-1} \cdot j!}{2^{j+1}}. \tag{29}$$

The coefficient formulas in Corollary 2.1 are then applied to obtain the following new series results for the next few special cases of the alternating zeta function constants in (28) [5]:

$$\begin{aligned}
\zeta^*(1) &= \sum_{j=1}^{\infty} \frac{H_j}{2 \cdot 2^j} \equiv \mathrm{Log}(2) \\
\zeta^*(2) &= \sum_{j=1}^{\infty} \frac{\left(H_j^2 + H_j^{(2)}\right)}{4 \cdot 2^j} \equiv \frac{1}{2} \cdot \frac{\pi^2}{6} \\
\zeta^*(3) &= \sum_{j=1}^{\infty} \frac{\left(H_j^3 + 3H_j H_j^{(2)} + 2H_j^{(3)}\right)}{12 \cdot 2^j} \equiv \frac{3}{4} \cdot \zeta(3) \\
\zeta^*(4) &= \sum_{j=1}^{\infty} \frac{\left(H_j^4 + 6H_j^2 H_j^{(2)} + 3\left(H_j^{(2)}\right)^2 + 8H_j H_j^{(3)} + 6H_j^{(4)}\right)}{48 \cdot 2^j} \equiv \frac{7}{8} \cdot \frac{\pi^4}{90}.
\end{aligned}$$

Notice that since the exponential generating function for the Stirling numbers of the first kind, $\left[{j+1 \atop k+1}\right]/j!$, is given by

$$\sum_{j \geq 0} \left[{j+1 \atop k+1}\right] \frac{z^j}{j!} = \frac{(-1)^k}{k! \cdot (1-z)} \mathrm{Log}\left(\frac{1}{1-z}\right)^k,$$

[4] The alternating zeta function, $\zeta^*(s)$, is defined in the alternate notation of $\bar{\zeta}(s)$ for the function in the reference [4, §7]. The series for $\zeta^*(s)$ is also commonly denoted by the *Dirichlet eta function*, $\eta(s)$, also as noted in the reference [7, eq. (3); §2].

[5] These formulas are compared to the Bell polynomial expansions of the identity in (13) cited in [3, §3]

we may also write the left-hand-side sums in the previous equations in terms of powers of $\mathrm{Log}(2)$ and partial Euler-like sums involving weighted terms of harmonic numbers. For example, the series for $\zeta^*(s)$ for $3 \leq s \leq 5$ are expanded as [1, §2] [6]

$$\begin{aligned}
\zeta^*(3) &= \frac{1}{6}\mathrm{Log}(2)^3 + \sum_{j\geq 0} \frac{H_j H_j^{(2)}}{2^{j+1}} && \approx 0.901543 \\
\zeta^*(4) &= \frac{1}{24}\mathrm{Log}(2)^4 + \sum_{j\geq 0} \frac{H_j^2 H_j^{(2)}}{2^{j+2}} + \sum_{j\geq 0} \frac{H_j H_j^{(3)}}{2^{j+2}} && \approx 0.947033 \\
\zeta^*(5) &= \frac{1}{120}\mathrm{Log}(2)^5 + \sum_{j\geq 0} \frac{H_j^3 H_j^{(2)}}{12\cdot 2^{j}} + \sum_{j\geq 0} \frac{H_j^{(2)} H_j^{(3)}}{6\cdot 2^{j}} + \sum_{j\geq 0} \frac{H_j H_j^{(4)}}{2^{j+2}} && \approx 0.972120.
\end{aligned}$$

Other identities for the partial sums of the right-hand-side sums in the previous equations are obtained through the `Sigma` package.

Example 4.3 (Fourier Series for the Periodic Bernoulli Polynomials)**.** The *Bernoulli polynomials*, $B_n(x)$, have the exponential generating function [9, §24.2]

$$\sum_{n\geq 0} \frac{B_n(x)}{n!} z^n = \frac{ze^{xz}}{e^z - 1}.$$

These polynomials also satisfy Fourier series of the following forms when $k \geq 0$ and where $\{x\}$ denotes the *fractional part* of $x \in \mathbb{R}$ [9, *cf.* §24.8(i)]:

$$\begin{aligned}
\frac{B_{2k+2}(\{x\})}{(2k+2)!} &= \frac{2(-1)^{k+1}}{(2\pi)^{2k+2}} \times \sum_{n\geq 0} \frac{(-1)^n}{n^{2k+2}} \cos\left[2\pi n\left(x - \frac{1}{2}\right)\right] \\
&= \frac{(-1)^{k+1}}{(2\pi)^{2k+2}} \sum_{j\geq 1} \left\{ {2k+4 \atop j} \right\}_* \left[\frac{\left(e^{2\pi\imath(x-1/2)}\right)^j}{\left(1+e^{2\pi\imath(x-1/2)}\right)^{j+1}} + \frac{\left(e^{-2\pi\imath(x-1/2)}\right)^j}{\left(1+e^{-2\pi\imath(x-1/2)}\right)^{j+1}} \right] \\
\frac{B_{2k+1}(\{x\})}{(2k+1)!} &= \frac{2(-1)^{k}}{(2\pi)^{2k+1}} \times \sum_{n\geq 0} \frac{(-1)^n}{n^{2k+1}} \sin\left[2\pi n\left(x - \frac{1}{2}\right)\right] \\
&= \frac{(-1)^{k}}{(2\pi)^{2k+1}\cdot\imath} \sum_{j\geq 1} \left\{ {2k+3 \atop j} \right\}_* \left[\frac{\left(e^{2\pi\imath(x-1/2)}\right)^j}{\left(1+e^{2\pi\imath(x-1/2)}\right)^{j+1}} - \frac{\left(e^{-2\pi\imath(x-1/2)}\right)^j}{\left(1+e^{-2\pi\imath(x-1/2)}\right)^{j+1}} \right]
\end{aligned}$$

Several particular examples of the series for the *periodic Bernoulli polynomial* variants, $B_k(\{x\})/k!$, expanded by the previous equations are given by

$$B_1(\{5/4\}) = \left\{\frac{5}{4}\right\} - \frac{1}{2}$$

$$= \frac{(\imath+1)}{4\pi\imath} \times \sum_{j\geq 0} \frac{H_j}{2^j}\left((1-\imath)^j + \imath(1+\imath)^j\right)$$

$$\frac{B_2\left(\left\{\frac{5}{4}\right\}\right)}{2} = \left\{\frac{5}{4}\right\}^2 - \left\{\frac{5}{4}\right\} - \frac{1}{6}$$

$$= -\frac{(\imath+1)}{16\pi^2} \times \sum_{j\geq 0} \frac{H_j^2 + H_j^{(2)}}{2^j}\left((1-\imath)^j - \imath(1+\imath)^j\right)$$

$$\frac{B_3\left(\left\{\frac{11}{4}\right\}\right)}{6} = \left\{\frac{11}{4}\right\}^3 - \frac{3}{2}\left\{\frac{11}{4}\right\}^2 + \frac{1}{2}\left\{\frac{11}{4}\right\}$$

$$= -\frac{(\imath+1)}{48\pi^2\cdot\imath} \times \sum_{j\geq 0}\left(H_j^3 + 3H_jH_j^{(2)} + 2H_j^{(3)}\right)\frac{(\imath^j - \imath)}{(1+\imath)^{j+1}}.$$

More generally, for $k = 1, 2$ and any $x \in \mathbb{R}$ we may write the periodic Bernoulli polynomials in the forms of

$$B_1(\{x\}) = \frac{1}{2\pi\imath}\operatorname{Log}\left(\frac{1-e^{2\pi\imath\cdot x}}{1-e^{-2\pi\imath\cdot x}}\right)$$

$$\frac{B_2(\{x\})}{2} = -\frac{1}{4\pi^2}\sum_{b=\pm 1}\left(\operatorname{Log}(1-e^{2\pi\imath\cdot bx})^2 + 2\operatorname{Li}_2\left(\frac{1}{2}(1+b\imath\cot(\pi x))\right)\right).$$

4.2. **Almost Linear Recurrence Relations for the k-Order Harmonic Numbers.** We first define the sequences, $M_{k+1}^{(d)}(n)$, for integers $k > 2$, $d \geq 1$, and $n \geq 0$ as

$$M_{k+1}^{(d)}(n) = \sum_{1\leq m\leq d}\begin{bmatrix} d \\ m \end{bmatrix} H_n^{(k+1-m)}. \tag{30}$$

We have an alternate sum for these terms proved in the following proposition.

Proposition 4.4 (An Alternate Sum Identity)**.** *For integers $k > 2$, $d \geq 1$, and $n \geq 0$, we have that the harmonic number sums, $M_{k+1}^{(d)}(n)$, in* (30) *satisfy*

$$M_{k+1}^{(d)}(n) = \sum_{1\leq j\leq n}\binom{n}{j}\left\{\begin{matrix} k+2 \\ j \end{matrix}\right\}_* \frac{(-1)^j}{(j+d)}\cdot\frac{(n+d)!}{n!}. \tag{31}$$

Proof. We first use the *RISC Mathematica* package `Guess`[6] to suggest a short proof that both (30) and (31) satisfy the same homogeneous recurrence relation given by

$$M_{k+1}^{(d)}(n) - M_{k+1}^{(d)}(n+1) + (n+2)M_{k+2}^{(d)}(n+1) - (n+2)M_{k+2}^{(d)}(n) = 0,$$

though many other variants of this recurrence are formulated similarly. We then deduce from this observation that the two formulas are equivalent representations of the harmonic number sums in (30). □

[6] https://www.risc.jku.at/research/combinat/software/ergosum/RISC/Guess.html.

We use the definition in (30) for multiple cases of $d \in \mathbb{Z}^+$ to obtain new almost linear recurrence relations between the r-order harmonic numbers over r with *"remainder"* terms given in terms of the sums in (31) of Proposition 4.4. The next corollary provides several particular examples.

Corollary 4.5 (New Almost Linear Recurrence Relations for the Harmonic Numbers)**.** *For $n \geq 0$, $k \in \mathbb{Z}^+$, and any fixed $m \in \mathbb{R}$, we have the following "almost linear" recurrence relations for the harmonic number sequences involving so-termed "remainder" terms given by the sequences defined in* (31)*:*

$$\begin{aligned}
H_n^{(k)} &= H_n^{(k-2)} - 3M_{k+1}^{(2)}(n) + M_{k+1}^{(3)}(n) \\
2H_n^{(k)} &= -3H_n^{(k-1)} - H_n^{(k-2)} - M_{k+1}^{(3)}(n) \\
7H_n^{(k)} &= -12H_n^{(k-1)} + 6H_n^{(k-2)} - H_n^{(p-3)} - M_{k+1}^{(2)}(n) + M_{k+1}^{(4)}(n) \\
5H_n^{(k)} &= -9H_n^{(k-1)} - 5H_n^{(k-2)} - H_n^{(p-3)} - M_{k+1}^{(2)}(n) + M_{k+1}^{(3)}(n) - M_{k+1}^{(4)}(n) \\
H_n^{(k)} &= 2H_n^{(k-2)} - H_n^{(p-3)} + M_{k+1}^{(2)}(n) - 4M_{k+1}^{(3)}(n) + M_{k+1}^{(4)}(n) \\
H_n^{(k)} &= (1-m)H_n^{(k-2)} + mH_n^{(p-4)} - (12m+3)M_{k+1}^{(2)}(n) + (24m+1)M_{k+1}^{(3)}(n) \\
&\quad - 10mM_{k+1}^{(4)}(n) + mM_{k+1}^{(5)}(n).
\end{aligned}$$

Proof. We are able to prove these recurrences as special cases of more general equations obtained from (30) by Gaussian elimination. Specifically, for constants $a_i, b_i, c_i \in \mathbb{R}$ and $d \neq 0$, the following more general recurrence relations follow from (30):

$$\begin{aligned}
dH_n^{(k)} &= a_1 H_n^{(k-1)} + (a_1 + d)\, H_n^{(k-2)} + (2a_1 + 3d)\, M_{k+1}^{(2)}(n) + (a_1 + d)\, M_{k+1}^{(3)}(n) \\
&= b_1 H_n^{(k-1)} + b_2 H_n^{(k-2)} - (b_1 - b_2 + d)\, H_n^{(k-3)} - (6b_1 - 4b_2 + 7d)\, M_{k+1}^{(2)}(n) \\
&\quad + (6b_1 - 5b_2 + 6d)\, M_{k+1}^{(3)}(n) - (b_1 - b_2 + d)\, M_{k+1}^{(4)}(n) \\
&= c_1 H_n^{(k-1)} + c_2 H_n^{(k-2)} + c_3 H_n^{(k-3)} + (c_1 - c_2 + c_3 + d)\, H_n^{(k-4)} \\
&\quad - (14c_1 - 12c_2 + 8c_3 + 15d)\, M_{k+1}^{(2)}(n) + (25c_1 - 24c_2 + 19c_3 + 25d)\, M_{k+1}^{(3)}(n) \\
&\quad - (10c_1 - 10c_2 + 9c_3 + 10d)\, M_{k+1}^{(4)}(n) + (c_1 - c_2 + c_3 + d)\, M_{k+1}^{(5)}(n).
\end{aligned}$$

Each of the recurrences listed in the corollaries above follow as special cases of these results. ☐

Remark 4.6 (Applications of the Corollary)**.** The limiting behavior of the recurrences given in the corollary provide new almost linear relations between the integer-order zeta constants and remainder terms expanded in the form of (31). The new recurrences given in Corollary 4.5 suggest an inductive approach to the limiting behaviors of these harmonic number sequences, and to the properties of the zeta function constants, $\zeta(2k+1)$, for integers $k \geq 1$. The catch with this approach is finding non-trivial approximations and limiting behaviors for the remainder terms, $M_{k+1}^{(d)}(n)$.

While the zeta function constants, $\zeta(2k)$ for $k \geq 1$, are known in closed-form through the *Bernoulli numbers* and rational multiples of powers of π, comparatively little is known about properties of the odd-indexed zeta constants, $\zeta(2k+1)$, when $k \geq 1$, with the exception of *Apéry's constant*, $\zeta(3)$, which is known to be irrational. We do however know that infinitely-many of these constants are irrational, and that at least one of the constants, $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, or $\zeta(11)$, must be irrational [15]. Statements of recurrence relations between the zeta functions over the positive integers of this type are apparently new, and offer new inductively-phrased insights to the properties of these constants as considered in the special case example below.

Example 4.7 (Generating Functions for the Remainder Term Sequences)**.** We can obtain the following coefficient forms of the exponential and ordinary generating functions for the remainder terms, $M_{k+1}^{(d)}(n)$, both directly from (31) and by applying the the binomial transform to the corresponding generating functions for the sequences, $\left\{ {k \atop j} \right\}_*$, respectively denoted by $\widetilde{S}_{k,*}(z)$ and $\widehat{S}_{k,*}(z)$:

$$\frac{M_{k+1}^{(d)}(n)}{n!} = [z^n]\left(\sum_{0 \leq i \leq d} \binom{d}{i}^2 (d-i)! z^i D_z^{(i)}\left[\frac{e^z}{(-z)^d}\int_0^{-z} t^{d-1}\widehat{S}_{k+2,*}(t)dt\right]\right)$$

$$\begin{aligned} M_{k+1}^{(d)}(n) &= \frac{(n+d)!}{n!} \cdot [z^n]\left(\frac{e^{z/(1-z)}}{1-z}\int_0^{-z/(1-z)} t^{d-1}\widetilde{S}_{k+2,*}(t)dt\right) \\ &= [z^n]\left(\sum_{0 \leq i \leq d} \binom{d}{i}^2 (d-i)! z^i D_z^{(i)}\left[\frac{e^{z/(1-z)}}{1-z}\left(\frac{-z}{1-z}\right)^{-d}\int_0^{-\frac{z}{1-z}} t^{d-1}\widetilde{S}_{k+2,*}(t)dt\right]\right). \end{aligned}$$

The forms of these generating functions imply relations between the exponential series functions, $\widehat{\mathrm{Li}}_s(z) = \sum_{n \geq 1} \frac{z^n}{n^s n!}$, and the exponential harmonic number generating functions, as well as between the zeta function constants, $\zeta(k)$, for integers $k \geq 2$ by Corollary 4.5. For example, we may relate the first two cases of the zeta function constants, $\zeta(2k+1)$, over the odd positive integers by asymptotically estimating the limiting behavior of the sums $M_6^{(2)}(n)$ and $M_6^{(3)}(n)$ as[7]

$$\zeta(5) = \zeta(3) - \lim_{n \to \infty}\left(3M_6^{(2)}(n) - M_6^{(3)}(n)\right)$$

[7] We also note that the following formula is obtained from the series in (7) by a reverse binomial transform operation for $k \geq 0$ and $j \geq 1$:

$$\left\{ {k+2 \atop j} \right\}_* = \frac{(-1)^j}{(j-1)!}[z^j]\,\mathrm{Li}_{k+1}\left(-\frac{z}{1-z}\right).$$

$$= \zeta(3) + \lim_{n\to\infty} [z^n] \left(\frac{(3z-1)}{(1-z)^4} \widetilde{S}_{8,*}\left(-\frac{z}{1-z}\right) - \frac{z(3z+1)}{(1-z)^5} \widetilde{S}_{8,*}'\left(-\frac{z}{1-z}\right) + \frac{z^2}{(1-z)^6} \widetilde{S}_{8,*}''\left(-\frac{z}{1-z}\right) \right).$$

Other special case relations between zeta function constants are constructed similarly from Corollary 4.5.

5. Conclusions

Summary of Results. The generalized coefficients implicitly defined through the transformation result in (3) satisfy a number of properties and relations analogous to those of the Stirling numbers of the second kind. The form of these implicit transformation coefficients satisfy a non-triangular, two-index recurrence relation given in (4) that can effectively be viewed as the Stirling number recurrence from (1) *"in reverse."* The coefficients defined recursively through (4) can alternately be viewed as a generalization of the Stirling numbers of the second kind in the context of (2) for $k \in \mathbb{Z} \setminus \mathbb{N}$.

There are a plethora of additional harmonic number identities and recurrence relations that are derived from the identities given in Section 2. We may also use the binomial transform with the new polylogarithm function series in (27) to give new proofs of well-known functional equations satisfied by the dilogarithm and trilogarithm functions. Since the truncated polylogarithm series and ordinary harmonic number generating functions are always rational, we may adapt these generalized series expansions to enumerate *Euler-sum-like* series with weighted coefficients of the form $H_n^{(\pi_1)} \cdots H_n^{(\pi_k)}/n^s$ [12, §6.3].

Generalizations. The interpretation of the transformation coefficients in (4) as the finite sum in (10) motivates several generalizations briefly outlined below. For example, given any non-zero sequence, $\langle f(n) \rangle$, we may define a formal series transformation for the corresponding sums

$$\sum_{n\geq 1} \frac{g_n}{f(n)^k} z^n = \sum_{j\geq 1} \left\{ {k+2 \atop j} \right\}_{f,*} z^j G^{(j)}(z),$$

where the modified coefficients, $\left\{ {k+2 \atop j} \right\}_{f,*}$, are given by the sum

$$\left\{ {k+2 \atop j} \right\}_{f,*} = \frac{1}{j!} \sum_{m=1}^{j} \binom{j}{m} \frac{(-1)^{j-m}}{f(m)^k}.$$

When $f(n) = \alpha n + \beta$, we can derive a number of similar identities to the relations established in this article in terms of the partial sums of the modified Hurwitz zeta function, $\Phi(z, s, \alpha, \beta)$ [5, *cf.* §3].

Suppose that $p/q \in \mathbb{Q}^+$ and let the rational-order series transformation with respect to z be defined as

$$\mathrm{QT}_{p/q}\left[F(z)\right] := \sum_{n=0}^{\infty} n^{p/q} \cdot [z^n]F(z) \cdot z^n.$$

One topic for further exploration is generalizing the first transformation involving the Stirling numbers of the second kind in (2) to analogous finite sum expansions that generate the positive rational-order series in the previous equation. If $|z| < 1$, the function $\mathrm{Li}_{s+1}(z)$ is given by

$$\mathrm{Li}_{s+1}(z) = \frac{z \cdot (-1)^s}{s!} \int_0^1 \frac{\mathrm{Log}(t)^s}{(1-tz)}\, dt,$$

which is evaluated termwise with respect to z as [2, eq. (4); §2]

$$\frac{1}{n^{s+1}} = \frac{(-1)^s}{s!} \int_0^1 t^{n-1} \cdot \mathrm{Log}(t)^s\, dt.$$

Other possible approaches to formulating the transformations of these series include considering series involving *fractional derivatives* described briefly in the references [9, §1.15(vi)-(vii)]. Another alternate approach is to consider a shifted series in powers $(n \pm 1)^{p/q}$ that then employs an expansion over non-negative integral powers of n with coefficients in terms of binomial coefficients, though the resulting transformations in this case are no longer formulated as finite sums as in the formula from (2) when the exponent of p/q assumes values over the non-integer, positive rational numbers.

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School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332
E-mail address: `maxieds@gmail.com`