NUMBER OF SINGLETONS IN INVOLUTIONS OF LARGE SIZE: A CENTRAL RANGE AND A LARGE DEVIATION ANALYSIS

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ABSTRACT. In this paper, we analyze the asymptotic number $I(m,n)$ of involutions of large size n with m singletons. We consider a central region and a non-central region. In the range $m = n - n^{\alpha}, 0 < \alpha < 1$, we analyze the dependence of $I(m,n)$ on α. This paper fits within the framework of Analytic Combinatorics.

1. INTRODUCTION

During the last few years, we have been interested in asymptotic properties of some permutation parameters. For instance, we analyzed the number of inversions in [9] (in cooperation with Prodinger), of cycles (related to the Stirling numbers of the first kind) in [7], of rises (related to Eulerian numbers) in [8]. We extended the Gaussian approximation with more terms and we also considered some large deviation expansions. In this paper, we turn to another property: the number of singletons in involutions, an involution is a permutation σ such that σ^2 is the identity permutation. This corresponds to cycles of size 1 and 2, see Bóna [1] for details. We will use the saddle point method: see Flajolet and Sedgewick [2, ch. VIII] for a nice introduction.

We denote by I_n the total number of involutions of size n and by $I(m,n)$ the number of all involutions of size n with m singletons. We define a random variable J_n by the relation

$$\mathbb{P}(J_n = m) = \frac{I(m,n)}{I_n}.$$

This corresponds to the number of singletons in an involution chosen (uniformly) at random among all involutions of size n. In [2, ch. VIII, p.560], using the saddle point technique, Flajolet and Sedgewick give the first terms of the asymptotic expansion of I_n, also obtained by Knuth [6] (see also Moser and Wyman [10]). In Section 2, we provide a more detailed expansion of I_n. In [2, ch. VIII, p.692], the authors provide the first terms of the mean and variance of J_n. In Section 2, we consider a detailed analysis of all moments of J_n. In [2, ch. VIII, p.692], the authors prove the dominant Gaussian asymptotic of J_n by using together the saddle point technique and a generalized quasi-powers technique (see Sachkov [11], Hwang [4], [5]). In Section 3, we give a detailed

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analysis of the asymptotic distribution of J_n. In Section 4, we consider a large deviation range: $m = n - n^\alpha, 0 < \alpha < 1$. In this section, we will use multiseries expansions: multiseries are in effect power series (in which the powers may be non-integral but must tend to infinity) and the variables are elements of a scale. The scale is a set of variables of increasing order. The series is computed in terms of the variable of maximum order, the coefficients of which are given in terms of the next-to-maximum order, etc. This is more precise than mixing different terms.

Note finally that our approach can be used in generalizations of the involution: we can deal with cycles of any chosen size and deal with singletons or other specific cycles.

2. The moments

We will prove the following theorem:

Theorem 2.1. *The asymptotic expansion of the factorial moments of J_n is given by*

$$\mathbb{E}(J_n^{\underline{\ell}}) = n^{\ell/2}\left[1 - \frac{\ell}{2\sqrt{n}} - \frac{\ell(\ell-4)}{8n} + \frac{\ell\left(7 - 18\,\ell + 5\,\ell^2\right)}{48n^{3/2}} + \mathcal{O}\left(\frac{1}{n^2}\right)\right].$$

We have the classical (exponential) generating functions of I_n and $I(m,n)$:

$$f_1(z) = \sum_{n=0}^{\infty} I_n \frac{z^n}{n!} = e^{z+z^2/2},$$
$$f_2(z,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} I(m,n)\frac{z^n}{n!}y^m = e^{zy+z^2/2},$$
$$f_{3,m}(z) = \sum_{n=0}^{\infty} I(m,n)\frac{z^n}{n!} = e^{z^2/2}\frac{z^m}{m!}.$$

From $f_{3,m}(z)$, we have

$$\frac{I(m,n)}{n!} = \frac{1}{2^{(n-m)/2}((n-m)/2)!m!}. \tag{1}$$

We define $m^{\underline{\ell}} := \prod_{j=0}^{\ell-1}(m-j)$ as the ℓ-th falling factorial of m.

We have

$$\mathbb{E}(J_n^{\underline{\ell}}) = \sum_{m=0}^{\infty} m^{\underline{\ell}}\mathbb{P}(J_n = m) = \frac{S_\ell}{I_n},$$
$$S_\ell := \sum_{m=0}^{\infty} m^{\underline{\ell}} I(m,n) = n!\,[z^n]\,\frac{\partial^\ell}{\partial y^\ell} f_2(z,y)\bigg|_{y=1} = n!\,[z^n]z^\ell e^{z+z^2/2} = n![z^{n-\ell}]e^{z+z^2/2},$$
$$\mathbb{E}(J_n^{\underline{\ell}}) = \frac{n!}{I_n}\frac{I_{n-\ell}}{(n-\ell)!}.$$

Now we turn to an asymptotic expansion of I_n.

Let Ω denote the circle $\rho e^{\mathbf{i}\theta}$. By Cauchy's theorem, it follows that (integral performed counter-clockwise)

$$\begin{aligned} I_n/n! &= \frac{1}{2\pi\mathbf{i}}\int_\Omega \frac{f_1(z)}{z^{n+1}}\,dz \\ &= \frac{1}{\rho^n}\frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(\rho e^{\mathbf{i}\theta})e^{-n\mathbf{i}\theta}\,d\theta \qquad \text{using } z=\rho e^{\mathbf{i}\theta} \\ &= \frac{1}{\rho^n}\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(\ln(f_1(\rho e^{\mathbf{i}\theta})) - n\mathbf{i}\theta\right)d\theta \\ &= \frac{1}{\rho^n}\frac{f_1(\rho)}{2\pi}\int_{-\pi}^{\pi}\exp\left[\mathbf{i}\kappa_1\theta - n\mathbf{i}\theta - \frac{1}{2}\kappa_2\theta^2 - \frac{\mathbf{i}}{6}\kappa_3\theta^3 + \cdots\right]d\theta, \end{aligned} \tag{2}$$

where

$$\kappa_i := \frac{\partial^i}{\partial u^i}\ln(f_1(\rho e^u))|_{u=0}\,.$$

Now we have $\kappa_1 = \rho+\rho^2$ and we set $\kappa_1 - n = 0$ such that the saddle point is the root (of smallest modulus) of $\rho^2+\rho-n=0$. (From now on, we only provide a few terms in our expansions, but of course we use more terms in our computations). A direct computation gives

$$\begin{aligned} \rho &= -\frac{1}{2}+\frac{1}{2}\sqrt{4n+1} \\ &= \sqrt{n}-\frac{1}{2}+\frac{1}{8\sqrt{n}}-\frac{1}{128\,n^{3/2}}+\frac{1}{1024\,n^{5/2}}-\frac{5}{32768\,n^{7/2}}+\mathcal{O}\left(\frac{1}{n^{9/2}}\right), \\ \ln(\rho) &= \frac{\ln(n)}{2}-\frac{1}{2\sqrt{n}}+\frac{1}{48\,n^{3/2}}-\frac{3}{1280\,n^{5/2}}+\frac{5}{14336\,n^{7/2}}-\frac{35}{589824\,n^{9/2}}+\mathcal{O}\left(\frac{1}{n^5}\right). \end{aligned}$$

See Good [3] for a neat description of this technique.

The dominant part of (2) gives

$$\frac{f_1(\rho)}{\rho^n} = \exp(E_1),$$

$$\begin{aligned} E_1 &:= \rho+\frac{\rho^2}{2}-n\ln(\rho) = \frac{n}{2}+\frac{\rho}{2}-n\ln(\rho) \\ &= \frac{(1-\ln(n))\,n}{2}+\sqrt{n}-\frac{1}{4}+\frac{1}{24\sqrt{n}}-\frac{1}{640n^{3/2}}+\mathcal{O}\left(\frac{1}{n^2}\right), \end{aligned}$$

with the substitution $\rho^2 = n-\rho$. (This substitution will be frequently used in the sequel.)

Now we turn to the integral. We have

$$\kappa_2 = -\rho+2n,$$

and more generally

$$\kappa_j = -(2^{j-1}-1)\rho + 2^{j-1}n.$$

We choose a splitting value θ_0 such that $\kappa_2\theta_0^2 \to \infty$, and $\kappa_3\theta_0^3 \to 0$, as $n \to \infty$. If we choose $\theta_0 = n^\beta$, we must have $n^{2\beta+1} \to \infty, n^{3\beta+1} \to 0$. For instance, we can use $\theta_0 = n^{-5/12}$. We must prove that the integral

$$K_n = \int_{\theta_0}^{2\pi-\theta_0} \exp\left(\ln(f_1(\rho e^{\mathbf{i}\theta})) - n\mathbf{i}\theta\right) d\theta$$

is such that $|K_n|$ is exponentially small (tail pruning). This is done in [2, ch. VIII, p.559]. Now we use the classical trick of setting

$$\sum_{j=2}^{\infty} \kappa_j(\mathbf{i}\theta)^j/j! = \frac{1}{2}\left[(n-\rho)(e^{2\mathbf{i}\theta}-1-2\mathbf{i}\theta)\right] + \rho(e^{\mathbf{i}\theta}-1-\mathbf{i}\theta) = -u^2/2.$$

Computing θ as a series in u, this gives, by Lagrange inversion,

(3)
$$\theta = \sum_{i=1}^{\infty} a_i \frac{u^i}{n^{i/2}},$$

with, for instance (we use more coefficients in our computations),

$$a_1 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{8\sqrt{n}} - \frac{\sqrt{2}}{64\,n} - \frac{3\sqrt{2}}{256\,n^{3/2}} + \frac{11\sqrt{2}}{4096\,n^2} + \mathcal{O}\left(\frac{1}{n^{5/2}}\right),$$
$$a_2 = -\frac{\mathbf{i}}{6} - \frac{\mathbf{i}}{24\sqrt{n}} + \frac{\mathbf{i}}{48\,n} + \frac{\mathbf{i}}{192\,n^{3/2}} - \frac{\mathbf{i}}{192\,n^2} + \mathcal{O}\left(\frac{1}{n^{5/2}}\right).$$

This expansion is valid in the dominant integration domain

$$|u| \le \frac{\sqrt{n}\,\theta_0}{a_1} = \mathcal{O}\left(n^{1/12}\right).$$

Setting $d\theta = \frac{d\theta}{du}du$, we integrate on $u = [-\infty, \infty]$:

$$\frac{1}{2\pi\mathbf{i}}\int_{-\infty}^{\infty} e^{-u^2/2}\frac{d\theta}{du}du.$$

The extension of the range (tail completion) is justified in [2, ch. VIII, p.560]. The same justification is applicable in the next sections. The integration gives

$$\frac{1}{2\sqrt{\pi}\sqrt{n}}F_1, \text{ with}$$
$$F_1 := 1 + \frac{1}{4\sqrt{n}} - \frac{19}{96\,n} - \frac{13}{384\,n^{3/2}} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

This leads to the proposition

Proposition 2.2.

$$\frac{I_n}{n!} \sim \frac{1}{2\sqrt{\pi}\sqrt{n}} F_1 \exp(E_1)$$
$$= \frac{1}{2\sqrt{\pi}\sqrt{n}} \left[1 + \frac{1}{4\sqrt{n}} - \frac{19}{96\,n} - \frac{13}{384\,n^{3/2}} + \mathcal{O}\left(\frac{1}{n^2}\right)\right] \times$$
$$\times \exp\left[\frac{(1-\ln(n))\,n}{2} + \sqrt{n} - \frac{1}{4} + \frac{1}{24\sqrt{n}} - \frac{1}{640n^{3/2}} + \mathcal{O}\left(\frac{1}{n^2}\right)\right].$$

which extends previous results.

Now we turn to $\frac{I_{n-\ell}}{(n-\ell)!}$. We successively have, for $\ell = \mathcal{O}(1)$,

$$\rho_\ell + \rho_\ell^2 - n + \ell = 0,$$

$\rho_\ell^2 = n - \ell - \rho_\ell$ is used as a next substitution,

$$\rho_\ell = \sqrt{n} - \frac{1}{2} + \frac{(1-4\,\ell)}{8\sqrt{n}} - \frac{(-1+4\,\ell)^2}{128n^{3/2}} - \frac{(-1+4\,\ell)^3}{1024n^{5/2}}$$
$$- \frac{5\,(-1+4\,\ell)^4}{32768n^{7/2}} + \mathcal{O}\left(\frac{1}{n^{9/2}}\right),$$

$$\ln(\rho_\ell) = \frac{\ln(n)}{2} - \frac{1}{2\sqrt{n}} - \frac{\ell}{2n} + \frac{1-12\,\ell}{48n^{3/2}} - \frac{\ell^2}{4n^2} - \frac{3-40\,\ell+240\,\ell^2}{1280n^{5/2}} - \frac{\ell^3}{6n^3}$$
$$- \frac{-5+84\,\ell-560\,\ell^2+2240\,\ell^3}{14336n^{7/2}} - \frac{\ell^4}{8n^4}$$
$$- \frac{35-720\,\ell+6048\,\ell^2-26880\,\ell^3+80640\,\ell^4}{589824n^{9/2}} + \mathcal{O}\left(\frac{1}{n^5}\right),$$

$$\frac{f_1(\rho_\ell)}{\rho_\ell^n} = \exp(E_{1,\ell}),$$

$$E_{1,\ell} := \frac{n}{2} + \frac{\rho_\ell}{2} - \frac{\ell}{2} - (n-\ell)\ln(\rho_\ell),$$

$$\kappa_{j,\ell} = -(2^{j-1}-1)\rho_\ell + 2^{j-1}(n-\ell),$$

θ is again given by (3),

$$a_{1,\ell} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{8\sqrt{n}} + \frac{\sqrt{2}\,(-1+16\,\ell)}{64n} + \frac{\sqrt{2}\,(32\,\ell-3)}{256n^{3/2}} + \mathcal{O}\left(\frac{1}{n^2}\right),$$

$$F_{1,\ell} := 1 + \frac{1}{4\sqrt{n}} + \frac{-19+48\,\ell}{96n} + \frac{96\,\ell-13}{384n^{3/2}} + \mathcal{O}\left(\frac{1}{n^2}\right),$$

$$\frac{I_{n-\ell}}{(n-\ell)!} \sim \frac{1}{2\sqrt{\pi}\sqrt{n}} F_{1,\ell} \exp(E_{1,\ell}).$$

Note that setting $\ell = 0$, we recover of course I_n. The detailed expansions of ρ_ℓ and $\ln(\rho_\ell)$ are used in E_1 and $E_{1,\ell}$.

We are now ready to compute $\mathbb{E}(J_n^\ell) = \frac{S_\ell}{I_n}$. We derive

$$\exp(E_{1,\ell} - E_1) = n^{\ell/2} T_1,$$

$$T_1 = 1 - \frac{1}{2\sqrt{n}} - \frac{\ell^2}{8n} + \frac{\ell\,(5\,\ell - 1)\,(\ell - 1)}{48n^{3/2}} + \frac{\ell^2\,(-4 + 24\,\ell + \ell^2)}{384n^2} - \frac{\ell^3\,(10 - 60\,\ell + 41\,\ell^2)}{3840n^{5/2}} + \mathcal{O}\left(\frac{1}{n^3}\right),$$

$$\frac{F_{1,\ell}}{F_1} = T_2,$$

$$T_2 = 1 + \frac{\ell}{2n} + \frac{\ell}{8n^{3/2}} + \mathcal{O}\left(\frac{1}{n^2}\right),$$

$$T_1 T_2 = 1 - \frac{\ell}{2\sqrt{n}} - \frac{\ell\,(\ell - 4)}{8n} + \frac{\ell\,(7 - 18\,\ell + 5\,\ell^2)}{48n^{3/2}} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

This leads to Theorem 2.1.

The first moments of J_n are now immediate:

$$M = \mathbb{E}(J_n) = \frac{S_1}{I_n} = \sqrt{n} - \frac{1}{2} + \frac{3}{8\sqrt{n}} - \frac{1}{8n} + \mathcal{O}\left(\frac{1}{n^{3/2}}\right),$$

$$\mathbb{E}(J_n(J_n - 1)) = \frac{S_2}{I_n},$$

$$\sigma^2 = \frac{S_2}{I_n} + M - M^2 = \sqrt{n} - 1 + \frac{5}{8\sqrt{n}} + \mathcal{O}\left(\frac{1}{n}\right),$$

$$\sigma = \sqrt[4]{n} - \frac{1}{2\sqrt[4]{n}} + \frac{3}{16n^{3/4}} + \mathcal{O}\left(\frac{1}{n}\right).$$

All moments can be similarly mechanically obtained.

3. Distribution of J_n

We consider the central range $M - 2\sigma < m < M + 2\sigma$. The distribution of J_n is given by the following theorem:

Theorem 3.1. *The asymptotic distribution of J_n in the central range is given by the local limit theorem:*

$$\mathbb{P}(J_n = m) = 2\frac{1}{\sqrt{2\pi}n^{1/4}} e^{-x^2/2} \left[1 + \frac{x\,(x^2 - 3)}{6\sqrt[4]{n}} + \frac{30 + 27\,x^2 - 12\,x^4 + x^6}{72}\frac{1}{\sqrt{n}} + \frac{x\,(810 - 2115\,x^2 + 999\,x^4 - 135\,x^6 + 5\,x^8)}{6480n^{3/4}} + \mathcal{O}\left(\frac{1}{n}\right)\right]. \tag{4}$$

We have $m = M + x\sigma,\ x = \Theta(1)$. We first analyze $\frac{I(m,n)}{n!}$.

3.1. **Approach 1: Asymptotic expansion of $\frac{I(m,n)}{n!}$ using the saddle point technique.** We derive

$$\kappa_i := \frac{\partial^i}{\partial u^i} \ln(f_{3,m}(\rho e^u))|_{u=0},$$
$$\kappa_1 = \rho^2 + m,$$
$$\rho^2 = n - m \text{ (we use that as a substitution in the sequel)},$$
$$\kappa_j = (n-m)2^{j-1},$$
$$\rho = \sqrt{n} - \frac{1}{2} - \frac{x}{2\sqrt[4]{n}} + \frac{1}{8\sqrt{n}} - \frac{1+x^2}{8n} - \frac{x}{32n^{5/4}} + \mathcal{O}\left(\frac{1}{n^{3/2}}\right),$$
$$\ln(\rho) = \frac{\ln(n)}{2} - \frac{1}{2\sqrt{n}} - \frac{x}{2n^{3/4}} - \frac{x}{4n^{5/4}} - \frac{5+12x^2}{48n^{3/2}} - \frac{3x}{32n^{7/4}} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

The dominant part of (2) gives

$$\frac{f_{3,m}(\rho)}{\rho^n} = \exp(E_2), \tag{5}$$
$$E_2 := \frac{\rho^2}{2} + m\ln(\rho) - \ln(m!) - n\ln(\rho),$$

due to Stirling's formula we know that

$$\ln(\ell!) = -\ell + \ell\ln(\ell) + \frac{\ln(2\pi\ell)}{2} + \frac{1}{12\ell} + \mathcal{O}\left(\frac{1}{\ell^2}\right), \tag{6}$$

hence

$$E_2 = \frac{(1-\ln(n))\,n}{2} + \sqrt{n} - \frac{1}{4} - \frac{x^2}{2} - \frac{\ln(n)}{4} - \frac{\ln(2)}{2} - \frac{\ln(\pi)}{2} + \frac{x\left(x^2-3\right)}{6\sqrt[4]{n}}$$
$$- \frac{-5-6x^2+2x^4}{24\sqrt{n}} + \frac{x\left(-10-15x^2+3x^4\right)}{60n^{3/4}} + \mathcal{O}\left(\frac{1}{n}\right).$$

Let us mention that in the expansion of E_2 we uncovered the $-\frac{x^2}{2}$ exponent of the density of a normal distribution appearing in the local law.

Now we turn to the integral. Proceeding as previously, we have

$$\frac{1}{2}(n-m)(e^{2\mathbf{i}\theta}-1-2\mathbf{i}\theta) = -\frac{u^2}{2},$$

θ is again given by (3),

$$a_1 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{4\sqrt{n}} + \frac{\sqrt{2}x}{4n^{3/4}} + \frac{\sqrt{2}}{16n} + \frac{\sqrt{2}x}{4n^{5/4}} + \frac{\sqrt{2}\left(1+3x^2\right)}{16n^{3/2}} + \frac{7\sqrt{2}x}{64n^{7/4}} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

$$a_2 = -\frac{\mathbf{i}}{6} - \frac{\mathbf{i}}{6\sqrt{n}} - \frac{\mathbf{i}x}{6n^{3/4}} - \frac{\mathbf{i}}{12n} - \frac{\mathbf{i}x}{4n^{5/4}} - \frac{\mathbf{i}\left(8x^2+3\right)}{48n^{3/2}} - \frac{17\,\mathbf{i}x}{96n^{7/4}} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

This integration gives

$$\frac{1}{2\sqrt{\pi}\sqrt{n}}F_2,$$

$$F_2 := 1 + \frac{1}{2\sqrt{n}} + \frac{x}{2n^{3/4}} - \frac{1}{24n} + \frac{x}{2n^{5/4}} + \frac{-1+3x^2}{8n^{3/2}} - \frac{x}{32n^{7/4}} + \mathcal{O}\left(\frac{1}{n^2}\right),$$

$$\frac{I(m,n)}{n!} = 2\left[\frac{1}{2\sqrt{\pi}\sqrt{n}}F_2\exp(E_2)\right] = \frac{1}{\sqrt{\pi}\sqrt{n}}F_2\exp(E_2).$$

Note carefully the factor 2 in front of our expression: the tail pruning for the Gaussian asymptotics leads to consider

$$\Re[\ln(f_{3,m}(\rho e^{\mathbf{i}\theta})) - n\mathbf{i}\theta] = \Re[\rho^2 e^{2\mathbf{i}\theta}/2 + m\mathbf{i}\theta - n\mathbf{i}\theta + m\ln(\rho) - \ln(m!)]$$

$$= \frac{\rho^2\cos(2\theta)}{2} + m\ln(\rho) - \ln(m!)$$

which has two dominant peaks at 0 and π. So we must be more precise and analyze these two peaks: we have, with $n-m$ even,

$$\Re\left[\mathrm{e}^{1/2\,\rho^2\mathrm{e}^{2\,\mathrm{i}\theta}+\mathbf{i}(m-n)\theta}\right] = \mathrm{e}^{1/2\,\rho^2\cos(2\,\theta)}\cos\left(1/2\,\rho^2\sin\left(2\,\theta\right)+\theta\left(m-n\right)\right),$$

and, indeed, if we set $\theta = \pi + \delta$, we recover the same expression. The two peaks are equivalent. Hence a factor 2. The choice of θ_0 is the same as for I_n.

3.2. **Approach 2: Asymptotic expansion of $\frac{I(m,n)}{n!}$ using its explicit expression.** We use again $\ln(\ell!)$ as given by (6). We successively obtain

$$
\begin{aligned}
m &= \sqrt{n} + x\sqrt[4]{n} - \frac{1}{2} - \frac{x}{2\sqrt[4]{n}} + \frac{3}{8\sqrt{n}} + \frac{3x}{16n^{3/4}} + \mathcal{O}\left(\frac{1}{n}\right),\\
(n-m)/2 &= \frac{n}{2} - \frac{\sqrt{n}}{2} - \frac{x\sqrt[4]{n}}{2} + \frac{1}{4} + \frac{x}{4\sqrt[4]{n}} - \frac{3}{16\sqrt{n}} - \frac{3x}{32n^{3/4}} + \mathcal{O}\left(\frac{1}{n}\right),\\
\ln(m!) &= \left(-1 + \frac{\ln(n)}{2}\right)\sqrt{n} + \frac{x\ln(n)\sqrt[4]{n}}{2} + \frac{x^2}{2} + \frac{\ln(2)}{2} + \frac{\ln(\pi)}{2}\\
&\quad - \frac{x\left(x^2 + 3/2\ln(n)\right)}{6\sqrt[4]{n}} + \frac{-1 - 12x^2 + 2x^4 + 9/2\ln(n)}{24\sqrt{n}}\\
&\quad \frac{x\left(100 - 12x^4 + 45/2\ln(n) + 60x^2\right)}{240n^{3/4}} + \mathcal{O}\left(\frac{1}{n}\right),\\
\ln(((n-m)/2)!) &= \frac{-1-\ln(2)+\ln(n)}{2} + \frac{(\ln(2)-\ln(n))\sqrt{n}}{2} + \frac{x(\ln(2)-\ln(n))\sqrt[4]{n}}{2}\\
&\quad + \frac{1}{4} - \frac{\ln(2)}{4} + \frac{3\ln(n)}{4} + \frac{\ln((\pi)}{2} - \frac{x(\ln(2) - \ln(n) - 2)}{4\sqrt[4]{n}} + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}
$$

This leads to

$$
\begin{aligned}
\frac{I(m,n)}{n!} = \exp\Bigg[&\left(-\frac{\ln(n)}{2} + 1/2\right)n + \sqrt{n} - \frac{\ln(2)}{2} - \frac{1}{4} - \frac{x^2}{2}\\
&- \frac{3\ln(n)}{4} - \ln(\pi) + \frac{x\left(-3 + x^2\right)}{6\sqrt[4]{n}} + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\Bigg].
\end{aligned}
$$

This fits with the result of Approach 1 but more initial asymptotic precision is needed in Approach 2 in order to obtain the same final precision in the coefficient ($\mathcal{O}\left(\frac{1}{n^{3/4}}\right)$) as in Approach 1. We only obtain a $\mathcal{O}\left(\frac{1}{n^{1/4}}\right)$ precision. This is due to the fact that we use two $\ln(\ell!)$ asymptotics in Approach 2 instead of one in Approach 1.

3.3. **Distribution of J_n.** So, finally

$$\frac{I(m,n)}{I_n} = 2F_2/F_1 \exp(E_2 - E_1),$$
$$\exp(E_2 - E_1) = \frac{1}{\sqrt{2\pi} n^{1/4}} e^{-x^2/2} F_3,$$
$$F_3 := 1 + \frac{x\left(x^2-3\right)}{6\sqrt[4]{n}} + \frac{12 + 27\,x^2 - 12\,x^4 + x^6}{72\sqrt{n}} + \frac{x\left(-1620 - 2385\,x^2 + 999\,x^4 - 135\,x^6 + 5\,x^8\right)}{6480 n^{3/4}} + \mathcal{O}\left(\frac{1}{n}\right),$$
$$\frac{I(m,n)}{I_n} = \frac{2}{\sqrt{2\pi} n^{1/4}} e^{-x^2/2} F_4,$$
$$F_4 := F_2/F_1 F_3 = 1 + \frac{x\left(x^2-3\right)}{6\sqrt[4]{n}} + \frac{30 + 27\,x^2 - 12\,x^4 + x^6}{72} \frac{1}{\sqrt{n}} + \frac{x\left(810 - 2115\,x^2 + 999\,x^4 - 135\,x^6 + 5\,x^8\right)}{6480 n^{3/4}} + \mathcal{O}\left(\frac{1}{n}\right).$$

This leads to the local limit theorem 3.1

Of course more terms can be mechanically computed, but the expressions become much more intricate.

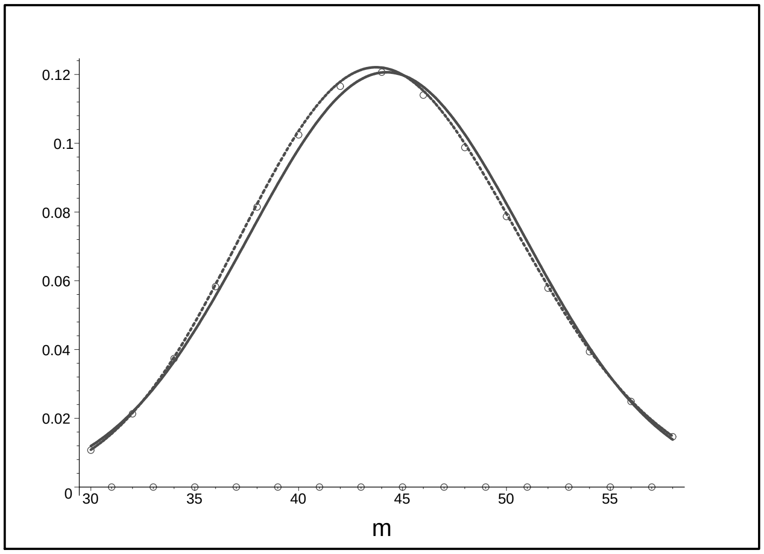
To check the quality of our asymptotics, we have chosen $n = 2000$. This gives $M = 44.22968229\ldots$, $\sigma = 6.613262555\ldots$, and a range $m \in [30, 58]$.

Figure 1 shows $I(m,n)/I_n$ (circle), a first-order asymptotic $2\frac{e^{-x^2/2}}{\sqrt{2\pi}\sigma}$ (line) and the asymptotic of Equ. (4) (dashed line). This fit is better. Note that m and n do have the same parity: $n - m$ is even. $I(m,n) = 0$ for m odd in our example.

Figure 2 gives the quotient of $I(m,n)/I_n$ and the asymptotic $2\frac{e^{-x^2/2}}{\sqrt{2\pi}\sigma}$ (box) as well as the quotient of $I(m,n)/I_n$ and the asymptotic Equ. (4) (circle). This last asymptotic is of course more precise than the first one.

4. Large deviation $m = n - n^\alpha$, $0 < \alpha < 1$

The multiseries' scale is here $n \gg n^\alpha \gg 1/\varepsilon$ if $\alpha > 1/2$ and $n \gg 1/\varepsilon \gg n^\alpha$ if $\alpha < 1/2$. It appears that the exact expression (1) is suitable in this case. We have the following theorem:



and

FIGURE 1. $I(m,n)/I_n$ (circle), a first-order asymptotic $2\frac{e^{-x^2/2}}{\sqrt{2\pi}\sigma}$ (line) and the asymptotic of Equ. (4) (dashed line)

Theorem 4.1. *The asymptotic expression of the $I(m,n)$ for large deviation $m = n - n^{\alpha}$, $0 < \alpha < 1$ is given, with $\varepsilon := n^{\alpha-1}$, by*

$$\begin{aligned}\frac{I(m,n)}{n!} &= \frac{1}{\sqrt{\pi n^{\alpha}}}\left(1 - \frac{1}{6n^{\alpha}} + \frac{1}{72n^{2\alpha}} + \mathcal{O}\left(\frac{1}{n^{3\alpha}}\right)\right) \times \\ &\times \frac{1}{\sqrt{2\pi n(1-\varepsilon)}}\left[1 - \frac{1}{12\,(1-\varepsilon)\,n} + \frac{1}{288\,(1-\varepsilon)^2\,n^2} + \mathcal{O}\left(\frac{1}{n^3}\right)\right] \times \\ &\times \exp\left[\left(\frac{1}{2} - \frac{\alpha\ln(n)}{2} + \ln(n)\right)n^{\alpha} + (-\ln(n)+1)\,n\right. \\ &\left. - \frac{n^{2\alpha-1}}{2} - \frac{n^{3\alpha-2}}{6} - \frac{n^{4\alpha-3}}{12} - \frac{n^{5\alpha-4}}{4} + \mathcal{O}\left(n^{6\alpha-5}\right)\right]. \end{aligned} \tag{7}$$

We set

$$\begin{aligned} m &= n(1-\varepsilon), \\ (n-m)/2 &= \frac{n^{\alpha}}{2}, \\ \ln(\ell!) &= -\ell + \ell\ln(\ell) + \frac{\ln(2\pi\ell)}{2} + \frac{1}{12\ell} - \frac{1}{360\ell^3} + \mathcal{O}\left(\frac{1}{\ell^4}\right). \end{aligned}$$

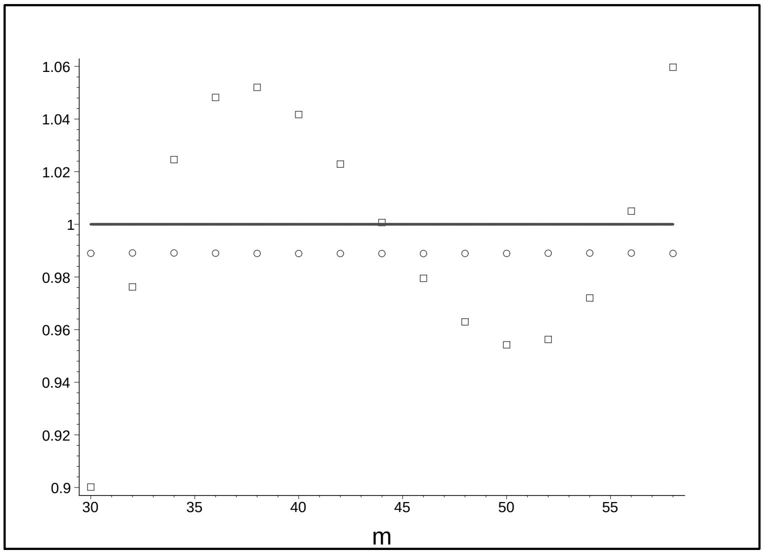


FIGURE 2. Quotient of $I(m,n)/I_n$ and the asymptotic $2\frac{e^{-x^2/2}}{\sqrt{2\pi}\sigma}$ (box) as well as the quotient of $I(m,n)/I_n$ and the asymptotic Equ. (4) (circle)

This leads to

$$\ln(m) = \ln(n) - \varepsilon - \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{3} - \frac{\varepsilon^4}{4} + \mathcal{O}\left(\varepsilon^5\right),$$
$$\ln((n-m)/2) = \alpha \ln(n) - \ln(2),$$
$$\ln(((n-m)/2)!) = -\frac{n^\alpha}{2} + \frac{n^\alpha\ (\alpha\ \ln(n) - \ln(2))}{2} + \ln(\pi\, n^\alpha) + \frac{1}{6n^\alpha} - \frac{1}{45n^{3\alpha}} + \mathcal{O}\left(\frac{1}{n^{4\alpha}}\right)$$

Now, we have

$$\ln(m!) = -n + n^\alpha + n(1-\varepsilon)\ln(m) + \frac{\ln(2\pi m)}{2} + \frac{1}{12m} - \frac{1}{360m^3} + \mathcal{O}\left(\frac{1}{m^4}\right).$$

We use the substitution

$$n\varepsilon^j = n^{j\alpha-(j-1)},$$

we have

$$n(1-\varepsilon)\ln(m) = n\ln(n) - n^{\alpha} - n^{\alpha}\ \ln(n) + \frac{n^{2\alpha-1}}{2} + \frac{n^{3\alpha-2}}{6} + \frac{n^{4\alpha-3}}{12} + \frac{n^{5\alpha-4}}{4} + \mathcal{O}\left(n^{6\alpha-5}\right),$$

$$\ln(m!) = -n + n\ln(n) - n^{\alpha}\ \ln(n) + \frac{n^{2\alpha-1}}{2} + \frac{n^{3\alpha-2}}{6} + \frac{n^{4\alpha-3}}{12} + \frac{n^{5\alpha-4}}{4} + \mathcal{O}\left(n^{6\alpha-5}\right) + \frac{\ln(2\,\pi\,n\,(1-\varepsilon))}{2} + \frac{1}{12n\,(1-\varepsilon)} - \frac{1}{360n^3\,(1-\varepsilon)^3} + \mathcal{O}\left(\frac{1}{n^4}\right),$$

This leads to

$$\begin{aligned}
\frac{I(m,n)}{n!} &= \exp\left[-\frac{n^{\alpha}\ln(2)}{2} - \ln(m!) - \ln(((n-m)/2)!)\right] \\
&= \exp\Bigg[-\frac{n^{\alpha}\ \ln(2)}{2} + n - n\ln(n) + n^{\alpha}\ \ln(n) - \frac{n^{2\alpha-1}}{2} - \frac{n^{3\alpha-2}}{6} - \frac{n^{4\alpha-3}}{12} - \frac{n^{5\alpha-4}}{4} \\
&\quad + \mathcal{O}\left(n^{6\alpha-5}\right) - \frac{\ln(2\,\pi\,n\,(1-\varepsilon))}{2} - \frac{1}{12n\,(1-\varepsilon)} + \frac{1}{260n^3\,(1-\varepsilon)^3} + \mathcal{O}\left(\frac{1}{n^4}\right) \\
&\quad + \frac{n^{\alpha}}{2} - \frac{n^{\alpha}\ (\alpha\ \ln(n) - \ln(2))}{2} - \ln(\pi\,n^{\alpha}) - \frac{1}{6n^{\alpha}} + \frac{1}{45n^{3\alpha}} + \mathcal{O}\left(\frac{1}{n^{4\alpha}}\right)\Bigg] \\
&= \exp(E_3)F_5, \\
F_5 &:= \frac{1}{\sqrt{\pi n^{\alpha}}}\left(1 - \frac{1}{6n^{\alpha}} + \frac{1}{72n^{2\alpha}} + \mathcal{O}\left(\frac{1}{n^{3\alpha}}\right)\right) \times \\
&\quad \times \frac{1}{\sqrt{2\pi n(1-\varepsilon)}}\left[1 - \frac{1}{12\,(1-\varepsilon)\,n} + \frac{1}{288\,(1-\varepsilon)^2\,n^2} + \mathcal{O}\left(\frac{1}{n^3}\right)\right], \\
E_3 &:= \left(\frac{1}{2} - \frac{\alpha\ \ln(n)}{2} + \ln(n)\right)n^{\alpha} + (-\ln(n) + 1)\,n \\
&\quad - \frac{n^{2\alpha-1}}{2} - \frac{n^{3\alpha-2}}{6} - \frac{n^{4\alpha-3}}{12} - \frac{n^{5\alpha-4}}{4} + \mathcal{O}\left(n^{6\alpha-5}\right).
\end{aligned}$$

This leads to Theorem 4.1.

Note that we prefer to keep two separate factors in (7): one in powers of n^{α} and one in powers of n instead of mixing them.

Let us analyze the importance of the terms in E_3. We have two sets: the set A of dominant terms, which stay in the exponent and the set B of small terms, leading to a coefficient of type $(1+\Delta)$, with Δ small. The property of each term may depend on α. In E_3, each term $n^{j\alpha-(j-1)}$ is in A if $j > \frac{1}{1-\alpha}$ and in B otherwise. We finally mention that our non-central range is not sacred: other types of ranges can be analyzed with similar methods.

To check the quality of our asymptotics, we have first chosen $n = 2000$ and a range $\alpha \in (0.125, 0.45)$. This corresponds to the range $m \in (1968, 1998)$.

Figure 3 gives $\ln(I(n,m)/n!)$ (circle) and ln(Equ.(7))(line) with the substitutions $n^\alpha = n - m, \varepsilon = (n-m)/n, \alpha = \ln(n-m)/\ln(n), n^{2\alpha-1} = (n-m)^2/n, \ldots$.

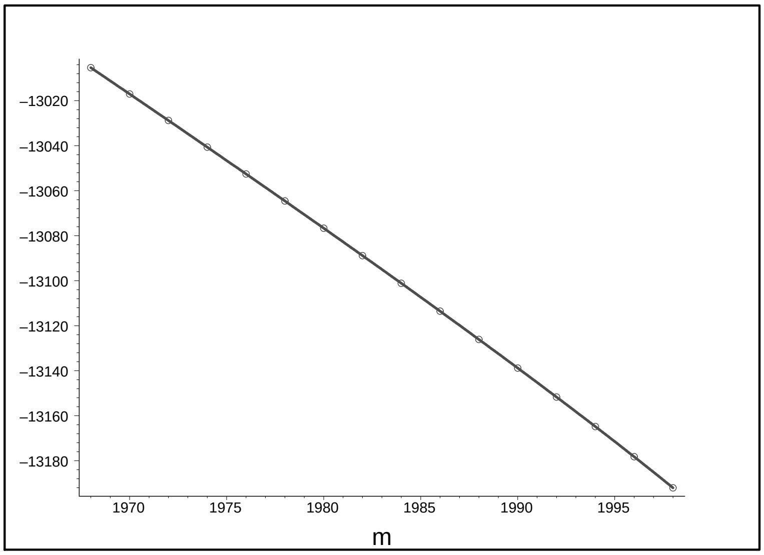


FIGURE 3. $n = 2000$, $\ln(I(n,m)/n!)$ (circle) and ln(Equ.(7)) (line)

Figure 4 gives the quotient of $\ln(I(n,m)/n!)$ and ln(Equ.(7)) (circle).

Another way for checking the quality is to fix α. We choose $\alpha = 1/4$ and $n \in (1950, 2000)$. Of course, we must use an integer value for m: $m = \lfloor n - n^\alpha + 1 \rfloor$. Hence, in (7), we use α as the root of $n - n^\alpha - m = 0$.

Figure 5 gives $\ln(I(n,m)/n!)$ (circle) and ln(Equ.(7)) (line). The relative error is of order 5.10^{-4}.

Figure 6 gives the quotient of $\ln(I(n,m)/n!)$ and ln(Equ.(7)) (circle).

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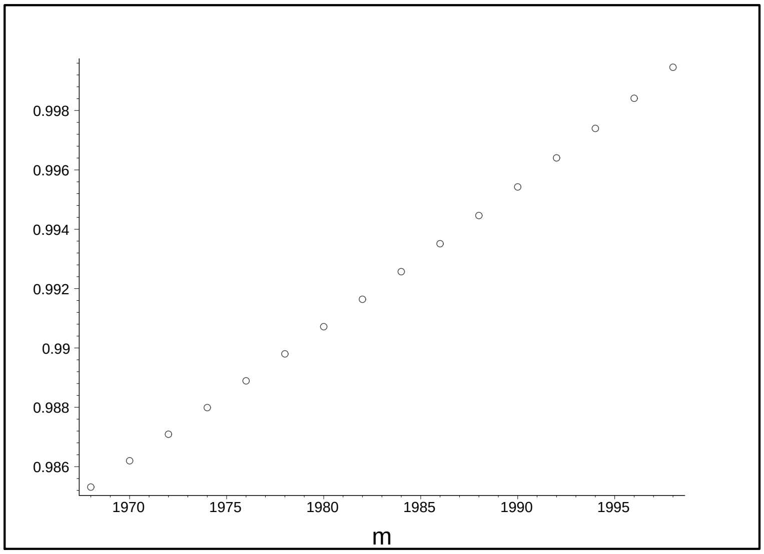

FIGURE 4. $n = 2000$, quotient of $\ln(I(n,m)/n!)$ and ln(Equ.(7)) (circle)

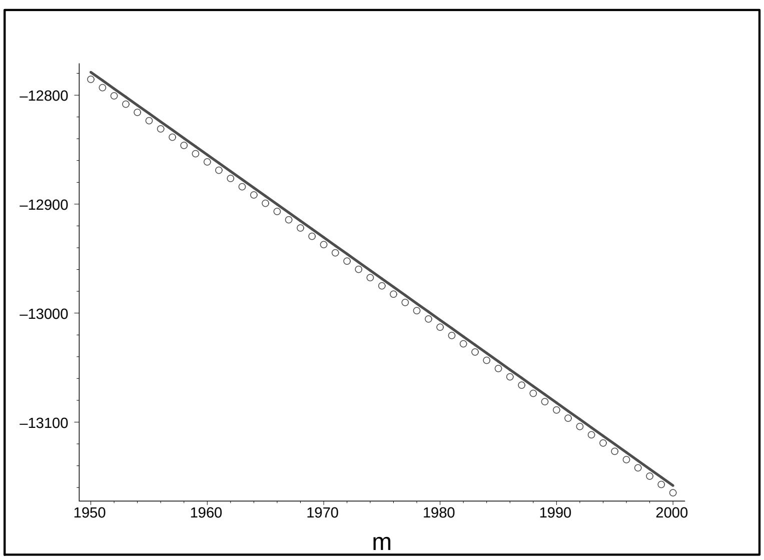

FIGURE 5. $\alpha = 1/4$, $\ln(I(n,m)/n!)$ (circle) and ln(Equ.(7)) (line)

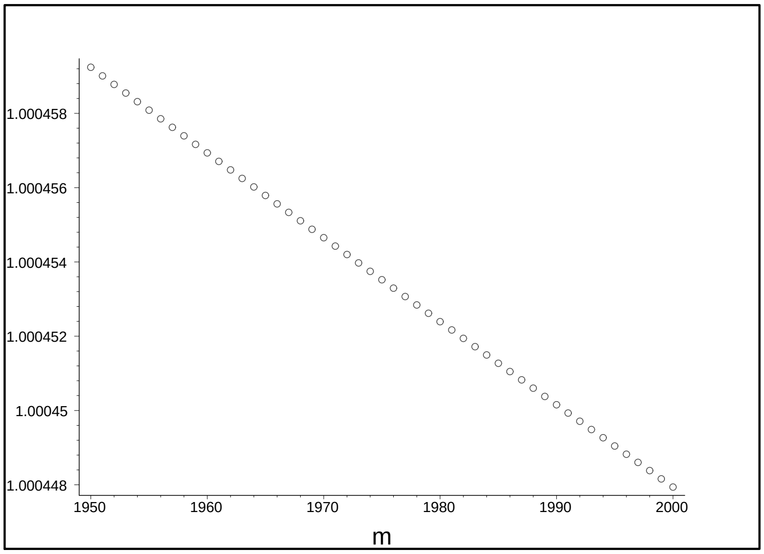

FIGURE 6. $\alpha = 1/4$, quotient of $\ln(I(n,m)/n!)$ and ln(Equ.(7)) (circle)

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