INVERSE-CONJUGATE COMPOSITIONS INTO PARTS OF SIZE AT MOST k

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Abstract. An inverse-conjugate composition of a positive integer m is an ordered partition of m whose conjugate coincides with its reversal. In this paper we consider inverse-conjugate compositions in which the part sizes do not exceed a given integer k. It is proved that the number of such inverse-conjugate compositions of $2n-1$ is equal to $2F_n^{(k-1)}$, where $F_n^{(k)}$ is a Fibonacci k-step number. We also give several connections with other types of compositions, and obtain some analogues of classical combinatorial identities.

1. Introduction

A composition of a positive integer n is a representation of n as a sequence of positive integers called parts which sum to n. For example, the compositions of 4 are: (4), $(3,1)$, $(1,3)$, $(2,2)$, $(2,1,1)$, $(1,2,1)$, $(1,1,2)$, $(1,1,1,1)$. It is known that there are 2^{n-1} unrestricted compositions of n (see for example [2]). The graphical representation of a composition, called a *zig-zag graph*, resembles the partition Ferrers graph except that the first dot of each part is aligned with the last part of its predecessor. For example, the *zig-zag graph* of the composition $(6,3,1,2,2)$ is shown in Figure 1.

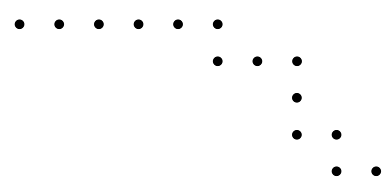

Figure 1

The conjugate of a composition is obtained by reading its graph by columns from left to right. Thus Figure 1 gives the conjugate of the composition $(6,3,1,2,2)$ as $(1,1,1,1,1,2,1,3,2,1)$.

Let C denote a composition of n. A k-composition is a composition with k parts, i.e. $C=(c_1,c_2,...,c_k)$. The conjugate of C is denoted by C' and the *inverse* of C is given by

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$\overline{C} = (c_k, c_{k-1}, ..., c_1)$. C is called *self-inverse* if $C = \overline{C}$, and *inverse-conjugate* if $C' = \overline{C}$. For example, $(2, 1, 3, 1)$ is an inverse-conjugate composition of 7.

Inverse-conjugate compositions have been studied by many researchers (see for example [5, 6, 4, 1]). It is known that these compositions are defined for only odd weights, and that there are 2^n inverse-conjugate compositions of $2n - 1$. (Note: 'weight' means 'positive integer').

We will consider inverse-conjugate compositions with parts of size not exceeding a fixed integer $k > 0$. Heubach-Mansour investigated a more general set of compositions in [3].

Recently Guo [1] imposed parity restrictions on inverse-conjugate compositions and proved that inverse-conjugate compositions using only odd parts exist for odd numbers of the type $4k + 1$ but not $4k + 3$. He found that the number of inverse-conjugate compositions of $4k + 1$ is given by 2^k, $k > 0$.

In 1975, Hoggatt-Bicknell [7] studied ordinary compositions with parts $\leq k$, and obtained the following result (also [2, p. 72]).

Theorem 1.1. *Let $C_k(n)$ be the number of compositions of a positive integer n using only the parts $1, 2, \ldots, k$. Then*

$$C_k(n) = F_{n+1}^{(k)}, \tag{1}$$

where $F_r^{(n)}$ is the Fibonacci n-step number [8, 2] *(see Equation* (2) *below).*

In Section 2, we obtain a general recurrence relation (Theorem 2.3), and an explicit formula for the number of inverse-conjugate compositions of $2n - 1$ with parts $\leq k$ in terms of the Fibonacci n-step number (Corollary 2.5). This is followed by proofs of the analogues of two classical identities inspired by the works of P. A. MacMahon (Theorems 2.7 and 2.10). Then in Section 3 we discuss an interesting combinatorial identity obtained by reversing the viewpoint of the main identity in the previous section (Theorem 3.1).

2. Inverse-conjugate compositions

We first collect few known results that will be used later.

The Fibonacci n-step numbers $F_r^{(n)}$ extend the ordinary Fibonacci numbers and are defined for any positive integer n, by

$$F_r^{(n)} = 0 \text{ for } r \leq 0,\ F_1^{(n)} = F_2^{(n)} = 1,$$

$$F_r^{(n)} = \sum_{i=1}^{n} F_{r-i}^{(n)}, \quad r > 2. \tag{2}$$

Note that the case $n = 1$ gives the sequence of ones, $F_r^{(1)} : 1, 1, 1, \ldots$ while the case $n = 2$ gives the Fibonacci numbers, that is ($F_r^{(2)} = F_r$),

$$F_1 = F_2 = 1,\ F_r = F_{r-1} + F_{r-2},\ r > 2.$$

It is not hard to deduce the generating function

$$\sum_{n=1}^{\infty} F_r^{(n)} x^n = \frac{x}{1-x-x^2-\cdots-x^n}. \tag{3}$$

Let $A = (a_1, a_2, ..., a_i)$ and $B = (b_1, b_2, ..., b_j)$ be compositions. The concatenation of the parts of A and B is defined by $A\big|B = (a_1, a_2, ..., a_i, b_1, b_2,$
$..., b_j)$. In particular for a nonnegative integer c, we have $A\big|(c) = (A, c)$ and $(c)\big|A = (c, A)$.
The *join* of A and B is defined by $A \uplus B = (a_1, a_2, ..., a_i + b_1, b_2, ..., b_j)$.

The following results may be found in [5].

Lemma 2.1. *An inverse-conjugate composition C (or its inverse) has the form:*

$$C = (1^{b_r-1}, b_1, 1^{b_{r-1}-2}, b_2, 1^{b_{r-2}-2}, b_3, \ldots, b_{r-1}, 1^{b_1-2}, b_r),\ b_i \geq 2. \tag{4}$$

Lemma 2.2. *If $C = (c_1, \ldots, c_k)$ is an inverse-conjugate composition of $n = 2k-1 > 1$, or its inverse, then there is an index j such that $c_1 + \cdots + c_j = k-1$ and $c_{j+1} + \cdots + c_k = k$ with $c_{j+1} > 1$.*

Moreover,

$$\overline{(c_1, \ldots, c_j)} = (c_{j+1} - 1, c_{j+2}, \ldots, c_k)' \tag{5}$$

Thus C can be written in the form

$$C = A\big|(1) \uplus B \quad \textit{such that} \quad B' = \overline{A}, \tag{6}$$

where A and B are generally different compositions of $k-1$.

As our first new result we present a fundamental recurrence relation.

Theorem 2.3. *Let $IC_k(N)$ be the number of inverse-conjugate compositions of N into parts of size $\leq k$. Then*

$$IC_k(2n-1) = \sum_{j=1}^{k-1} IC_k(2(n-j)-1), \quad n > k, \tag{7}$$

with $IC_k(1) = 1$ and $IC_k(2t-1) = 2^{t-1}$, $t = 2, \ldots, k$.

Proof. Let $N > 1$ be an odd integer. Then from (4) a composition C enumerated by $IC_k(N)$ has the form
$C = (1^{b_r-1}, b_1, 1^{b_{r-1}-2}, b_2, \ldots, b_{r-1}, 1^{b_1-2}, b_r)$, $2 \leq b_i \leq k$.
We can obtain a composition T enumerated by $IC_k(N - 2b_r + 2)$ by deleting the first $b_r - 1$ copies of 1 and replacing the last part b_r with 1:

$$C \longrightarrow (b_1, 1^{b_{r-1}-2}, b_2, \ldots, b_{r-1}, 1^{b_1-2}, 1) = T.$$

If we begin with $\overline{C}$, we may apply a similar transformation to obtain $\overline{T}$.

Conversely, let T be a composition enumerated by $IC_k(N-2b+2)$, $b \geq 2$. Then if the first part of T is 1, we create a new first part b by adding the 1 to $b-1$, and insert $b-1$ copies of 1 after the last big part. The resulting composition has first part > 1 and is clearly enumerated by $IC_k(N)$. If the first part of T is greater than 1, we insert $b-1$ copies of 1 on the left of the first part, and create a new last part by adding the previous last part 1 to $b-1$. This produces a composition enumerated by $IC_k(N)$ having first part 1.

The range of b, that is, $2 \leq b \leq k$, implies that

$$\begin{aligned} IC_k(N) = {} & IC_k(N-2) + IC_k(N-4) + \cdots + IC_k(N-2b+2) + \cdots \\ & + IC_k(N-2(k-1)), \end{aligned}$$

which gives the recurrence (7) on putting $N = 2n-1$.

The initial values follow from the fact that every inverse-conjugate composition of $2t-1$, $1 \leq t \leq k$, has all parts $\leq k$. Since the number of inverse-conjugate compositions of $2t-1$ is known to be 2^{t-1}, we have $IC_k(2t-1) = 2^{t-1}$. □

When $k = 2$ in Theorem 2.3, we obtain $IC_2(1) = 1$ and $IC_2(2n-1)$, $n > 1$.
When $k = 3$, the theorem reduces to

$$IC_3(2n+1) = IC_3(2n-1) + IC_3(2n-3), \; n > 2.$$

with $IC_3(1) = 1, \; IC_3(3) = 2, \; IC_3(5) = 4$.

We give some values of $IC_k(2n-1)$ for small n in Table 1.
Observe the interesting relation $IC_k(2k+1) - IC_k(2k-1) = 2^{k-1} - 2$, $k \geq 3$.

$2n-1$	1	3	5	7	9	11	13	15	17	19	21	23
$IC_1(2n-1)$	1	1	1	1	1	1	1	1	1	1	1	1
$IC_2(2n-1)$	1	2	2	2	2	2	2	2	2	2	2	2
$IC_3(2n-1)$	1	2	4	6	10	16	26	42	68	110	178	288
$IC_4(2n-1)$	1	2	4	8	14	26	48	88	162	298	548	1008
$IC_5(2n-1)$	1	2	4	8	16	30	58	112	216	416	802	1546
$IC_6(2n-1)$	1	2	4	8	16	32	62	122	240	472	928	1824
$IC_7(2n-1)$	1	2	4	8	16	32	64	126	250	496	984	1952
$IC_8(2n-1)$	1	2	4	8	16	32	64	128	254	506	1008	2008
$IC_9(2n-1)$	1	2	4	8	16	32	64	128	256	510	1018	2032

TABLE 1. Some values of $IC_k(2n-1)$

The following result may be deduced from Theorem 2.3.

Corollary 2.4. *The generating function for the number of inverse-conjugate compositions of $2n-1$ with parts of size $\leq k$ is given by*

$$\sum_{n=1}^{\infty} IC_k(2n-1)x^n = \frac{x(1+x+x^2+\cdots+x^{k-1})}{1-x-x^2-\cdots-x^{k-1}}.$$

From Corollary 2.4 and Equation (3) we obtain the following enumeration result.

Corollary 2.5. *We have*

$$IC_k(2n-1) = 2F_n^{(k-1)}, \quad n \geq k-1. \tag{8}$$

In particular the following simple formulas hold:

$IC_3(2n-1) = 2F_n, \quad n > 1.$

$IC_4(2n-1) = 2T_n, \quad n > 2,$

where $T_n = F_n^{(3)}$ is a Tribonacci number.

From Theorem 1.1 and (8) we easily obtain the following relation.

Corollary 2.6. *We have*

$$IC_{k+1}(2n+1) = 2C_k(n), \quad n \geq 1.$$

The fact that inverse-conjugate compositions of $2n-1$ are equinumerous with unrestricted compositions of n is implicit in MacMahon's classic text [4]. Here we obtain an analogous relation between certain compositions of n with parts $\leq k$ and inverse-conjugate compositions of $2n-1$.

Theorem 2.7. *Let $CC_k(n)$ be the number of compositions C of n when the parts of C and the parts of C' are $\leq k$. Then*

$$IC_k(2n-1) = CC_k(n),\ n > 1.$$

Proof. Using Lemma 2.2, it may be deduced that every inverse-conjugate composition $C = (c_1, \ldots, c_n)$ of $2n-1 > 1$ having parts $\leq k$ fulfills either of the following pairs of properties (see also [6]):

(1a) $c_1 + \cdots + c_j = n-1$ and $c_{j+1} + \cdots + c_n = n$ with $c_{j+1} > 1$, and

(1b) $\overline{(c_1, \ldots, c_j, 1)}' = (c_{j+1}, \ldots, c_n)$.

(2a) $c_1 + \cdots + c_j = n$ and $c_{j+1} + \cdots + c_n = n-1$ with $c_j > 1$, and

(2b) $\overline{(c_1, \ldots, c_{j-1}, c_j - 1)}' = (c_{j+1}, \ldots, c_n)$.

We describe a bijection θ between the sets of compositions enumerated by $IC_k(2n-1)$ and $CC_k(n)$.

For each inverse-conjugate composition C satisfying $(1a)$, define $B = \theta(C)$ to be the composition of n given by $(c_1, c_2, \cdots, c_j, 1)$, and for each satisfying $(2a)$ define $B = \theta(C)$ to be the composition of n given by $(c_1, c_2, \cdots, c_j)$, which is already a composition of n with parts $\leq k$.

Since $B' = \overline{(c_{j+1}, c_{j+2}, \cdots, c_n)}$ or $B' = \overline{(1, c_{j+1}, c_{j+2}, \cdots, c_n)}$, we see that B' also has parts $\leq k$.

Conversely, $\theta^{-1}(B)$ is obtained for any composition B enumerated by $CC_k(n)$ by extending it to a unique inverse-conjugate composition of $2n-1$ using property $(1b)$ or $(2b)$ depending on whether the last part of B is 1 or > 1. □

Example 2.8. *Let $n = 4$ and $k = 3$. Then the corresponding relations between the relevant compositions of 7 and 4 are as follows.*

$$(2,1,3,1) \longleftrightarrow (2,1,1),\ (1,3,1,2) \longleftrightarrow (1,3),$$
$$(3,2,1,1) \longleftrightarrow (3,1),\ (1,1,2,3) \longleftrightarrow (1,1,2),$$
$$(2,2,2,1) \longleftrightarrow (2,2),\ (1,2,2,2) \longleftrightarrow (1,2,1).$$

We recall a classical combinatorial identity of MacMahon [4]. Since the algebraic proof plays an important role in the proof of the next theorem we summarize it below.

Theorem 2.9. *(MacMahon) The number of inverse-conjugate compositions of an odd integer $N > 1$ equals the number of compositions of N which are self-inverse.*

Proof. This version of the proof first appeared in [5]. Let $N = 2n-1,\ n > 1$ and denote the set of objects enumerated by a function $f(n)$ by $\{f(n)\}$. Let the quantities in the theorem be denoted respectively by $IC(2n-1)$ and $SI(2n-1)$, so that $\{IC(2n-1)\}$ and $\{SI(2n-1)\}$ represent the enumerated sets.

We describe a bijection $\alpha : \{IC(2n-1)\} \to \{SI(2n-1)\}$ using Lemma 2.2 as follows. If $C \in \{IC(2n-1)\}$, then one can write $C = A|(1) \uplus B$ or $C = A \uplus (1)|B$ for certain compositions of $n-1$ satisfying $B' = \overline{A}$.

In the first case we use (5) to get $\alpha(C) = A|[(1) \uplus B]'$, which is a member of $\{SI(2n-1)\}$ of the type $A|(1)|\overline{A}$.

The second case, $C = A \uplus (1)|B$, implies that there is a part $v \geq 2$ such that $C = X|(v)|B$, where the weight of X is less than $n-1$ and B is a composition of $n-1$. Split off 1 from v and write $C = (X, v-1) \uplus (1, B)$. Now set $\alpha(C) = (X, v-1) \uplus (1, B)'$, which is a member of $\{SI(2n-1)\}$ of the type $Y|(d)|\overline{Y}$, with d an odd integer > 1

Conversely, if $T \in \{SI(2n-1)\}$, then $T = A|(d)|\overline{A}$, where A is a composition with weight $< n$.

If $d = 1$, then $T = A|(1)|\overline{A} = A|(1, \overline{A})$. Now set $\alpha^{-1}(T) = A|(1, \overline{A})'$ which is a member of $\{IC(2n-1)\}$ of type $A|(1) \uplus B$.

If $d > 1$, then we write $T = (A, \frac{d-1}{2}) \uplus (\frac{d+1}{2}, \overline{A})$. Now set $\alpha^{-1}(T) = (A, \frac{d-1}{2}) \uplus (\frac{d+1}{2}, \overline{A})'$ which is a member of $\{IC(2n-1)\}$ of type $A \uplus (1)|B$.

□

The next result asserts an analogous identity between the number of inverse-conjugate compositions and the number of self-inverse compositions with parts $\leq k$.

Theorem 2.10. *Let $SI_k(N)$ be the number of self-inverse compositions of N when only parts of size $\leq k$ are allowed in both a composition and its conjugate. Then*

$$IC_k(2n-1) = SI_k(2n-1) + SI_k(2n-k), \quad n > 1. \tag{9}$$

Proof. We invoke the bijection α used in the proof of Theorem 2.9. Let α_k be the restriction of α to $\{IC_k(2n-1)\}$. Then we notice that a self-inverse composition $T \in \alpha_k(\{IC_k(2n-1)\}) = Im(\alpha_k)$ may have a middle part $> k$ or contain at least $k-1$

copies of 1 in the center. This implies that such $T \notin \{SI_k(2n-1)\}$. (Note that if T contains $\geq k-1$ copies of 1 in the center, then T' has a middle part $> k$).

We remark that the middle part of $T = B|(d)|\overline{B}$ is an odd integer satisfying $1 \leq d \leq 2k-1$. To see this we refer to the definition of α.
The case $d = 1$ is clearly accounted for by pre-images of the form $A|(1) \uplus B$.
But when $d > 1$, the pre-image of T has the form $(a_1, \ldots, a_j) \uplus (1)|\overline{(a_1, \ldots, a_j)'}$, where $1 \leq a_j \leq k-1$. Since $T = (a_1, \ldots, a_j) \uplus (1, \overline{(a_1, \ldots, a_j)'}\ )'$ we deduce that $d = 2a_j + 1$. Thus $1 \leq a_j \leq k-1$ translates to $3 \leq d \leq 2k-1$ as required.

Thus the image set $Im(\alpha_k)$ splits into two disjoint subsets, namely
(i) $\{SI_k(2n-1)\}$, the set of compositions enumerated by $SI_k(2n-1)$;
(ii) $V(2n-1) := Im(\alpha_k) \setminus \{SI_k(2n-1)\}$.

It remains to obtain the cardinality of $V(2n-1)$. We claim that
$|V(2n-1)| = SI_k(2n-k)$.

Note that a composition $T \in V(2n-1)$ has the following property:
(iia) T has a middle part $> k$ and all other parts $\leq k$. There are at most $k-2$ copies of 1 between two consecutive parts > 1 elsewhere (with possible exception of initial or final string of 1's which may have up to $k-1$ copies).

or

(iib) T contains at least $k-1$ copies of 1 at the center and all other parts $\leq k$. There are at most $k-2$ copies of 1 between two consecutive parts > 1 elsewhere (with possible exception of initial or final string of 1's which may have up to $k-1$ copies).

Define a bijection

$$\beta : Im(\alpha_k) \longrightarrow \{SI_k(2n-1)\} \cup \{SI_k(2n-k)\}.$$

If $T \in \{SI_k(2n-1)\} \subset Im(\alpha_k)$, then $\beta(T) = T$.
But if $T \in V(2n-1)$, then the image is obtained as follows.
If property (iia) holds, then $T = (c_1, \ldots, c_s, d, c_s, \ldots, c_1)$, $d \geq k+1$, and

$$\beta(T) = (c_1, \ldots, c_s, d-k+1, c_s, \ldots, c_1).$$

If property (iib) holds, then $T = (c_1, \ldots, c_s, 1^f, c_s, \ldots, c_1)$, $c_s > 1$, $f \geq k-1$, and

$$\beta(T) = (c_1, \ldots, c_s, 1^{f-k+1}, c_s, \ldots, c_1).$$

The action of β on T is to deduct $k-1$ from a middle part $d > k$ or remove $k-1$ copies of 1 from the center, where necessary. So $\beta(T) \in \{IC_k(2n-k)\}$ whenever $T \in V(2n-1)$. This mapping is clearly reversible since $d-k+1 \geq 3$ or $f-k+1 \geq 1$ when k is odd, and $d-k+1 \geq 2$ or $f-k+1 \geq 0$ when k is even. Thus β is indeed a bijection. Hence $|V(2n-1)| = IC_k(2n-k)$.

Lastly the identity (9) is established via a bijection consisting of the composition of the two bijections α_k and β, that is

$$\beta\alpha_k : \{IC_k(2n-1)\} \longrightarrow \{SI_k(2n-1)\} \cup \{SI_k(2n-k)\}.$$

□

Example 2.11. *The full correspondence $\beta\alpha_k$ is illustrated in Table 2 for $2n-1=7$ when $k=3$, and $k=4$.*

$\{IC_3(7)\}$	$\xrightarrow{\alpha_3}$	$Im(\alpha_3)$	$\xrightarrow{\beta}$	$\{SI_3(7)\}$	$\{SI_3(5)\}$
(1,1,2,3)	$\mapsto$	(1,1,3,1,1)	$\mapsto$	(1,1,3,1,1)	
(1,2,2,2)	$\mapsto$	(1,2,1,2,1)	$\mapsto$	(1,2,1,2,1)	
(1,3,1,2)	$\mapsto$	(1,5,1)	$\mapsto$		(1,3,1)
(2,1,3,1)	$\mapsto$	(2,1,1,1,2)	$\mapsto$		(2,1,2)
(2,2,2,1)	$\mapsto$	(2,3,2)	$\mapsto$	(2,3,2)	
(3,2,1,1)	$\mapsto$	(3,1,3)	$\mapsto$	(3,1,3)	

$\{IC_4(7)\}$	$\xrightarrow{\alpha_4}$	$Im(\alpha_4)$	$\xrightarrow{\beta}$	$\{SI_4(7)\}$	$\{SI_4(4)\}$
(1,1,1,4)	$\mapsto$	(1,1,1,1,1,1,1)	$\mapsto$		(1,1,1,1)
(1,1,2,3)	$\mapsto$	(1,1,3,1,1)	$\mapsto$	(1,1,3,1,1)	
(1,2,2,2)	$\mapsto$	(1,2,1,2,1)	$\mapsto$	(1,2,1,2,1)	
(1,3,1,2)	$\mapsto$	(1,5,1)	$\mapsto$		(1,2,1)
(2,1,3,1)	$\mapsto$	(2,1,1,1,2)	$\mapsto$		(2,2)
(2,2,2,1)	$\mapsto$	(2,3,2)	$\mapsto$	(2,3,2)	
(3,2,1,1)	$\mapsto$	(3,1,3)	$\mapsto$	(3,1,3)	
(4,1,1,1)	$\mapsto$	(7)	$\mapsto$		(4)

TABLE 2. The Bijection $\beta\alpha_k$ for $2n-1=7$ and $k=3,4$.

3. A DUAL IDENTITY

In this section we prove a dual identity to (9) that expresses $SI_k(N)$ in terms of the $IC_k(2M-1)$, $2M-1<N$.

Theorem 3.1. *Let n be a positive integer. Then,*

$$SI_k(2n+k-2) = \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} IC_k(2n-3+2j),\ k \geq 2. \tag{10}$$

In particular we obtain

$SI_2(2n) = IC_2(2n-1)$;

$SI_4(2n+2) = IC_4(2n-1) + IC_4(2n+1)$;

$SI_6(2n+4) = IC_6(2n-1) + IC_6(2n+1) + IC_6(2n+3)$.

Also,

$SI_3(2n+1) = IC_3(2n-1)$;

$SI_5(2n+3) = IC_5(2n-1) + IC_5(2n+1)$;

$SI_7(2n+5) = IC_7(2n-1) + IC_7(2n+1) + IC_7(2n+3)$.

The following result follows immediately from Theorem 3.1 and Corollary 2.5.

Corollary 3.2. *For all integers $n > 0$, we have*

$$SI_k(2n+k-2) = 2\sum_{j=0}^{\lfloor\frac{k-2}{2}\rfloor} F_{n+j}^{(k-1)},\ k \geq 2. \tag{11}$$

We give two proofs of the theorem below, one algebraic, one combinatorial.

3.1. **An algebraic proof of Theorem 3.1.** We show that the stated identity is consistent with the recurrence (7) and the identity (9).
First, let k be an odd integer. Then using (10) we obtain

$$\begin{aligned}
&SI_k(2n+k-2) + SI_k(2n+2k-3)\\
&\quad= \sum_{j=1}^{\frac{k-1}{2}} IC_k(2n-3+2j) + \sum_{j=1}^{\frac{k-1}{2}} IC_k(2n+k-4+2j)\\
&\quad= (IC_k(2n-1) + IC_k(2n+1) + \cdots + IC_k(2n+k-4))\\
&\qquad+ (IC_k(2n+k-2) + IC_k(2n+k) + \cdots + IC_k(2n+2k-5))\\
&\quad= IC_k(2n+2k-3),
\end{aligned}$$

where the last equality follows from the recurrence (7). That is, we have shown that

$$SI_k(2n+k-2) + SI_k(2n+2k-3) = IC_k(2n+2k-3).$$

But this identity is precisely a restatement of (9) with the weight $2n+2k-3$ in place of $2n-1$.

Note that the foregoing proof cannot be used if k is an even integer because $IC_k(2n+k-2+2j)$ is not defined as $2n+k-2+2j$ is even for all j. However, we observe that for any positive integer t, the statement

$$SI_{2t}(2n+2t-2) = \sum_{j=1}^{t} IC_{2t}(2n-3+2j)$$

implies and is implied by

$$SI_{2t+1}(2n+2t-1) = \sum_{j=1}^{t} IC_{2t+1}(2n-3+2j).$$

For example, see the specific expansions of $SI_5(2n+3)$ and $SI_4(2n+2)$, or $SI_7(2n+5)$ and $SI_6(2n+4)$, given earlier.

By this correspondence we conclude that the above proof for an odd integer k implies a proof for the even integer $m = k-1$. Since k is arbitrary the result follows. □

3.2. **A combinatorial proof of Theorem 3.1.** We will use the properties of the map α_k and some notations defined in the proof of Theorem 2.10. Consider the map

$$\psi: \bigcup_{j=1}^{\lfloor k/2 \rfloor} \{IC_k(2n-3+2j)\} \longrightarrow \{SI_k(2n+k-2)\}. \tag{12}$$

We will prove that ψ is a bijection consisting of the composition of two bijections α_k and ρ, where the latter is described below.

If $T \in Im(\alpha_k)$, then $T = B|(d)|\overline{B}$, where d is odd with $1 \le d \le 2k-1$. Thus T has two forms corresponding to the form of the middle part:
(a) T has a middle part d, $2 \le d \le 2k-1$ and all other parts $\le k$. There are at most $k-2$ copies of 1 between two consecutive parts (with possible exception of initial or final string of 1's which may have up to $k-1$ copies).
(b) T contains v copies of 1 at the center, $1 \le v \le 2k-3$, with all parts $\le k$. There are at most $k-2$ copies of 1 between two consecutive parts > 1 elsewhere (with possible exception of initial or final string of 1's which may have up to $k-1$ copies).

We define a map ρ on $Im(\alpha_k)$ to change the middle terms of T so that the weight of $\rho(T)$ is $2n+k-2$ and the parts of $\rho(T)$ and $\rho(T)'$ are $\le k$. Note that since the weight of T is less than $2n+k-2$, we have $T \ne \rho(T)$. So ρ has no fixed points.

$$\rho: Im(\alpha_k) \longrightarrow \{SI_k(2n+k-2)\}. \tag{13}$$

Consider $T \in \alpha_k(\{IC_k(2n-3+2j)\})$, $1 \le j \le \lfloor \frac{k}{2} \rfloor$. Define $\rho(T)$ to have weight $2n-3+2j+(k-2j+1) = 2n+k-2$. Notice that the weight difference $k-2j+1$ has opposite parity to k.

If property (a) holds, let $T = (c_1, \dots, c_s, d, c_s, \dots, c_1)$, $2 \le d \le 2k-1$. Then

$$\rho(T) = (c_1, \dots, c_s, \tfrac{d+1}{2}, 1^{k-2j}, \tfrac{d+1}{2}, c_s, \dots, c_1). \tag{14}$$

If property (b) holds, let $T = (c_1, \dots, c_s, 1^v, c_s, \dots, c_1)$, $c_s > 1$, $1 \le v \le 2k-3$. Then

$$\rho(T) = (c_1, \dots, c_s, 1^{(v-1)/2}, k-2j+2, 1^{(v-1)/2}, c_s, \dots, c_1). \tag{15}$$

We see that $\rho(T) \in \{SI_k(2n+k-2)\}$ in (14) and (15); the $k-2j$ copies of 1 in (14) transform into the middle value $k-2j+2 \le k$ by conjugation, and vice-versa. A similar property in the opposite direction is inherited from pre-images T as indicated in the mutually conjugate types given under properties (a) and (b) above.

We show that ρ is one-to-one by describing ρ^{-1}. Any $D \in \{SI_k(2n+k-2)\}$ fulfills either of the following properties.
(c) D has a middle part b, $2 \le b \le k$ and all other parts $\le k$. There are at most $k-2$ copies of 1 between two consecutive parts (with possible exception of initial or final string of 1's which may have up to $k-1$ copies).

or
(d) D contains s copies of 1 at the center, $0 \le s \le k-2$, with all parts $\le k$. There are at most $k-2$ copies of 1 between two consecutive parts > 1 elsewhere (with possible exception of initial or final string of 1's which may have up to $k-1$ copies).

So if property (c) holds, let $D = (c_1, \ldots, c_t, 1^u, b, 1^u, c_t, \ldots, c_1)$, $0 \leq u \leq k-2$, $2 \leq b \leq k$. Then b is replaced with 1 and

$$\rho^{-1}(D) = (c_1, \ldots, c_t, 1^{2u+1}, c_t, \ldots, c_1). \tag{16}$$

If property (d) holds, let $D = (c_1, \ldots, c_t, b, 1^s, b, c_t, \ldots, c_1)$, $2 \leq b \leq k$, $0 \leq s \leq k-2$. Then 1^s is replaced by -1 and

$$\rho^{-1}(D) = (c_1, \ldots, c_t, 2b-1, c_t, \ldots, c_1). \tag{17}$$

Observe that $0 \leq u \leq k-2 \implies 1 \leq 2u+1 \leq 2k-3$, and $2 \leq b \leq k \implies 3 \leq 2b-1 \leq 2k-1$. That is, each $\rho^{-1}(D)$ in (16) and (17) possesses property (b) and (a) respectively. Thus uniquely $\rho^{-1}(D) \in Im(\alpha_k)$. Hence ρ is a bijection. Since α_k is a bijection, it follows from (12) that ψ is a bijection given by $\psi = \rho\alpha_k$.

This completes the proof of Theorem 3.1. □

Example 3.3. *We give an illustration of the bijections in the proof of Theorem 3.1. Let $k = 4$ and $n = 3$ so that* (10) *becomes $SI_4(8) = IC_4(5) + IC_4(7)$, the summands correspond to $j = 1$ and $j = 2$, respectively, in the assignments* (14) *and* (15)*. The details are shown in Table 3.*

$\{IC_4(5)\} \cup \{IC_4(7)\}$	$\alpha_4(\{IC_4(5)\} \cup \{IC_4(7)\})$	$\xrightarrow{\rho}$	$\{SI_4(8)\}$
(1,1,3)	(1,1,1,1,1)	$\mapsto$	(1,1,4,1,1)
(1,2,2)	(1,3,1)	$\mapsto$	(1,2,1,1,2,1)
(2,2,1)	(2,1,2)	$\mapsto$	(2,4,2)
(3,1,1)	(5)	$\mapsto$	(3,1,1,3)
(1,1,1,4)	(1,1,1,1,1,1,1)	$\mapsto$	(1,1,1,2,1,1,1)
(1,1,2,3)	(1,1,3,1,1)	$\mapsto$	(1,1,2,2,1,1)
(1,2,2,2)	(1,2,1,2,1)	$\mapsto$	(1,2,2,2,1)
(1,3,1,2)	(1,5,1)	$\mapsto$	(1,3,3,1)
(2,1,3,1)	(2,1,1,1,2)	$\mapsto$	(2,1,2,1,2)
(2,2,2,1)	(2,3,2)	$\mapsto$	(2,2,2,2)
(3,2,1,1)	(3,1,3)	$\mapsto$	(3,2,3)
(4,1,1,1)	(7)	$\mapsto$	(4,4)

TABLE 3. The Bijection $\psi = \rho\alpha_4 : \{IC_4(5)\} \cup \{IC_4(7)\} \to \{SI_4(8)\}$.

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