MULTISECTION METHOD FOR APÉRY–LIKE SERIES

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ABSTRACT. Applying the multisection series method to the MacLaurin series expansion of arcsin-function, we transform the Apéry–like series involving the central binomial coefficients into systems of linear equations. By resolving the linear systems (for example, by *Mathematica*), we establish numerous remarkable infinite series formulae for π and logarithm functions, including several recent results due to Almkvist *et al.* (2003) and Zheng (2008).

1. Introduction and Preliminaries

The central binomial coefficient is quite interesting in combinatorics and number theory. It has been involved in the following important topics on infinite series:

- In 1979, Apéry [2] (cf. [9] also) proved the irrationality of $\zeta(2)$ and $\zeta(3)$ by making use of the following infinite series identities:

$$\zeta(2) = 3\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}} \qquad \text{and} \qquad \zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

- In 1985, by employing the two MacLaurin series

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty}\binom{2n}{n}x^n \quad \text{and} \quad \frac{2x\arcsin x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty}\frac{(2x)^{2n}}{n\binom{2n}{n}} \tag{1}$$

  Lehmer [7] examined systematically the following infinite series

$$\sum_{n=0}^{\infty} A_n\binom{2n}{n} \qquad \text{and} \qquad \sum_{n=0}^{\infty}\frac{B_n}{\binom{2n}{n}}$$

  where the A_n and B_n are simple functions of n (for example, polynomials and exponential functions). More identities were derived by Elsner [6] and Zucker [11] in the same year.
- Recently, Almkvist *et al.* [1] and Zheng [10] found several expressions for π containing the multiple central binomial coefficient $\binom{2mn}{mn}$ and a polynomial $P_m(n)$ of

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degree m in n:

$$\sum_{n=0}^{\infty}\frac{P_m(n)}{\binom{2mn}{mn}}\alpha^n \quad \text{where} \quad \alpha\in\mathbb{R} \quad \text{and} \quad m\in\mathbb{N}.$$

Motivated by these works, we shall investigate the following more general expressions of Apéry–like series for π and logarithm function

$$\Omega_m^{\gamma}(x):=\sum_{n=0}^{\infty}\frac{\Lambda_m(n)}{\binom{2mn+2\gamma}{mn+\gamma}}x^n \quad \text{with} \quad \Lambda_m(n):=\sum_{k=0}^{m}\lambda_k\langle 2mn+2\gamma\rangle_k \tag{2}$$

where the falling factorial is defined, for an indeterminate x, by

$$\langle x\rangle_0=1 \quad \text{and} \quad \langle x\rangle_n=\prod_{k=0}^{n-1}(x-k) \quad \text{where} \quad n\in\mathbb{N}.$$

Several interesting multisection series for $\Omega_m^{\gamma}(x)$ are expressed in terms of π and logarithm functions, that not only recover, in particular for $\gamma=0$, the results obtained previously by Almkvist *et al.* [1] and Zheng [10], but also discover numerous remarkable new infinite identities involving the central binomial coefficients $\binom{2mn+2\gamma}{mn+\gamma}$ covering all the residue classes of γ modulo m. As exemplification, we anticipate below a few formulae (see Examples 1 and 3) that do not seem to have appeared previously in the literature:

$$\frac{4\pi}{9\sqrt{3}} = \sum_{n=0}^{\infty}\frac{10n-1}{\binom{4n}{2n}}, \tag{3}$$

$$\frac{2\pi}{27\sqrt{3}} = \sum_{n=0}^{\infty}\frac{10n^2-1}{\binom{4n+2}{2n+1}}; \tag{4}$$

$$\frac{9\pi}{4} = \sum_{n=0}^{\infty}\frac{-2+53n-24n^2}{\binom{4n}{2n}}4^n, \tag{5}$$

$$\frac{3\pi}{16} = \sum_{n=0}^{\infty}\frac{-8-11n+15n^2}{\binom{4n+2}{2n+1}}4^n. \tag{6}$$

As preliminaries, we illustrate below higher order derivatives of $\arcsin x$, multisection series method and special values for the function $\mathrm{hyp}(y)$.

1.1. **Higher order derivatives of $\arcsin x$.** For the reciprocals of central binomial coefficients, there exists the following interesting generating function [7, Equation 15]

$$h(x):=\mathcal{D}_x\frac{\arcsin x}{\sqrt{1-x^2}}=\frac{1}{1-x^2}+\frac{x\arcsin x}{\sqrt{(1-x^2)^3}}=\sum_{n=0}^{\infty}\frac{(2x)^{2n}}{\binom{2n}{n}} \text{ where } |x|<1. \tag{7}$$

Several variants of this expansion and applications to infinite series involving reciprocals of central binomial coefficients can be found in [3,4,6,8,11].

For the higher order derivatives defined by

$$\mathcal{D}_x^k h(ex) = e^k \mathcal{D}_y^{k+1} \frac{\arcsin y}{\sqrt{1-y^2}}\Big|_{y=ex}$$

one can verify by the induction principle that there exist $\{P_k, Q_k\}$ polynomials

$$\mathcal{D}_x^k h(ex) = \frac{e^k P_k(ex)}{(1-e^2x^2)^{k+1}} + \frac{e^k Q_k(ex)}{(1-e^2x^2)^{k+1}} \frac{\arcsin(ex)}{\sqrt{1-e^2x^2}} \tag{8}$$

that satisfy the following recurrence relations

$$P_{k+1}(y) = (1-y^2)P_k'(y) + (2k+2)yP_k(y) + Q_k(y), \tag{9}$$
$$Q_{k+1}(y) = (1-y^2)Q_k'(y) + (2k+3)yQ_k(y). \tag{10}$$

We display the first few terms of $P_k(y)$ polynomials as follows:

$$\begin{aligned}
P_0(y) &= 1,\\
P_1(y) &= 3y,\\
P_2(y) &= 4 + 11y^2,\\
P_3(y) &= 5y(11 + 10y^2),\\
P_4(y) &= 64 + 607y^2 + 274y^4,\\
P_5(y) &= 63y(33 + 104y^2 + 28y^4),\\
P_6(y) &= 9(256 + 5175y^2 + 8132y^4 + 1452y^6),\\
P_7(y) &= 9y(15159 + 101978y^2 + 95912y^4 + 12176y^6),\\
P_8(y) &= 9(16384 + 572519y^2 + 1922874y^4 + 1202984y^6 + 114064y^8),\\
P_9(y) &= 495y(27985 + 325640y^2 + 655332y^4 + 292256y^6 + 21472y^8).
\end{aligned}$$

Similarly, the first few terms of $Q_k(y)$ polynomials are produced as

$$\begin{aligned}
Q_0(y) &= y,\\
Q_1(y) &= 1 + 2y^2,\\
Q_2(y) &= 3y(3 + 2y^2),\\
Q_3(y) &= 3(3 + 24y^2 + 8y^4),\\
Q_4(y) &= 15y(15 + 40y^2 + 8y^4),\\
Q_5(y) &= 45(5 + 90y^2 + 120y^4 + 16y^6),\\
Q_6(y) &= 315y(35 + 210y^2 + 168y^4 + 16y^6),\\
Q_7(y) &= 315(35 + 1120y^2 + 3360y^4 + 1792y^6 + 128y^8),\\
Q_8(y) &= 2835y(315 + 3360y^2 + 6048y^4 + 2304y^6 + 128y^8),\\
Q_9(y) &= 14175(63 + 3150y^2 + 16800y^4 + 20160y^6 + 5760y^8 + 256y^{10}).
\end{aligned}$$

Furthermore, observe that the polynomial $P_k(y)$ has the same parity as its index k. Instead, $Q_k(y)$ has the opposite parity to its index k.

1.2. **Multisection series.** In classical combinatorics, there is the following useful expression of multisection series for formal power series (cf. Comtet [5, Page 84] for example). Let $f(x)$ be a formal power series with complex coefficients defined by $f(x) := \sum_{n=0}^{\infty} A_n x^n$. Then for a natural number m and an integer γ subject to $0 \le \gamma < m$, the following formula holds:

$$\sum_{n=0}^{\infty} A_{mn+\gamma} x^{mn+\gamma} = \frac{1}{m} \sum_{k=1}^{m} \omega_m^{-k\gamma} f(x\omega_m^k) \tag{11}$$

where $\omega_m := \exp(2\pi i/m)$ is the m-th root of unity. It is not hard to check that for an even function $g(x) := \sum_{n=0}^{\infty} B_n x^{2n}$, there holds similarly the formula

$$\sum_{n=0}^{\infty} B_{mn+\gamma} x^{2mn+2\gamma} = \frac{1}{m} \sum_{k=1}^{m} \omega_{2m}^{-2k\gamma} g(x\omega_{2m}^k) \quad \text{for} \quad m \in \mathbb{N}. \tag{12}$$

In fact, writing the right hand side as a double series and then interchanging the summation order, we have

$$\begin{aligned} \frac{1}{m} \sum_{k=1}^{m} \omega_{2m}^{-2k\gamma} g(x\omega_{2m}^k) &= \frac{1}{m} \sum_{k=1}^{m} \sum_{n=0}^{\infty} B_n x^{2n} \omega_{2m}^{2k(n-\gamma)} \\ &= \frac{1}{m} \sum_{n=0}^{\infty} B_n x^{2n} \sum_{k=1}^{m} \omega_{2m}^{2k(n-\gamma)}. \end{aligned}$$

Then the result follows from the almost trivial sum

$$\sum_{k=1}^{m} \omega_{2m}^{2k(n-\gamma)} = \begin{cases} m, & n \equiv \gamma \pmod{m}; \\ 0, & n \not\equiv \gamma \pmod{m}. \end{cases}$$

It should be pointed out that under the replacements $g \to h$ and $B_n \to 4^n/\binom{2n}{n}$, formula (12) remains valid because the h-series defined by (7) is even.

1.3. **Special function values.** Recall that $\Omega_m^{\gamma}(x)$ displayed in (2) results in a multisection series related to h-series given by (7). In order to evaluate $\Omega_m^{\gamma}(x)$ for specific integers m, γ and real number x, it will be necessary for us to know special values of the following function:

$$\text{hyp}(y) := \frac{\arcsin y}{y\sqrt{1-y^2}}. \tag{13}$$

They are tabulated below for the subsequent references:

Special Values for $\mathbf{hyp}\big(x\times e^{i\theta}\big)$

$x \backslash \theta$	π	$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{8}$
$\frac{1}{2}$	$\frac{2\pi}{3\sqrt{3}}$	$\frac{4}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2}$		
$\frac{1}{\sqrt{2}}$	$\frac{\pi}{2}$	$\frac{\ln(2+\sqrt{3})}{\sqrt{3}}$		
$\frac{\sqrt{3}}{2}$	$\frac{4\pi}{3\sqrt{3}}$	$\frac{2}{\sqrt{21}}\ln\frac{5+\sqrt{21}}{2}$	$\frac{4-2i}{15\sqrt{3}}\big\{\pi+3i\ln(2+\sqrt{3})\big\}$	
$\frac{1}{\sqrt[4]{2}}$		$\sqrt{\frac{2}{1+\sqrt{2}}}\ln\frac{1+\sqrt{1+\sqrt{2}}}{\sqrt[4]{2}}$		$\frac{\pi+2i\ln(1+\sqrt{2})}{2\sqrt{2}}$
$\frac{1}{\sqrt[4]{8}}$		$\sqrt{\frac{8}{1+2\sqrt{2}}}\ln\frac{1+\sqrt{1+2\sqrt{2}}}{\frac{1}{\sqrt[4]{8}}}$	$\frac{\sqrt{2}-i}{6\sqrt{2}}\big\{\pi+4i\ln(1+\sqrt{2})\big\}$	
$\frac{1}{\sqrt[4]{32}}$		$\sqrt{\frac{32}{1+4\sqrt{2}}}\ln\frac{1+\sqrt{1+4\sqrt{2}}}{\sqrt[4]{32}}$		$\frac{3-i}{10}\big\{\pi+2i\ln 2\big\}$

2. Summation Formulae: $m\in\mathbb{N}$

Now we are going to examine the series defined by

$$\Omega_m^{\gamma}\big((2x)^{2m}\big):=\sum_{n=0}^{\infty}\frac{\Lambda_m(n)}{\binom{2mn+2\gamma}{mn+\gamma}}(2x)^{2mn}.$$

The objective is to figure out the λ-parameters such that the series has closed expressions in terms of π and logarithm functions.

Suppose that m and γ are two nonnegative integers subject to $0\le\gamma\le m$. Combining (8) with (12), we can reformulate the following series

$$\begin{aligned}\Omega_m^{\gamma}\big((2x)^{2m}\big) &= \sum_{k=0}^{m}x^{k-2\gamma}\frac{\lambda_k}{4^{\gamma}}\mathcal{D}_x^k\sum_{n=0}^{\infty}\frac{(2x)^{2mn+2\gamma}}{\binom{2mn+2\gamma}{mn+\gamma}}\\ &= \sum_{k=0}^{m}\frac{\lambda_k x^{k-2\gamma}}{4^{\gamma}m}\mathcal{D}_x^k\sum_{\ell=1}^{m}\omega_{2m}^{-2\ell\gamma}h(x\omega_{2m}^{\ell})\\ &= \sum_{k=0}^{m}\frac{\lambda_k}{4^{\gamma}m}\sum_{\ell=1}^{m}\frac{(x\omega_{2m}^{\ell})^{k-2\gamma}P_k(x\omega_{2m}^{\ell})}{(1-x^2\omega_{2m}^{2\ell})^{k+1}}\\ &+ \sum_{k=0}^{m}\frac{\lambda_k}{4^{\gamma}m}\sum_{\ell=1}^{m}\frac{(x\omega_{2m}^{\ell})^{k-2\gamma}Q_k(x\omega_{2m}^{\ell})}{(1-x^2\omega_{2m}^{2\ell})^{k+1}}\frac{\arcsin(x\omega_{2m}^{\ell})}{\sqrt{1-x^2\omega_{2m}^{2\ell}}}.\end{aligned}$$

Interchanging the summation order, we get further the following equation:

$$\Omega_m^{\gamma}\big((2x)^{2m}\big) = \sum_{k=0}^{m}\frac{\lambda_k}{4^{\gamma}m}\sum_{\ell=1}^{m}\frac{(x\omega_{2m}^{\ell})^{k-2\gamma}P_k(x\omega_{2m}^{\ell})}{(1-x^2\omega_{2m}^{2\ell})^{k+1}}$$

$$+\sum_{\ell=1}^{m}\text{hyp}(x\omega_{2m}^{\ell})\sum_{k=0}^{m}\frac{\lambda_k}{4^{\gamma}m}\frac{(x\omega_{2m}^{\ell})^{1+k-2\gamma}Q_k(x\omega_{2m}^{\ell})}{(1-x^2\omega_{2m}^{2\ell})^{k+1}}.$$

From the last expression, we may formulate, by introducing $m+1$ constants $\{U_0, U_1, \cdots, U_m\}$ the following system of linear equations

$$\begin{aligned} U_0 &= \sum_{k=0}^{m}\frac{\lambda_k}{4^{\gamma}m}\sum_{\ell=1}^{m}\frac{(x\omega_{2m}^{\ell})^{k-2\gamma}}{(1-x^2\omega_{2m}^{2\ell})^{k+1}}P_k(x\omega_{2m}^{\ell}),\\ U_\ell &= \sum_{k=0}^{m}\frac{\lambda_k}{4^{\gamma}m}\frac{(x\omega_{2m}^{\ell})^{1+k-2\gamma}}{(1-x^2\omega_{2m}^{2\ell})^{k+1}}Q_k(x\omega_{2m}^{\ell}) \quad \text{where} \quad \ell=1,2,\cdots,m. \end{aligned}$$

Resolving this system of equations for the $m+1$ variables $\{\lambda_0, \lambda_1, \cdots, \lambda_m\}$ will result in the following general identity

$$\Omega_m^{\gamma}\big((2x)^{2m}\big) = U_0 + \sum_{\ell=1}^{m} U_\ell \times \text{hyp}(x\omega_{2m}^{\ell}) \quad \text{for} \quad m\in\mathbb{N}.$$

Denote by $\delta_{i,j}$ the Kronecker symbol which equals 1 for $i=j$ and 0 otherwise. For simplicity, we may specify $U_\ell = \delta_{\ell,k}$ with the only one term different from zero, that corresponds to a known special value $\text{hyp}(x\omega_{2m}^{k})$ (displayed in §1.3). Then the solution of the last linear system will lead to the following simplified formula $\Omega_m^{\gamma}\big((2x)^{2m}\big) = \text{hyp}(x\omega_{2m}^{k})$ which expresses a mutlisection series in terms of π and/or logarithm functions.

Based on this analysis, some *Mathematica* commands are written in order to resolve the systems of linear equations and then to determine the $\Lambda_m(n)$ polynomials. This leads us to find hundreds of infinite series identities. What is remarkable is that most of our formulae involve the central binomial coefficients $\binom{2mn+2\gamma}{mn+\gamma}$ covering all the residue classes γ modulo m, while the known formulae hitherto (for example those in [1, 10]) concern only one single class $\gamma=0$ of binomial coefficients $\binom{2mn}{mn}$.

In order to facilitate readers for comparison and reference, thirty–six carefully selected examples are displayed below, where for the sake of brevity, the Kronecker symbol $\delta_{i,j}$ will be utilized with $\delta_{i,j}=1$ for $i=j$ and otherwise $\delta_{i,j}=0$ for $i\neq j$.

2.1. Formulae corresponding to $m=2$.

Example 1 ($m=2$ and $x=1/2:\ U_2=1 \quad \text{and} \quad U_0=U_1=0$)**.**

(a) $$\frac{4\pi}{9\sqrt{3}} = \sum_{n=0}^{\infty}\frac{\Lambda_2(n)}{\binom{4n}{2n}} \quad \textit{where} \quad \Lambda_2(n) = -1+10n.$$

(b) $$\frac{2\pi}{27\sqrt{3}} = \sum_{n=0}^{\infty}\frac{\Lambda_2(n)}{\binom{4n+2}{2n+1}} \quad \textit{where} \quad \Lambda_2(n) = -1+10n^2.$$

Example 2 ($m = 2$ and $x = 1/2 :\ U_1 = 1 \quad \text{and} \quad U_0 = U_2 = 0$)**.**

(a) $$\frac{72}{5\sqrt{5}} \ln \frac{1+\sqrt{5}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_2(n)}{\binom{4n}{2n}} \quad \textit{where} \quad \Lambda_2(n) = -13 - 134n + 192n^2.$$

(b) $$\frac{12}{5\sqrt{5}} \ln \frac{1+\sqrt{5}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_2(n)}{\binom{4n+2}{2n+1}} \quad \textit{where} \quad \Lambda_2(n) = 7 - 8n - 42n^2.$$

Example 3 ($m = 2$ and $x = 1/\sqrt{2} :\ U_2 = 1 \quad \text{and} \quad U_0 = U_1 = 0$)**.**

(a) $$\frac{9\pi}{4} = \sum_{n=0}^{\infty} \frac{\Lambda_2(n)}{\binom{4n}{2n}} 4^n \quad \textit{where} \quad \Lambda_2(n) = -2 + 53n - 24n^2.$$

(b) $$\frac{3\pi}{16} = \sum_{n=0}^{\infty} \frac{\Lambda_2(n)}{\binom{4n+2}{2n+1}} 4^n \quad \textit{where} \quad \Lambda_2(n) = -8 - 11n + 15n^2.$$

Example 4 ($m = 2$ and $x = 1/\sqrt{2} :\ U_1 = 1 \quad \text{and} \quad U_0 = U_2 = 0$)**.**

(a) $$\sqrt{3} \ln \tfrac{1+\sqrt{3}}{\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_2(n)}{\binom{4n}{2n}} 4^n \quad \textit{where} \quad \Lambda_2(n) = -7 - 35n + 24n^2.$$

(b) $$\frac{\ln \frac{1+\sqrt{3}}{\sqrt{2}}}{2\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_2(n)}{\binom{4n+2}{2n+1}} 4^n \quad \textit{where} \quad \Lambda_2(n) = 5 + 2n - 6n^2.$$

Example 5 ($m = 2$ and $x = \sqrt{3}/2 :\ U_2 = 1 \quad \text{and} \quad U_0 = U_1 = 0$)**.**

(a) $$72\sqrt{3}\pi = \sum_{n=0}^{\infty} \frac{\Lambda_2(n)}{\binom{4n}{2n}} 9^n \quad \textit{where} \quad \Lambda_2(n) = 99 + 1930 - 448n^2.$$

(b) $$4\sqrt{3}\pi = \sum_{n=0}^{\infty} \frac{\Lambda_2(n)}{\binom{4n+2}{2n+1}} 9^n \quad \textit{where} \quad \Lambda_2(n) = -315 - 488n + 154n^2.$$

Example 6 ($m = 2$ and $x = \sqrt{3}/2 :\ U_1 = 1 \quad \text{and} \quad U_0 = U_2 = 0$)**.**

(a) $$\frac{216\sqrt{3}}{\sqrt{7}} \ln \tfrac{\sqrt{3}+\sqrt{7}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_2(n)}{\binom{4n}{2n}} 9^n \ \textit{where}\ \Lambda_2(n) = -927 - 3482n + 896n^2.$$

(b) $$\frac{12\sqrt{3}}{\sqrt{7}} \ln \tfrac{\sqrt{3}+\sqrt{7}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_2(n)}{\binom{4n+2}{2n+1}} 9^n \ \textit{where}\ \Lambda_2(n) = 243 + 176n - 70n^2.$$

2.2. Formulae corresponding to $m = 4$.

Example 7 ($m = 4$ and $x = 1/2 :\ U_k = \delta_{4,k} \quad \text{for} \quad 0 \le k \le 4$)**.**

(a) $$\frac{9800\pi}{27\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n}{4n}} \qquad \textit{where}$$

$$\Lambda_4(n) = -1149 + 15202n - 8576n^2 - 65632n^3 + 174080n^4.$$

(b) $$\frac{700\pi}{27\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+2}{4n+1}} \qquad \textit{where}$$

$$\Lambda_4(n) = -195 + 1042n + 12320n^2 - 51568n^3 + 70720n^4.$$

(c) $$\frac{140\pi}{27\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+4}{4n+2}} \qquad where$$

$$\Lambda_4(n) = 261 + 1656n + 11266n^2 + 4896n^3 - 46240n^4.$$

(d) $$\frac{70\pi}{27\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+6}{4n+3}} \qquad where$$

$$\Lambda_4(n) = -645 - 1588n + 12422n^2 + 55480n^3 + 54400n^4.$$

Example 8 ($m = 4$ and $x = 1/2 : \ U_k = \delta_{2,k} \quad \text{for} \quad 0 \leq k \leq 4$)**.**

(a) $$2352\sqrt{5}\ln\tfrac{1+\sqrt{5}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n}{4n}} \qquad where$$

$$\Lambda_4(n) = -14753 - 744486n + 6254048n^2 - 16392544n^3 + 11663360n^4.$$

(b) $$168\sqrt{5}\ln\tfrac{1+\sqrt{5}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+2}{4n+1}} \qquad where$$

$$\Lambda_4(n) = 4945 - 6646n - 8400n^2 + 979184n^3 - 1463360n^4.$$

(c) $$\frac{168}{\sqrt{5}}\ln\tfrac{1+\sqrt{5}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+4}{4n+2}} \qquad where$$

$$\Lambda_4(n) = -2967 - 19416n - 63478n^2 + 94176n^3 + 443360n^4.$$

(d) $$\frac{84}{\sqrt{5}}\ln\tfrac{1+\sqrt{5}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+6}{4n+3}} \qquad where$$

$$\Lambda_4(n) = 2335 - 3588n - 65914n^2 - 152536n^3 - 103360n^4.$$

Example 9 ($m = 4$ and $x = 1/\sqrt{2} : \ U_k = \delta_{4,k} \quad \text{for} \quad 0 \leq k \leq 4$)**.**

(a) $$\frac{3675\pi}{8} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n}{4n}} 16^n \qquad where$$

$$\Lambda_4(n) = -1096 + 22223n - 66684n^2 + 120272n^3 - 48640n^4.$$

(b) $$\frac{525\pi}{32} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+2}{4n+1}} 16^n \qquad where$$

$$\Lambda_4(n) = -920 - 3741n - 1510n^2 - 34136n^3 + 26720n^4.$$

(c) $$\frac{105\pi}{32} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+4}{4n+2}} 16^n \qquad where$$

$$\Lambda_4(n) = +2424 + 9465n + 17628n^2 + 8112n^3 - 26240n^4.$$

(d) $$\frac{105\pi}{128} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+6}{4n+3}} 16^n \qquad where$$

$$\Lambda_4(n) = -2920 - 8905n - 6594n^2 + 9416n^3 + 13280n^4.$$

Example 10 ($m = 4$ and $x = 1/\sqrt{2}:\ U_k = \delta_{2,k}\quad \text{for}\quad 0 \le k \le 4$)**.**

$$\text{(a)}\quad \frac{1225}{2\sqrt{3}}\ln(2+\sqrt{3}) = \sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n}{4n}}16^n \qquad \textit{where}$$
$$\Lambda_4(n) = -5049 - 66158n + 192184n^2 - 637792n^3 + 343040n^4.$$

$$\text{(b)}\quad \frac{175}{4\sqrt{3}}\ln(2+\sqrt{3}) = \sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n+2}{4n+1}}16^n \qquad \textit{where}$$
$$\Lambda_4(n) = 3825 + 14752n + 24680n^2 + 87392n^3 - 83840n^4.$$

$$\text{(c)}\quad \frac{35}{4\sqrt{3}}\ln(2+\sqrt{3}) = \sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n+4}{4n+2}}16^n \qquad \textit{where}$$
$$\Lambda_4(n) = -3897 - 14748n - 22496n^2 - 960n^3 + 32000n^4.$$

$$\text{(d)}\quad \frac{35}{8\sqrt{3}}\ln(2+\sqrt{3}) = \sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n+6}{4n+3}}16^n \qquad \textit{where}$$
$$\Lambda_4(n) = 1185 + 632n - 6496n^2 - 6848n^3 + 2560n^4.$$

Example 11 ($m = 4$ and $x = \sqrt{3}/2:\ U_k = \delta_{4,k}\quad \text{for}\quad 0 \le k \le 4$)**.**

$$\text{(a)}\ 2\sqrt{3}\pi = \sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n}{4n}}\frac{81^n}{264600} \qquad \textit{where}$$
$$\Lambda_4(n) = 226719 + 33194778n - 36578144n^2 + 156768672n^3 - 38348800n^4.$$

$$\text{(b)}\ 2\sqrt{3}\pi = \sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n+2}{4n+1}}\frac{81^n}{6300} \qquad \textit{where}$$
$$\Lambda_4(n) = -2845125 - 13960274n - 22188240n^2 - 39451504n^3 + 12801600n^4.$$

$$\text{(c)}\ 2\sqrt{3}\pi = \sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n+4}{4n+2}}\frac{81^n}{1260} \qquad \textit{where}$$
$$\Lambda_4(n) = 8289459 + 31656696n + 46986622n^2 + 26660640n^3 - 12796000n^4.$$

$$\text{(d)}\ 2\sqrt{3}\pi = \sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n+6}{4n+3}}\frac{81^n}{70} \qquad \textit{where}$$
$$\Lambda_4(n) = -2167155 - 6593196n - 6907342n^2 - 1538856n^3 + 1422400n^4.$$

Example 12 ($m = 4$ and $x = \sqrt{3}/2:\ U_k = \delta_{2,k}\quad \text{for}\quad 0 \le k \le 4$)**.**

$$\text{(a)}\ \frac{\sqrt{3}}{\sqrt{7}}\ln\frac{5+\sqrt{21}}{2} = \sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n}{4n}}\frac{81^n}{113400} \qquad \textit{where}$$
$$\Lambda_4(n) = -2291733 - 20898606n - 2975552n^2 - 105996384n^3 + 28313600n^4.$$

$$\text{(b)}\ \frac{\sqrt{3}}{\sqrt{7}}\ln\frac{5+\sqrt{21}}{2} = \sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n+2}{4n+1}}\frac{81^n}{2700} \qquad \textit{where}$$

$\Lambda_4(n) = 885735 + 3997078n + 6888640n^2 + 8671088n^3 - 3035200n^4.$

(c) $\frac{\sqrt{3}}{\sqrt{7}} \ln \frac{5+\sqrt{21}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+4}{4n+2}} \frac{81^n}{540}$ *where*

$\Lambda_4(n) = -1005777 - 3578520n - 4757834n^2 - 1909536n^3 + 1114400n^4.$

(d) $\frac{\sqrt{3}}{\sqrt{7}} \ln \frac{5+\sqrt{21}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+6}{4n+3}} \frac{81^n}{30}$ *where*

$\Lambda_4(n) = -49095 - 235740n - 375614n^2 - 165624n^3 + 89600n^4.$

Example 13 ($m = 4$ and $x = \sqrt{3}/2 : \; U_k = \delta_{1,k}$ for $0 \le k \le 4$)**.**

(a) $\frac{\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n}{4n}} \frac{81^n}{793800}$ *where*

$\Lambda_4(n) = -844911 + 61504038n - 114762544n^2 + 520764192n^3 - 126156800n^4.$

(b) $\frac{\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+2}{4n+1}} \frac{81^n}{18900}$ *where*

$\Lambda_4(n) = -7098435 - 35747294n - 58837400n^2 - 84430624n^3 + 28649600n^4.$

(c) $\frac{\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+4}{4n+2}} \frac{81^n}{3780}$ *where*

$\Lambda_4(n) = 15204753 + 55874328n + 79924918n^2 + 45444288n^3 - 21935200n^4.$

(d) $\frac{\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+6}{4n+3}} \frac{81^n}{210}$ *where*

$\Lambda_4(n) = -3917565 - 12299436n - 13678786n^2 - 3512160n^3 + 2884000n^4.$

Example 14 ($m = 4$ and $x = \sqrt{3}/2 : \; U_k = \delta_{1,k}$ for $0 \le k \le 4$)**.**

(a) $\frac{\ln(2+\sqrt{3})}{2\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n}{4n}} \frac{81^n}{595350}$ *where*

$\Lambda_4(n) = 421119 - 1003077n + 20670326n^2 - 4798968n^3 - 492800n^4.$

(b) $\frac{\ln(2+\sqrt{3})}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+2}{4n+1}} \frac{81^n}{14175}$ *where*

$\Lambda_4(n) = 468990 + 2482201n + 4653225n^2 + 8169146n^3 - 2633400n^4.$

(c) $\frac{\ln(2+\sqrt{3})}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+4}{4n+2}} \frac{81^n}{2835}$ *where*

$\Lambda_4(n) = -1557387 - 5896137n - 8290472n^2 - 4146252n^3 + 2130800n^4.$

(d) $\frac{\ln(2+\sqrt{3})}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+6}{4n+3}} \frac{81^n}{315}$ *where*

$$\Lambda_4(n) = 494145 + 1383888n + 1361663n^2 + 300630n^3 - 287000n^4.$$

Example 15 ($m = 4$ and $x = 1/\sqrt[4]{8} : \; U_k = \delta_{1,k} \quad \text{for} \quad 0 \le k \le 4$)**.**

(a) $$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n}{4n}} \frac{4^n}{3675} \qquad \textit{where}$$
$$\Lambda_4(n) = -13066 + 60817n - 529936n^2 - 398352n^3 + 924672n^4.$$

(b) $$\frac{\pi}{8} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+2}{4n+1}} \frac{4^n}{525} \qquad \textit{where}$$
$$\Lambda_4(n) = -3550 - 11965n + 6308n^2 - 520560n^3 + 547008n^4.$$

(c) $$\frac{\pi}{16} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+4}{4n+2}} \frac{4^n}{105} \qquad \textit{where}$$
$$\Lambda_4(n) = 4836 + 18321n + 41833n^2 - 8976n^3 - 177744n^4.$$

(d) $$\frac{\pi}{32} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+6}{4n+3}} \frac{4^n}{105} \qquad \textit{where}$$
$$\Lambda_4(n) = -6350 - 19672n + 3589n^2 + 112944n^3 + 128688n^4.$$

Example 16 ($m = 4$ and $x = 1/\sqrt[4]{8} : \; U_k = \delta_{1,k} \quad \text{for} \quad 0 \le k \le 4$)**.**

(a) $$\frac{\ln(1+\sqrt{2})}{\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n}{4n}} \frac{4^n}{3675} \qquad \textit{where}$$
$$\Lambda_4(n) = -221 - 10198n - 392036n^2 + 1523568n^3 - 956928n^4.$$

(b) $$\frac{\ln(1+\sqrt{2})}{2\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+2}{4n+1}} \frac{4^n}{525} \qquad \textit{where}$$
$$\Lambda_4(n) = -1655 - 14300n - 44552n^2 + 109080n^3 - 672n^4.$$

(c) $$\frac{\ln(1+\sqrt{2})}{2\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+4}{4n+2}} \frac{4^n}{105} \qquad \textit{where}$$
$$\Lambda_4(n) = -3201 - 17706n - 46436n^2 + 2544n^3 + 147840n^4.$$

(d) $$\frac{\ln(1+\sqrt{2})}{4\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+6}{4n+3}} \frac{4^n}{105} \qquad \textit{where}$$
$$\Lambda_4(n) = 5965 + 10124n - 27368n^2 - 99672n^3 - 84000n^4.$$

2.3. **Formulae corresponding to $m = 6$.**

Example 17 ($m = 6$ and $x = 1/2 : \; U_k = \delta_{6,k} \quad \text{for} \quad 0 \le k \le 6$)**.**

(a) $$\frac{\pi}{\sqrt{3}} = \frac{2187}{2371600} \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n}{6n}} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_6(n) = & -3700+17093n+1017665n^2-9169710n^3 \\ & +31935420n^4-47121048n^5+28304640n^6.\end{aligned}$$

(b) $$\frac{\pi}{\sqrt{3}}=\frac{2187}{107800}\sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+2}{6n+1}} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_6(n) = & -280+4817n+42935n^2-497190n^3 \\ & +2146590n^4-3591432n^5+2653560n^6.\end{aligned}$$

(c) $$\frac{\pi}{\sqrt{3}}=\frac{2187}{215600}\sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+4}{6n+2}} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_6(n) = & \ 1330+37929n+447470n^2-1093725n^3 \\ & -3254490n^4+20272356n^5-16805880n^6.\end{aligned}$$

(d) $$\frac{\pi}{\sqrt{3}}=\frac{2187}{15400}\sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+6}{6n+3}} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_6(n) = & -2190-8443n+72380n^2+8715n^3 \\ & -1573200n^4+234468n^5+7076160n^6.\end{aligned}$$

(e) $$\frac{\pi}{\sqrt{3}}=\frac{2187}{12320}\sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+8}{6n+4}} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_6(n) = & \ 10920+62701n+395992n^2+1233957n^3 \\ & -1923354n^4-13050180n^5-13267800n^6.\end{aligned}$$

(f) $$\frac{\pi}{\sqrt{3}}=\frac{2187}{30800}\sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+10}{6n+5}} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_6(n) = & -67200-295297n+1451480n^2+17882775n^3 \\ & +60067530n^4+83570292n^5+41572440n^6.\end{aligned}$$

Example 18 ($m=6$ and $x=1/2: \ U_k=\delta_{3,k} \quad \text{for} \quad 0\le k\le 6$)**.**

(a) $$\frac{4}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2}=\frac{1}{6403320}\sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n}{6n}} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_6(n) = & -46871540-16483103431n+321864419085n^2-2039427277470n^3 \\ & +5432797555020n^4-6453587573784n^5+2788233477120n^6.\end{aligned}$$

(b) $$\frac{4}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2}=\frac{1}{291060}\sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+2}{6n+1}} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_6(n) = & \ 11184320-76799159n+963395295n^2+18501001830n^3 \\ & -125539827750n^4+265480346664n^5-176135352120n^6.\end{aligned}$$

(c) $$\frac{4}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2}=\frac{1}{21560}\sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+4}{6n+2}} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_6(n) = & -3281250-35093889n-109147550n^2+1077643845n^3 \\ & -349845030n^4-14262392196n^5+20679193080n^6.\end{aligned}$$

(d) $\frac{4}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2} = \frac{1}{13860}\sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+6}{6n+3}}$ *where*

$$\Lambda_6(n) = 8532910 + 31268947n - 25380960n^2 + 816458805n^3 + 4127617980n^4 - 4113527652n^5 - 20575704240n^6.$$

(e) $\frac{4}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2} = \frac{1}{11088}\sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+8}{6n+4}}$ *where*

$$\Lambda_6(n) = -15099560 - 107763149n - 509132280n^2 - 401658885n^3 + 6267532410n^4 + 19565401284n^5 + 16274283480n^6.$$

(f) $\frac{4}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2} = \frac{1}{3080}\sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+10}{6n+5}}$ *where*

$$\Lambda_6(n) = 1974560 - 19730767n - 296982880n^2 - 1112802615n^3 - 1602765810n^4 - 714561588n^5 + 103488840n^6.$$

Example 19 ($m = 6$ and $x = 1/\sqrt{2}$: $U_k = \delta_{6,k}$ for $0 \le k \le 6$)**.**

(a) $\frac{\pi}{4} = \sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n}{6n}}\frac{2^{6n}}{4002075}$ *where*

$$\Lambda_6(n) = -2916880 + 74109649n - 508307874n^2 + 1857555720n^3 - 3052842480n^4 + 2707034256n^5 - 841954176n^6.$$

(b) $\frac{\pi}{16} = \sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+2}{6n+1}}\frac{2^{6n}}{363825}$ *where*

$$\Lambda_6(n) = -901600 - 5895665n - 11520183n^2 - 242054730n^3 + 586110600n^4 - 935193600n^5 + 446696208n^6.$$

(c) $\frac{\pi}{16} = \sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+4}{6n+2}}\frac{2^{6n}}{13475}$ *where*

$$\Lambda_6(n) = 1001280 + 7499781n + 24566366n^2 + 23770080n^3 + 3066480n^4 + 163527984n^5 - 147946176n^6.$$

(d) $\frac{\pi}{64} = \sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+6}{6n+3}}\frac{2^{6n}}{17325}$ *where*

$$\Lambda_6(n) = -4586240 - 27290933n - 63984669n^2 - 85827330n^3 - 115908840n^4 - 22734432n^5 + 222327504n^6.$$

(e) $\frac{\pi}{64} = \sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+8}{6n+4}}\frac{2^{6n}}{3465}$ *where*

$$\Lambda_6(n) = 5706400 + 28970659n + 62510586n^2 + 74709000n^3 + 21467376n^4 - 99502992n^5 - 111041280n^6.$$

(f) $\frac{\pi}{256} = \sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+10}{6n+5}}\frac{2^{6n}}{1925}$ *where*

$$\Lambda_6(n) = -2098880 - 9300689n - 16185191n^2 - 8394210n^3 +18085320n^4 + 35178624n^5 + 18534096n^6.$$

Example 20 ($m = 6$ and $x = 1/\sqrt{2}: U_k = \delta_{3,k}$ for $0 \le k \le 6$).

(a) $$\frac{\ln(2+\sqrt{3})}{2\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n}{6n}} \frac{2^{6n}}{444675} \quad \textit{where}$$
$$\Lambda_6(n) = -3329035 - 86772839n + 636599154n^2 - 4996923480n^3 +10676599200n^4 - 12572782416n^5 + 4872099456n^6.$$

(b) $$\frac{\ln(2+\sqrt{3})}{4\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+2}{6n+1}} \frac{2^{6n}}{40425} \quad \textit{where}$$
$$\Lambda_6(n) = 2036335 + 16053680n + 62193666n^2 + 227012220n^3 -548393760n^4 + 1252532160n^5 - 705602016n^6.$$

(c) $$\frac{\ln(2+\sqrt{3})}{36\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+4}{6n+2}} \frac{2^{6n}}{13475} \quad \textit{where}$$
$$\Lambda_6(n) = -727545 - 5138422n - 14673252n^2 - 15241560n^3 -15485040n^4 - 72085248n^5 + 80885952n^6.$$

(d) $$\frac{\ln(2+\sqrt{3})}{8\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+6}{6n+3}} \frac{2^{6n}}{1925} \quad \textit{where}$$
$$\Lambda_6(n) = 2808335 + 15959792n + 36911796n^2 + 51457680n^3 +57151440n^4 - 10655712n^5 - 110388096n^6.$$

(e) $$\frac{\ln(2+\sqrt{3})}{4\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+8}{6n+4}} \frac{2^{6n}}{385} \quad \textit{where}$$
$$\Lambda_6(n) = -2477195 - 11861312n - 24120912n^2 - 22647384n^3 +9165744n^4 + 45461088n^5 + 31026240n^6.$$

(f) $$\frac{\ln(2+\sqrt{3})}{72\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+10}{6n+5}} \frac{2^{6n}}{1925} \quad \textit{where}$$
$$\Lambda_6(n) = -1888075 - 9316664n - 19486896n^2 - 18713640n^3 +7674480n^4 + 37489824n^5 + 24929856n^6.$$

Example 21 ($m = 6$ and $x = \sqrt{3}/2: U_k = \delta_{6,k}$ for $0 \le k \le 6$).

(a) $$2\sqrt{3}\pi = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n}{6n}} \frac{3^{6n}}{1728896400} \quad \textit{where}$$
$$\Lambda_6(n) = -1985198220 + 321143302547n - 979270437137n^2 +7078874467590n^3 - 7138799437500n^4 +9790385494968n^5 - 2221193415168n^6.$$

(b) $$2\sqrt{3}\pi = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+2}{6n+1}} \frac{3^{6n}}{78586200} \quad \textit{where}$$

$$\Lambda_6(n) = -52810742880 - 488466005045n - 1797839252323n^2 -4833369579510n^3 - 94072466970n^4 -7569018124200n^5 + 2221422505848n^6.$$

(c) $$2\sqrt{3}\pi = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+4}{6n+2}} \frac{3^{6n}}{1940400} \qquad \textit{where}$$

$$\Lambda_6(n) = 46119536190 + 337725667759n + 1012209624354n^2 +1509548831045n^3 + 1216905606330n^4 +1188368095356n^5 - 493640960904n^6.$$

(d) $$2\sqrt{3}\pi = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+6}{6n+3}} \frac{3^{6n}}{138600} \qquad \textit{where}$$

$$\Lambda_6(n) = -46197874530 - 275887147781n - 673032724088n^2 -868400626515n^3 - 667171940220n^4 -231572770884n^5 + 164548199088n^6.$$

(e) $$2\sqrt{3}\pi = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+8}{6n+4}} \frac{3^{6n}}{110880} \qquad \textit{where}$$

$$\Lambda_6(n) = 230872171320 + 1171442262787n + 2454497967592n^2 +2724556150539n^3 + 1583186976666n^4 +134056254852n^5 - 329094216360n^6.$$

(f) $$\frac{2\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+10}{6n+5}} \frac{3^{6n}}{30800} \qquad \textit{where}$$

$$\Lambda_6(n) = -56459592000 - 249580425477n - 453667545808n^2 -420748778845n^3 - 173082550710n^4 +21672868932n^5 + 36566508888n^6.$$

Example 22 ($m = 6$ and $x = \sqrt{3}/2 : \ U_k = \delta_{3,k} \quad \text{for} \quad 0 \le k \le 6$)**.**

(a) $$\frac{\sqrt{3}}{\sqrt{7}} \ln \frac{\sqrt{3}+\sqrt{7}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n}{6n}} \frac{3^{6n}}{1481911200} \qquad \textit{where}$$

$$\Lambda_6(n) = -34596575820 - 533695161689n - 842105544421n^2 -11900542931370n^3 + 8643435415380n^4 -19979329763016n^5 + 4990808100096n^6.$$

(b) $$\frac{\sqrt{3}}{\sqrt{7}} \ln \frac{\sqrt{3}+\sqrt{7}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+2}{6n+1}} \frac{3^{6n}}{67359600} \qquad \textit{where}$$

$$\Lambda_6(n) = 37910187000 + 335578757315n + 1220272245061n^2 +2531514689910n^3 + 827643254850n^4 +3610978642440n^5 - 1140481041336n^6.$$

(c) $$\frac{\sqrt{3}}{\sqrt{7}} \ln \frac{\sqrt{3}+\sqrt{7}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+4}{6n+2}} \frac{3^{6n}}{1663200} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_6(n) = &-12872300490 - 90747609173n - 259815159798n^2\\&-375462757015n^3 - 313976759790n^4\\&-246446239572n^5 + 110835525528n^6.\end{aligned}$$

(d) $$\frac{\sqrt{3}}{\sqrt{7}}\ln\frac{\sqrt{3}+\sqrt{7}}{2} = \sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+6}{6n+3}}\frac{3^{6n}}{118800} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_6(n) = &\ 5961430710 + 34106274607n + 79929292396n^2\\&+99901655745n^3 + 72291564360n^4\\&+18958432428n^5 - 16573074336n^6.\end{aligned}$$

(e) $$\frac{\sqrt{3}}{\sqrt{7}}\ln\frac{\sqrt{3}+\sqrt{7}}{2} = \sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+8}{6n+4}}\frac{3^{6n}}{95040} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_6(n) = &-4151687400 - 15010449329n - 19337874776n^2\\&-5918173065n^3 + 13455162594n^4\\&+11912382612n^5 - 3523632840n^6.\end{aligned}$$

(f) $$\frac{\sqrt{3}}{\sqrt{7}}\ln\frac{\sqrt{3}+\sqrt{7}}{2} = \sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+10}{6n+5}}\frac{3^{6n}}{8800} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_6(n) = &-9673685280 - 45595406049n - 88386231416n^2\\&-88684206185n^3 - 41879771430n^4\\&+2151261684n^5 + 8540355096n^6.\end{aligned}$$

2.4. Formulae corresponding to $m = 8$.

Example 23 ($m = 8$ and $x = 1/2:\ U_k = \delta_{8,k}$ for $0 \le k \le 8$)**.**

(a) $$\frac{\pi}{3\sqrt{3}} = \frac{243}{400800400}\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n}{8n}} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = &-1933785 - 192513798n + 7456633664n^2\\&-85351111776n^3 + 464969971200n^4 - 1369404997632n^5\\&+2263044489216n^6 - 1976704303104n^7 + 732996567040n^8.\end{aligned}$$

(b) $$\frac{\pi}{3\sqrt{3}} = \frac{243}{200400200}\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+2}{8n+1}} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = &-1641465 + 47136114n + 227567280n^2\\&-8838858784n^3 + 82875672320n^4 - 348608963584n^5\\&+799088353280n^6 - 931100164096n^7 + 469575925760n^8.\end{aligned}$$

(c) $$\frac{\pi}{3\sqrt{3}} = \frac{243}{3083080}\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+4}{8n+2}} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = &-30051 + 1600578n + 29242512n^2\\&-192702608n^3 + 6526912n^4 + 4910338048n^5\\&-15490850816n^6 + 18802868224n^7 - 5726535680n^8.\end{aligned}$$

(d) $\frac{\pi}{3\sqrt{3}} = \frac{243}{700700}\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+6}{8n+3}}$ *where*

$$\begin{aligned}\Lambda_8(n) = {} & -146475 - 255030n + 19408096n^2 \\ & -80610768n^3 - 593503680n^4 + 3028677120n^5 \\ & +2287939584n^6 - 21238874112n^7 + 22906142720n^8.\end{aligned}$$

(e) $\frac{\pi}{3\sqrt{3}} = \frac{243}{280280}\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+8}{8n+4}}$ *where*

$$\begin{aligned}\Lambda_8(n) = {} & 688905 + 6797016n + 55889226n^2 \\ & +65350880n^3 - 1027847968n^4 + 158455808n^5 \\ & +12732575744n^6 + 672923648n^7 - 37222481920n^8.\end{aligned}$$

(f) $\frac{\pi}{3\sqrt{3}} = \frac{243}{500500}\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+10}{8n+5}}$ *where*

$$\begin{aligned}\Lambda_8(n) = {} & -8865045 - 58422096n + 29816118n^2 \\ & +570402688n^3 - 5575082720n^4 - 25216653824n^5 \\ & +22806618112n^6 + 193007091712n^7 + 191838945280n^8.\end{aligned}$$

(g) $\frac{\pi}{3\sqrt{3}} = \frac{243}{100100}\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+12}{8n+6}}$ *where*

$$\begin{aligned}\Lambda_8(n) = {} & 7120575 + 47459868n + 279528070n^2 \\ & +1453993992n^3 + 816054720n^4 - 23370283008n^5 \\ & -88925429760n^6 - 125737009152n^7 - 62991892480n^8.\end{aligned}$$

(h) $\frac{\pi}{3\sqrt{3}} = \frac{243}{350350}\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+14}{8n+7}}$ *where*

$$\begin{aligned}\Lambda_8(n) = {} & -42567525 - 260741916n + 822099702n^2 \\ & +22567437928n^3 + 140918713600n^4 + 428448246016n^5 \\ & +695079282688n^6 + 576327073792n^7 + 191838945280n^8.\end{aligned}$$

Example 24 ($m = 8$ and $x = 1/2:\ U_k = \delta_{4,k}$ for $0 \le k \le 8$)**.**

(a) $\frac{4}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2} = \frac{1}{146091745800}\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n}{8n}}$ *where*

$$\begin{aligned}\Lambda_8(n) = {} & -1549655887185 - 5880767610779766n + 179522310820446208n^2 \\ & -1872275344222817632n^3 + 9310071672968174080n^4 \\ & -24858150358857539584n^5 + 36436297732771643392n^6 \\ & -27580615489364819968n^7 + 8406317620348846080n^8.\end{aligned}$$

(b) $\frac{4}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2} = \frac{1}{4869724860}\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+2}{8n+1}}$ *where*

$$\begin{aligned}\Lambda_8(n) = &\ 326820388545 - 17484090415842n + 285301531354960n^2\\ &+4099952517902752n^3 - 66074623276901120n^4\\ &+332947221800983552n^5 - 767709823280046080n^6\\ &+834491584215973888n^7 - 345323879548846080n^8.\end{aligned}$$

(c) $$\frac{4}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2} = \frac{1}{374594220}\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+4}{8n+2}} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = &\ -111435648345 - 1996862229306n - 3039286264848n^2\\ &+180356245513808n^3 - 1056664837583040n^4\\ &-2820727686160384n^5 + 26908145193689088n^6\\ &-55073983067127808n^7 + 35285658748846080n^8.\end{aligned}$$

(d) $$\frac{4}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2} = \frac{1}{51081030}\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+6}{8n+3}} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = &\ 65289544425 + 429388795410n + 1415857983296n^2\\ &+30446814552112n^3 + 25355978258240n^4\\ &-740718440860160n^5 + 547443887427584n^6\\ &+4987389918281728n^7 - 7100365948846080n^8.\end{aligned}$$

(e) $$\frac{4}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2} = \frac{1}{3783780}\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+8}{8n+4}} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = &\ -18225094125 - 167451012760n - 739418471538n^2\\ &+1251016242336n^3 + 14575886867360n^4\\ &-39513755330560n^5 - 224066121101312n^6\\ &+156481076854784n^7 + 800939049615360n^8.\end{aligned}$$

(f) $$\frac{4}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2} = \frac{1}{4054050}\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+10}{8n+5}} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = &\ 60204164895 + 351806939856n + 351357375102n^2\\ &+3519490906432n^3 + 47405206569120n^4\\ &+101486452785664n^5 - 304099366060032n^6\\ &-1228292379410432n^7 - 1060660348846080n^8.\end{aligned}$$

(g) $$\frac{4}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2} = \frac{1}{810810}\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+12}{8n+6}} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = &\ -17418433725 - 148376797236n - 901532462290n^2\\ &-2064289688984n^3 + 13090276955840n^4\\ &+97786944802816n^5 + 251657500221440n^6\\ &+293258376085504n^7 + 129860649615360n^8.\end{aligned}$$

(h) $$\frac{4}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2} = \frac{1}{2837835}\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+14}{8n+7}} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = &-33054966945 - 681099826284n - 7778663317042n^2\\ &-32560154877208n^3 - 15740204757760n^4\\ &+238976350479104n^5 + 678171759491072n^6\\ &+727014122897408n^7 + 281496451153920n^8.\end{aligned}$$

Example 25 ($m = 8$ and $x = 1/\sqrt{2}: \ U_k = \delta_{8,k} \quad \text{for} \quad 0 \le k \le 8$)**.**

(a) $$\frac{\pi}{16} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n}{8n}} \frac{2^{8n}}{91307341125} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = &-17560206720 + 587642691753n - 6540957265624n^2\\ &+37805153134336n^3 - 119095249692160n^4 + 227983337193472n^5\\ &-248671109349376n^6 + 149578194878464n^7 - 36338946539520n^8.\end{aligned}$$

(b) $$\frac{\pi}{64} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+2}{8n+1}} \frac{2^{8n}}{1217431215} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = &-439689600 - 2856576219n - 9053259988n^2\\ &-531424559008n^3 + 2547243517952n^4 - 8665229805568n^5\\ &+13531703607296n^6 - 11489870479360n^7 + 3794487214080n^8.\end{aligned}$$

(c) $$\frac{\pi}{64} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+4}{8n+2}} \frac{2^{8n}}{468242775} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = &\ 7889374080 + 92974290543n + 483011700984n^2\\ &+881468247296n^3 + 2103658314240n^4 + 9384147648512n^5\\ &-25653840543744n^6 + 38467136651264n^7 - 18910669701120n^8.\end{aligned}$$

(d) $$\frac{\pi}{256} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+6}{8n+3}} \frac{2^{8n}}{127702575} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = &-14739580800 - 139484702715n - 560810573516n^2\\ &-1345234616992n^3 - 2127867875840n^4 - 343840860160n^5\\ &+138963091456n^6 - 9774209302528n^7 + 9460949975040n^8.\end{aligned}$$

(e) $$\frac{\pi}{256} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+8}{8n+4}} \frac{256}{4729725} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = &\ 7351612800 + 60016095785n + 212784771048n^2\\ &+421022832384n^3 + 469847544320n^4 + 402526208000n^5\\ &+712925151232n^6 + 52706410496n^7 - 1576513044480n^8.\end{aligned}$$

(f) $$\frac{\pi}{1024} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+10}{8n+5}} \frac{2^{8n}}{2027025} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = &-5738221440 - 41162757273n - 127483604796n^2\\ &-223251968096n^3 - 259413765120n^4 - 234448074752n^5\\ &-20619853824n^6 + 457273901056n^7 + 473007390720n^8.\end{aligned}$$

(g) $$\frac{\pi}{1024} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+12}{8n+6}} \frac{2^{8n}}{2027025} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & 22306838400 + 143033270913n + 399843190600n^2 \\ & +643082277632n^3 + 619511610880n^4 + 80770629632n^5 \\ & -976902717440n^6 - 1550170980352n^7 - 788256522240n^8.\end{aligned}$$

(h) $\dfrac{\pi}{4096} = \sum_{n=0}^{\infty} \dfrac{\Lambda_8(n)}{\binom{16n+14}{8n+7}} \dfrac{2^{8n}}{14189175}$ *where*

$$\begin{aligned}\Lambda_8(n) = {} & -71731820160 - 416574864711n - 1043258464348n^2 \\ & -1367408224672n^3 - 479853391360n^4 + 1872980350976n^5 \\ & +4019482492928n^6 + 3508768735232n^7 + 1182652170240n^8.\end{aligned}$$

Example 26 ($m = 8$ and $x = 1/\sqrt{2}: \ U_k = \delta_{4,k} \quad \text{for} \quad 0 \le k \le 8$)**.**

(a) $\dfrac{\ln(2+\sqrt{3})}{2\sqrt{3}} = \sum_{n=0}^{\infty} \dfrac{\Lambda_8(n)}{\binom{16n}{8n}} \dfrac{2^{8n}}{10145260125}$ *where*

$$\begin{aligned}\Lambda_8(n) = {} & -144568544085 - 7908072975036n + 130241279432288n^2 \\ & -1555482884920832n^3 + 6775005483653120n^4 \\ & -17893179784331264n^5 + 24374196295565312n^6 \\ & -17854321805754368n^7 + 5152344102666240n^8.\end{aligned}$$

(b) $\dfrac{\ln(2+\sqrt{3})}{4\sqrt{3}} = \sum_{n=0}^{\infty} \dfrac{\Lambda_8(n)}{\binom{16n+2}{8n+1}} \dfrac{2^{8n}}{676350675}$ *where*

$$\begin{aligned}\Lambda_8(n) = {} & 77553796005 + 973375101552n + 6238699336160n^2 \\ & +28066082901248n^3 - 96220044820480n^4 \\ & +615594224795648n^5 - 1183820775424000n^6 \\ & +1228924022423552n^7 - 469465049333760n^8.\end{aligned}$$

(c) $\dfrac{\ln(2+\sqrt{3})}{4\sqrt{3}} = \sum_{n=0}^{\infty} \dfrac{\Lambda_8(n)}{\binom{16n+4}{8n+2}} \dfrac{2^{8n}}{10405395}$ *where*

$$\begin{aligned}\Lambda_8(n) = {} & -13635464913 - 149062057128n - 686396438208n^2 \\ & -1562987883520n^3 - 3501002416128n^4 \\ & -6120865988608n^5 + 19699744899072n^6 \\ & -38422032941056n^7 + 21848716738560n^8.\end{aligned}$$

(d) $\dfrac{\ln(2+\sqrt{3})}{8\sqrt{3}} = \sum_{n=0}^{\infty} \dfrac{\Lambda_8(n)}{\binom{16n+6}{8n+3}} \dfrac{2^{8n}}{14189175}$ *where*

$$\begin{aligned}\Lambda_8(n) = {} & 85877900325 + 786829113120n + 3105349778752n^2 \\ & +7155022805504n^3 + 9907204894720n^4 \\ & +2640937779200n^5 + 5742925447168n^6 \\ & +37020212658176n^7 - 43748771758080n^8.\end{aligned}$$

(e) $\dfrac{\ln(2+\sqrt{3})}{4\sqrt{3}} = \sum_{n=0}^{\infty} \dfrac{\Lambda_8(n)}{\binom{16n+8}{8n+4}} \dfrac{2^{8n}}{1576575}$ *where*

$$\begin{aligned}\Lambda_8(n) = \ & -122315613525 - 974609195040n - 3359961165824n^2 \\ & -6432440870912n^3 - 7264338636800n^4 \\ & -6991362129920n^5 - 9560578850816n^6 \\ & +2908028403712n^7 + 21917167779840n^8.\end{aligned}$$

(f) $$\frac{\ln(2+\sqrt{3})}{8\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+10}{8n+5}} \frac{2^{8n}}{1126125} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = \ & 146921332005 + 1018523775744n + 3054129735168n^2 \\ & +5258450628608n^3 + 6092526366720n^4 \\ & +4641031847936n^5 - 2109473292288n^6 \\ & -11726233796608n^7 - 9441965506560n^8.\end{aligned}$$

(g) $$\frac{\ln(2+\sqrt{3})}{8\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+12}{8n+6}} \frac{2^{8n}}{675675} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = \ & 5801085675 + 108589652544n + 478353163520n^2 \\ & +1117357643776n^3 + 2240546754560n^4 \\ & +3652033970176n^5 + 1996071895040n^6 \\ & -2846565072896n^7 - 3033236766720n^8.\end{aligned}$$

(h) $$\frac{\ln(2+\sqrt{3})}{16\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+14}{8n+7}} \frac{2^{8n}}{1576575} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = \ & -1337841499995 - 8074149117312n - 21216536125696n^2 \\ & -30862006626304n^3 - 19585646940160n^4 \\ & +24018500452352n^5 + 75287812898816n^6 \\ & +75740934569984n^7 + 27983641313280n^8.\end{aligned}$$

Example 27 ($m = 8$ and $x = \sqrt{3}/2: \ U_k = \delta_{8,k} \quad \text{for} \quad 0 \le k \le 8$)**.**

(a) $$\frac{2\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n}{8n}} \frac{3^{8n}}{1065008826882000} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = \ & -946942228844955 + 83900303976032622n - 528745709649647616n^2 \\ & +489318076643617 5584n^3 - 10748084909946513920n^4 \\ & +28252278541982179328n^5 - 23812225635724918784n^6 \\ & +18215968405422473216n^7 - 3683222191446425600n^8.\end{aligned}$$

(b) $$\frac{2\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+2}{8n+1}} \frac{3^{8n}}{11833431409800} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = \ & -2997230391077895 - 42570601437242898n - 258147196508495280n^2 \\ & -1050913834433875552n^3 - 836953902850589440n^4 \\ & -5322445988932989952n^5 + 3161571986361098240n^6 \\ & -4844246887438286848n^7 + 1227751047417036800n^8.\end{aligned}$$

(c) $$\frac{2\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+4}{8n+2}} \frac{3^{8n}}{60684263640} \qquad \textit{where}$$

$$\Lambda_8(n) = 984107674523169 + 11263376278746954n + 55070268255437904n^2 +147788038935170480n^3 + 248843925172529344n^4 +293766869074509824n^5 + 36000373518721024n^6 +241099806344019968n^7 - 81849990466764800n^8.$$

(d) $$\frac{2\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+6}{8n+3}} \frac{3^{8n}}{41375634300} \qquad \textit{where}$$

$$\Lambda_8(n) = -18055581168748725 - 171369586614629610n - 699175637498713536n^2 -1607187975259095152n^3 - 2279841792412179520n^4 -1954627949567014400n^5 - 1055766299322241024n^6 -796248870590578688n^7 + 409250060553420800n^8.$$

(e) $$\frac{2\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+8}{8n+4}} \frac{3^{8n}}{3064861800} \qquad \textit{where}$$

$$\Lambda_8(n) = 18053691819230025 + 147303585547653240n + 519371875458835434n^2 +1033288355628236512n^3 + 1265455632228225760n^4 +987653495934085120n^5 + 524479147239878656n^6 +128999653927026688n^7 - 136416668814540800n^8.$$

(f) $$\frac{2\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+10}{8n+5}} \frac{3^{8n}}{364864500} \qquad \textit{where}$$

$$\Lambda_8(n) = -15647036372795835 - 112257631364100048n - 348923442095829846n^2 -613897843808980416n^3 - 670716895548869920n^4 -469473373875405312n^5 - 192557144290852864n^6 +2472357256790016n^7 + 45472228091494400n^8.$$

(g) $$\frac{2\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+12}{8n+6}} \frac{3^{8n}}{72972900} \qquad \textit{where}$$

$$\Lambda_8(n) = 12169665841496625 + 78037683923217924n + 217294341302668890n^2 +343344848068605496n^3 + 336107453324767040n^4 +202702199260880896n^5 + 56833271872593920n^6 -15981516714377216n^7 - 15157406787174400n^8.$$

(h) $$\frac{2\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+14}{8n+7}} \frac{3^{8n}}{28378350} \qquad \textit{where}$$

$$\Lambda_8(n) = -8692880975894115 - 50446242288648324n - 127237975225876662n^2 -181709395348896968n^3 - 158254955693596160n^4 -79475060117578496n^5 - 12072655415195648n^6 +10379655156809728n^7 + 5052471505715200n^8.$$

Example 28 ($m = 8$ and $x = \sqrt{3}/2: \ U_k = \delta_{4,k} \quad \text{for} \quad 0 \le k \le 8$)**.**

(a) $$\frac{\sqrt{3}}{\sqrt{7}} \ln \frac{\sqrt{3}+\sqrt{7}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n}{8n}} \frac{3^{8n}}{304288236252000} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = & -17598324323222055 - 399988319650914618n - 1424666557479391296n^2 \\ & -2317700041579417129 6n^3 + 2558896260978066688 0n^4 \\ & -1670881118423568056 32n^5 + 1355310928572595240 96n^6 \\ & -1349271463986209751 04n^7 + 3004311955717488640 0n^8.\end{aligned}$$

(b) $$\frac{\sqrt{3}}{\sqrt{7}} \ln \frac{\sqrt{3}+\sqrt{7}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+2}{8n+1}} \frac{3^{8n}}{676196080560} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = & \ 1142326278070425 + 15750431823338766n + 92169404012494032n^2 \\ & +303737159618722592n^3 + 442326295255453952n^4 \\ & +115732695495996723 2n^5 - 4700965644849725 44n^6 \\ & +1106807495413268480n^7 - 303586734309376000n^8.\end{aligned}$$

(c) $$\frac{\sqrt{3}}{\sqrt{7}} \ln \frac{\sqrt{3}+\sqrt{7}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+4}{8n+2}} \frac{3^{8n}}{86691805200} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = & -2783870604528615 - 30837281604102054n - 145501866071461872n^2 \\ & -380529928492669648n^3 - 622852526912266560n^4 \\ & -660489348750708736n^5 - 168050350255054848n^6 \\ & -492819721256108032n^7 + 179191755354931200n^8.\end{aligned}$$

(d) $$\frac{\sqrt{3}}{\sqrt{7}} \ln \frac{\sqrt{3}+\sqrt{7}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+6}{8n+3}} \frac{3^{8n}}{11821609800} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = & \ 4760212257941175 + 44150553108184590n + 176076197310769056n^2 \\ & +395403782635150352n^3 + 545700680640337600n^4 \\ & +459955889916239360n^5 + 256327005075656704n^6 \\ & +160916053450784768n^7 - 90068263947468800n^8.\end{aligned}$$

(e) $$\frac{\sqrt{3}}{\sqrt{7}} \ln \frac{\sqrt{3}+\sqrt{7}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+8}{8n+4}} \frac{3^{8n}}{875674800} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = & -2458805369159475 - 19562412999440520n - 67221665573292846n^2 \\ & -130259484124991648n^3 - 155645863989771680n^4 \\ & -119390575630592000n^5 - 60861540284862464n^6 \\ & -11161120256294912n^7 + 15168839414579200n^8.\end{aligned}$$

(f) $$\frac{\sqrt{3}}{\sqrt{7}} \ln \frac{\sqrt{3}+\sqrt{7}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+10}{8n+5}} \frac{3^{8n}}{20849400} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = & \ 129554766794565 + 858169653704208n + 2448437197642506n^2 \\ & +3935952396074496n^3 + 3917175156389600n^4 \\ & +2422362292449792n^5 + 670602955710464n^6 \\ & -236188919955456n^7 - 162460804710400n^8.\end{aligned}$$

(g) $$\frac{\sqrt{3}}{\sqrt{7}} \ln \frac{\sqrt{3}+\sqrt{7}}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+12}{8n+6}} \frac{3^{8n}}{20849400} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & 1026221430482325 + 6974899861698612n + 20502090900169410n^2\\ & +34083096082428568n^3 + 35149048592490560n^4\\ & +22784322268653568n^5 + 7590091174830080n^6\\ & -1269804089704448n^7 - 1851219882803200n^8.\end{aligned}$$

(h) $\dfrac{\sqrt{3}}{\sqrt{7}} \ln \dfrac{\sqrt{3}+\sqrt{7}}{2} = \sum_{n=0}^{\infty} \dfrac{\Lambda_8(n)}{\binom{16n+14}{8n+7}} \dfrac{3^{8n}}{8108100}$ *where*

$$\begin{aligned}\Lambda_8(n) = {} & -3673392339712095 - 22119522242195412n - 57814129810869726n^2\\ & -85584560153285704n^3 - 77703187199357440n^4\\ & -41613154206308608n^5 - 7998181102987264n^6\\ & +4831301025972224n^7 + 2739061184921600n^8.\end{aligned}$$

Example 29 ($m = 8$ and $x = \sqrt{3}/2: \ U_k = \delta_{2,k} \quad \text{for} \quad 0 \le k \le 8$)**.**

(a) $\dfrac{2\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} \dfrac{\Lambda_8(n)}{\binom{16n}{8n}} \dfrac{3^{8n}}{319502648064600}$ *where*

$$\begin{aligned}\Lambda_8(n) = {} & 1269138955648095 + 65815965905219946n + 163695939874324032n^2\\ & +224734874688439072n^3 + 272862115596378112 0n^4\\ & +247788104658087526 4n^5 + 245123629242961100 8n^6\\ & +268168631871458508 8n^7 - 105239216736174080 0n^8.\end{aligned}$$

(b) $\dfrac{2\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} \dfrac{\Lambda_8(n)}{\binom{16n+2}{8n+1}} \dfrac{3^{8n}}{17750147114700}$ *where*

$$\begin{aligned}\Lambda_8(n) = {} & -3570666537370905 - 46765175380286382n - 259903297808728560n^2\\ & -1167364731188323808n^3 + 2367411544002265 60n^4\\ & -7772825694165905408n^5 + 5833086463029575680n^6\\ & -6453273269238628352n^7 + 1501449013244723200n^8.\end{aligned}$$

(c) $\dfrac{2\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} \dfrac{\Lambda_8(n)}{\binom{16n+4}{8n+2}} \dfrac{3^{8n}}{91026395460}$ *where*

$$\begin{aligned}\Lambda_8(n) = {} & 1871889719002191 + 21613146769656342n + 106934527964382048n^2\\ & +291232744488937328n^3 + 499332461630285248n^4\\ & +595189557415319552n^5 + 620933253443829 76n^6\\ & +505424810481483776n^7 - 169388389439897600n^8.\end{aligned}$$

(d) $\dfrac{2\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} \dfrac{\Lambda_8(n)}{\binom{16n+6}{8n+3}} \dfrac{3^{8n}}{62063451450}$ *where*

$$\begin{aligned}\Lambda_8(n) = {} & -37825332701028075 - 359533984207273110n - 1467947215143507168n^2\\ & -3372048361584325136n^3 - 4771744432095296320n^4\\ & -4088959093882664960n^5 - 2224491325711900672n^6\\ & -1636338842285441024n^7 + 849590946489958400n^8.\end{aligned}$$

(e) $\dfrac{2\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} \dfrac{\Lambda_8(n)}{\binom{16n+8}{8n+4}} \dfrac{3^{8n}}{919458540}$ *where*

$$\begin{aligned}\Lambda_8(n) = {} & 7200135834802155 + 58598783255624616n + 206053324029987966n^2 \\ & +408906721802074720n^3 + 499801146648214432n^4 \\ & +389344923510796288n^5 + 205750199167012864n^6 \\ & +49993379563307008n^7 - 5348976968007680 0n^8.\end{aligned}$$

(f) $$\frac{2\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+10}{8n+5}} \frac{3^{8n}}{547296750} \qquad \text{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & -30390089316738165 - 218035855906636752n - 678012483406256154n^2 \\ & -1193774443409004384n^3 - 1305204895558285280n^4 \\ & -914472382629746688n^5 - 376692962335502336n^6 \\ & +3518792495529984n^7 + 89023473310105600n^8.\end{aligned}$$

(g) $$\frac{2\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+12}{8n+6}} \frac{3^{8n}}{109459350} \qquad \text{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & 25035572780670375 + 161135230032837036n + 450280971547542750n^2 \\ & +713923302199200184n^3 + 701495333337850880n^4 \\ & +425366410997392384n^5 + 120697810899230720n^6 \\ & -33029826692808704n^7 - 31924224498073600n^8.\end{aligned}$$

(h) $$\frac{2\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+14}{8n+7}} \frac{3^{8n}}{42567525} \qquad \text{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & -19227430150253325 - 111347369463459708n - 279958811965296954n^2 \\ & -398135088157369976n^3 - 344558784014500160n^4 \\ & -170717032475988992n^5 - 24159905899544576n^6 \\ & +22915987550273536n^7 + 10704479577702400n^8.\end{aligned}$$

Example 30 ($m = 8$ and $x = \sqrt{3}/2 : \ U_k = \delta_{2,k} \quad \text{for} \quad 0 \le k \le 8$)**.**

(a) $$\frac{\sqrt{3}}{4}\ln(2+\sqrt{3}) = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n}{8n}} \frac{3^{8n}}{53250441344100} \qquad \text{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & -60857959324365 - 3257421973654797n - 50209358056219 74n^2 \\ & -219002657480722304n^3 + 1774715785457585 60n^4 \\ & -188523663257954124 8n^5 + 159837593305856614 4n^6 \\ & -172131508387183001 6n^7 + 38816736802242560 0n^8.\end{aligned}$$

(b) $$\frac{\sqrt{3}}{4}\ln(2+\sqrt{3}) = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+2}{8n+1}} \frac{3^{8n}}{2958357852450} \qquad \text{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & 357300424624185 + 4998897955132839n + 29109271110018120n^2 \\ & +105346682300985916n^3 + 102017222335782880n^4 \\ & +369337818953271616n^5 - 142769249679311360n^6 \\ & +304953103978491904n^7 - 83949831345766400n^8.\end{aligned}$$

(c) $$\frac{\sqrt{3}}{4}\ln(2+\sqrt{3}) = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+4}{8n+2}} \frac{3^{8n}}{15171065910} \qquad \text{where}$$

$$\begin{aligned}\Lambda_8(n) = & -63777708460557 - 721317346535529n - 3491246917775106n^2 \\ & -9287239125521896n^3 - 15785354924836896n^4 \\ & -19615009150086784n^5 - 962286036052992n^6 \\ & -16636798483750912n^7 + 5475242955571200n^8.\end{aligned}$$

(d) $$\frac{\sqrt{3}}{4}\ln(2+\sqrt{3}) = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+6}{8n+3}} \frac{3^{8n}}{10343908575} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = & \; 1401940330547400 + 13389307998403095n + 55035169418070261n^2 \\ & +127524227890842622n^3 + 182069933558236640n^4 \\ & +156408059737877920n^5 + 84996183314572544n^6 \\ & +65581034325096448n^7 - 3328176493895680 0n^8.\end{aligned}$$

(e) $$\frac{\sqrt{3}}{4}\ln(2+\sqrt{3}) = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+8}{8n+4}} \frac{3^{8n}}{153243090} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = & -297569238088185 - 2429345167818957n - 8563417322357142n^2 \\ & -17017479156859720n^3 - 20813812050631264n^4 \\ & -16243648688718976n^5 - 8602250843296768n^6 \\ & -2057115589697536n^7 + 2224303077785600n^8.\end{aligned}$$

(f) $$\frac{\sqrt{3}}{4}\ln(2+\sqrt{3}) = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+10}{8n+5}} \frac{3^{8n}}{91216125} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = & \; 1209236142439080 + 8641586480818329n + 26756030673902583n^2 \\ & +46907983726518918n^3 + 51078921291638560n^4 \\ & +35589842802598176n^5 + 14461189431830272n^6 \\ & -251909808826368n^7 - 3425955121971200n^8.\end{aligned}$$

(g) $$\frac{\sqrt{3}}{4}\ln(2+\sqrt{3}) = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+12}{8n+6}} \frac{3^{8n}}{18243225} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = & -892809587698875 - 5729557491907497n - 15978898544493000n^2 \\ & -25300410940826068n^3 - 24831535877593760n^4 \\ & -15053350166507968n^5 - 4299740141949440n^6 \\ & +1150364372676608n^7 + 1138043077427200n^8.\end{aligned}$$

(h) $$\frac{\sqrt{3}}{4}\ln(2+\sqrt{3}) = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+14}{8n+7}} \frac{3^{8n}}{14189175} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = & \; 1623797403903675 + 9572843237512932n + 24516397762338741n^2 \\ & +35558643617532854n^3 + 31536149292280640n^4 \\ & +16269331358718368n^5 + 2710600603445504n^6 \\ & -2044470151481344n^7 - 1039912059289600n^8.\end{aligned}$$

Example 31 ($m = 8$ and $x = 1/\sqrt[4]{2}: \; U_k = \delta_{1,k} \quad \text{for} \quad 0 \le k \le 8$).

(a) $$\frac{\pi}{2^4\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n}{8n}} \frac{2^{12n}}{91307341125} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = &-8643962880 + 497377500327n - 3920074288736n^2\\ &+29906187699584n^3 - 75644859207680n^4\\ &+174088564686848n^5 - 162090491838464n^6\\ &+110015509692416n^7 - 23780697047040n^8.\end{aligned}$$

(b) $\dfrac{\pi}{2^7\sqrt{2}} = \sum_{n=0}^{\infty} \dfrac{\Lambda_8(n)}{\binom{16n+2}{8n+1}} \dfrac{2^{12n}}{6087156075}$ *where*

$$\begin{aligned}\Lambda_8(n) = &-9264568320 - 129235661283n - 778004293100n^2\\ &-3511093480832n^3 - 735357539840n^4\\ &-21952815890432n^5 + 18858971299840n^6\\ &-22552172822528n^7 + 6323039109120n^8.\end{aligned}$$

(c) $\dfrac{\pi}{2^7\sqrt{2}} = \sum_{n=0}^{\infty} \dfrac{\Lambda_8(n)}{\binom{16n+4}{8n+2}} \dfrac{2^{12n}}{93648555}$ *where*

$$\begin{aligned}\Lambda_8(n) = &\ 9077225472 + 103971761775n + 509171753808n^2\\ &+1355077327744n^3 + 2298810759168n^4\\ &+2903958077440n^5 - 384234160128n^6\\ &+3229452599296n^7 - 1258794516480n^8.\end{aligned}$$

(d) $\dfrac{\pi}{2^9\sqrt{2}} = \sum_{n=0}^{\infty} \dfrac{\Lambda_8(n)}{\binom{16n+6}{8n+3}} \dfrac{2^{12n}}{25540515}$ *where*

$$\begin{aligned}\Lambda_8(n) = &-16638289920 - 157898552709n - 643946137480n^2\\ &-1481675500928n^3 - 2107709621248n^4\\ &-1765981995008n^5 - 945035542528n^6\\ &-986452066304n^7 + 629397258240n^8.\end{aligned}$$

(e) $\dfrac{\pi}{2^{10}\sqrt{2}} = \sum_{n=0}^{\infty} \dfrac{\Lambda_8(n)}{\binom{16n+8}{8n+4}} \dfrac{2^{12n}}{14189175}$ *where*

$$\begin{aligned}\Lambda_8(n) = &\ 62418585600 + 509311502355n + 1796096500064n^2\\ &+3573090534272n^3 + 4366813122560n^4\\ &+3415902371840n^5 + 1917348478976n^6\\ &+445852418048n^7 - 787186974720n^8.\end{aligned}$$

(f) $\dfrac{\pi}{2^{13}\sqrt{2}} = \sum_{n=0}^{\infty} \dfrac{\Lambda_8(n)}{\binom{16n+10}{8n+5}} \dfrac{2^{12n}}{3378375}$ *where*

$$\begin{aligned}\Lambda_8(n) = &-13522360320 - 97011149541n - 301493147492n^2\\ &-530350884992n^3 - 580269928960n^4\\ &-409158549504n^5 - 163458121728n^6\\ &+28408020992n^7 + 65556971520n^8.\end{aligned}$$

(g) $\dfrac{\pi}{2^{13}\sqrt{2}} = \sum_{n=0}^{\infty} \dfrac{\Lambda_8(n)}{\binom{16n+12}{8n+6}} \dfrac{2^{12n}}{6081075}$ *where*

$$\begin{aligned}\Lambda_8(n) = & \ 94629427200 + 606817090617n + 1689902590960n^2 \\ & +2671604367488n^3 + 2614671861760n^4 \\ & +1546925785088n^5 + 330292264960n^6 \\ & -281980960768n^7 - 196796743680n^8.\end{aligned}$$

(h) $$\frac{\pi}{2^{15}\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+14}{8n+7}} \frac{2^{12n}}{14189175} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = & -101569628160 - 589498782567n - 1486674313816n^2 \\ & -2118711382144n^3 - 1820019174400n^4 \\ & -840152793088n^5 + 1999241216n^6 \\ & +239656239104n^7 + 98398371840n^8.\end{aligned}$$

The formula (a) in the last example is obtained by Zheng [10, Example 4.2].

Example 32 ($m = 8$ and $x = 1/\sqrt[4]{2}: \ U_k = \delta_{1,k} \quad \text{for} \quad 0 \le k \le 8$)**.**

(a) $$\frac{\ln(1+\sqrt{2})}{\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n}{8n}} \frac{2^{12n}}{91307341125} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = & \ 37293786495 - 2265848796168n + 17208323645824n^2 \\ & -135367347168256n^3 + 331260153118720n^4 \\ & -784745376120832n^5 + 705943123787776n^6 \\ & -501758027628544n^7 + 112356344463360n^8.\end{aligned}$$

(b) $$\frac{\ln(1+\sqrt{2})}{2\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+2}{8n+1}} \frac{2^{12n}}{6087156075} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = & \ 165699203985 + 2309247312144n + 13884747187840n^2 \\ & +62677691324416n^3 + 12235348787200n^4 \\ & +392210897698816n^5 - 339960701911040n^6 \\ & +402249087975424n^7 - 112356344463360n^8.\end{aligned}$$

(c) $$\frac{\ln(1+\sqrt{2})}{2\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+4}{8n+2}} \frac{2^{12n}}{93648555} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = & -162915680781 - 1866255779808n - 9141052769280n^2 \\ & -24334753673216n^3 - 41305275432960n^4 \\ & -52214539747328n^5 + 6810903773184n^6 \\ & -58107216527360n^7 + 22657294663680n^8.\end{aligned}$$

(d) $$\frac{\ln(1+\sqrt{2})}{4\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+6}{8n+3}} \frac{2^{12n}}{127702575} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = & \ 2989317956625 + 28369250263680n + 115696086236672n^2 \\ & +266197226143744n^3 + 378619609088000n^4 \\ & +317130192977920n^5 + 169538767290368n^6 \\ & +176952249942016n^7 - 112948244643840n^8.\end{aligned}$$

(e) $$\frac{\ln(1+\sqrt{2})}{2\sqrt{2}}=\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+8}{8n+4}}\frac{2^{12n}}{14189175}\qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n)=\;&-8960938596225-73113942969600n-257823631158272n^2\\&-512881909170176n^3-626809040076800n^4\\&-490384601907200n^5-275399938408448n^6\\&-64124264382464n^7+113060987535360n^8.\end{aligned}$$

(f) $$\frac{\ln(1+\sqrt{2})}{4\sqrt{2}}=\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+10}{8n+5}}\frac{2^{12n}}{10135125}\qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n)=\;&23320402490385+167318069061888n+520048500243456n^2\\&+914913206829056n^3+1001123479879680n^4\\&+705883403190272n^5+281880406523904n^6\\&-49073625235456n^7-113060987535360n^8.\end{aligned}$$

(g) $$\frac{\ln(1+\sqrt{2})}{4\sqrt{2}}=\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+12}{8n+6}}\frac{2^{12n}}{2027025}\qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n)=\;&-18115301479275-116147319842304n-323387021849600n^2\\&-511114737319936n^3-500076716687360n^4\\&-295795463028736n^5-63156361625600n^6\\&+53924253925376n^7+37644046172160n^8.\end{aligned}$$

(h) $$\frac{\ln(1+\sqrt{2})}{8\sqrt{2}}=\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+14}{8n+7}}\frac{2^{12n}}{405405}\qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n)=\;&3333439442619+19353755760384n+48836458734592n^2\\&+69658246807552n^3+59911135756288n^4\\&+27707475755008n^5-35190210560n^6\\&-7887304785920n^7-3241358131200n^8.\end{aligned}$$

Example 33 **(**$m=8$ and $x=1/\sqrt[4]{8}:\ U_k=\delta_{2,k}\quad \text{for}\quad 0\le k\le 8$**).**

(a) $$\frac{\pi}{8}=\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n}{8n}}\frac{16^n}{91307341125}\qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n)=\;&-161344489320+25614625793643n-770729148487564n^2\\&+7862280573528976n^3-39504767215782400n^4\\&+105888232822392832n^5-150593996131188736n^6\\&+106176922020020224n^7-28720407804641280n^8.\end{aligned}$$

(b) $$\frac{\pi}{16}=\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+2}{8n+1}}\frac{16^n}{6087156075}\qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n)=\;&-40105671480-384962359983n-3069126475660n^2\\&-44962268285872n^3+298235928609920n^4\\&-1511592786737152n^5+3998040747868160n^6\\&-5151658053664768n^7+2421793774632960n^8.\end{aligned}$$

(c) $$\frac{\pi}{32}=\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+4}{8n+2}}\frac{16^n}{468242775}\qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & 3179577240+29487166029n+199699899462n^2\\ & -2053005796072n^3-4078800401760n^4\\ & +42326907800576n^5-61494863818752n^6\\ & +16348516155392n^7+3058880348160n^8.\end{aligned}$$

(d) $$\frac{\pi}{128}=\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+6}{8n+3}}\frac{16^n}{127702575}\qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & -2348816400-23289828495n-75846862343n^2\\ & -644726217436n^3-2401518444080n^4\\ & +8020529968640n^5+10000148236288n^6\\ & -55651743023104n^7+44447157780480n^8.\end{aligned}$$

(e) $$\frac{\pi}{32}=\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+8}{8n+4}}\frac{16^n}{4729725}\qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & 11996665800+99448219775n+371686651812n^2\\ & +624647762256n^3-424759914880n^4\\ & +1185632430080n^5+14605167689728n^6\\ & -367033778176n^7-41893617991680n^8.\end{aligned}$$

(f) $$\frac{\pi}{64}=\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+10}{8n+5}}\frac{16^n}{2027025}\qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & -18305413560-130277164977n-387231116964n^2\\ & -667794867664n^3-1543579299840n^4\\ & -3462273230848n^5+3226496385024n^6\\ & +24041686237184n^7+23384681349120n^8.\end{aligned}$$

(g) $$\frac{\pi}{128}=\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+12}{8n+6}}\frac{16^n}{2027025}\qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & 31401631800+200872589931n+584291935270n^2\\ & +1122853271144n^3+745480701280n^4\\ & -6409666149376n^5-24960823316480n^6\\ & -35336891367424n^7-17735494533120n^8.\end{aligned}$$

(h) $$\frac{\pi}{512}=\sum_{n=0}^{\infty}\frac{\Lambda_8(n)}{\binom{16n+14}{8n+7}}\frac{16^n}{14189175}\qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & -103937193360-613234491309n-1453171767397n^2\\ & +356143042052n^3+16232067946640n^4\\ & +58224395124224n^5+99822832535552n^6\\ & +85083343044608n^7+28792603607040n^8.\end{aligned}$$

Example 34 ($m = 8$ and $x = 1/\sqrt[4]{8}:\ U_k = \delta_{2,k} \quad \text{for} \quad 0 \le k \le 8$)**.**

(a) $$\frac{\ln(1+\sqrt{2})}{\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n}{8n}} \frac{16^n}{91307341125} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & 132176123415 + 19506257644464n - 588203637070672n^2 \\ & +7119443059042048n^3 - 39020538792390400n^4 \\ & +121545619422631936n^5 - 206696043521916928n^6 \\ & +179055553418493952n^7 - 60623055948349440n^8.\end{aligned}$$

(b) $$\frac{\ln(1+\sqrt{2})}{2\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+2}{8n+1}} \frac{16^n}{6087156075} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & -17388445095 - 420339509628n + 944832494480n^2 \\ & +43585437293888n^3 - 546355905598720n^4 \\ & +2947670271300608n^5 - 6804422147891200n^6 \\ & +6928838373343232n^7 - 2502643970211840n^8.\end{aligned}$$

(c) $$\frac{\ln(1+\sqrt{2})}{2\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+4}{8n+2}} \frac{16^n}{93648555} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & -6083240373 - 74633108688n - 375950908368n^2 \\ & -190239013888n^3 - 852591188736n^4 \\ & -6459549902848n^5 + 47285416378368n^6 \\ & -126876282388480n^7 + 95934732042240n^8.\end{aligned}$$

(d) $$\frac{\ln(1+\sqrt{2})}{4\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+6}{8n+3}} \frac{16^n}{127702575} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & 7293231225 + 31872059580n - 74146288496n^2 \\ & +645310839488n^3 + 6651598723840n^4 \\ & -3986345681920n^5 - 29138914402304n^6 \\ & +50572360220672n^7 - 38235199242240n^8.\end{aligned}$$

(e) $$\frac{\ln(1+\sqrt{2})}{2\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+8}{8n+4}} \frac{16^n}{4729725} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & -6729004275 - 64170810920n - 287843549616n^2 \\ & -470008172928n^3 + 1198371408640n^4 \\ & +1148376248320n^5 - 13704299413504n^6 \\ & -6653952131072n^7 + 33179109949440n^8.\end{aligned}$$

(f) $$\frac{\ln(1+\sqrt{2})}{4\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+10}{8n+5}} \frac{16^n}{10135125} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = {} & 105945772905 + 746407091664n + 2131339010448n^2 \\ & +3710240035328n^3 + 10495824633600n^4 \\ & +25833173129216n^5 - 13542919815168n^6 \\ & -143230893948928n^7 - 141579088035840n^8.\end{aligned}$$

(g) $$\frac{\ln(1+\sqrt{2})}{4\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+12}{8n+6}} \frac{16^n}{2027025} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = \; & -78019142355 - 506961709416n - 1541852718320n^2 \\ & -2918598465664n^3 - 881353268480n^4 \\ & +19214916816896n^5 + 64368329359360n^6 \\ & +85445800361984n^7 + 41354731192320n^8.\end{aligned}$$

(h) $$\frac{\ln(1+\sqrt{2})}{8\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+14}{8n+7}} \frac{16^n}{14189175} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = \; & 317088196425 + 1691112879504n + 2580229649872n^2 \\ & -8176837690112n^3 - 60957371767040n^4 \\ & -180739522955264n^5 - 291031977377792n^6 \\ & -243173097340928n^7 - 82022435389440n^8.\end{aligned}$$

Example 35 ($m = 8$ and $x = 1/\sqrt[4]{32}:\ U_k = \delta_{1,k}$ for $0 \le k \le 8$)**.**

(a) $$\pi = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n}{8n}} \frac{16^{-n}}{91307341125} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = \; & -869897157255 - 3524219363487888n + 112466777263118189n^2 \\ & -1242789726208374386n^3 + 6693196178751930680n^4 \\ & -19768094496651298112n^5 + 32808347163463348736n^6 \\ & -28892659596072587264n^7 + 10530503748472012800n^8.\end{aligned}$$

(b) $$\pi = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+2}{8n+1}} \frac{16^{-n}}{2434862430} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = \; & -31888037475 + 11029268952492n - 137665162230431n^2 \\ & -1968918306120206n^3 + 36303851970633184n^4 \\ & -204571104162140576n^5 + 536268876501175552n^6 \\ & -679668438248683520n^7 + 332081830164480000n^8.\end{aligned}$$

(c) $$\pi = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+4}{8n+2}} \frac{16^{-n}}{3745942200} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = \; & -2357021835 + 1416452012784n + 89977498903737n^2 \\ & +312275144607958n^3 - 6507509169068040n^4 \\ & +41870139018050 56n^5 + 110556738165931008n^6 \\ & -361139911920508928n^7 + 288405095254425600n^8.\end{aligned}$$

(d) $$\pi = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+6}{8n+3}} \frac{16^{-n}}{255405150} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = \; & 16225276725 + 1021658183430n + 17701727448137n^2 \\ & -63192337663196n^3 - 434333155893700n^4 \\ & +2211288737692720n^5 + 63629985819008n^6 \\ & -19612973861080064n^7 + 17713948653772800n^8.\end{aligned}$$

(e) $\pi = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+8}{8n+4}} \frac{16^{-n}}{56756700}$ *where*

$$\begin{aligned}\Lambda_8(n) = \ & 37369831275 + 840414655380n + 15097490896399n^2 \\ & +5670116889262n^3 - 456189127408280n^4 \\ & -105343164584000n^5 + 4195636645935616n^6 \\ & -2055719522127872n^7 - 1697802388930560 0n^8.\end{aligned}$$

(f) $\pi = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+10}{8n+5}} \frac{16^{-n}}{16216200}$ *where*

$$\begin{aligned}\Lambda_8(n) = \ & -16145685045 + 299876649696n + 13925352355647n^2 \\ & +52611558029002n^3 - 359857182535200n^4 \\ & -1802199790299296n^5 + 4241722726279 68n^6 \\ & +10139314481072128n^7 + 10666506747494400n^8.\end{aligned}$$

(g) $\pi = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+12}{8n+6}} \frac{16^{-n}}{64864800}$ *where*

$$\begin{aligned}\Lambda_8(n) = \ & 753411980475 + 8706166866756n + 206966802505855n^2 \\ & +1479671613214334n^3 - 95018740868120n^4 \\ & -32548351229564416n^5 - 117296001757199360n^6 \\ & -163153200501121024n^7 - 81110105154355200n^8.\end{aligned}$$

(h) $\pi = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+14}{8n+7}} \frac{16^{-n}}{113513400}$ *where*

$$\begin{aligned}\Lambda_8(n) = \ & -465207237495 - 1113386805402n + 911464216704829n^2 \\ & +14019464108700316n^3 + 84025330872079660n^4 \\ & +254130120556257232n^5 + 412246533152304256n^6 \\ & +342074608255151104n^7 + 113954848384819200n^8.\end{aligned}$$

The formula (a) in the last example can be found in [1, Example 3.3] and [10, Example 3.2].

Example 36 ($m = 8$ and $x = 1/\sqrt[4]{32}: \ U_k = \delta_{1,k} \quad \text{for} \quad 0 \le k \le 8$)**.**

(a) $\ln 2 = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n}{8n}} \frac{16^{-n}}{11687339664000}$ *where*

$$\begin{aligned}\Lambda_8(n) = \ & -216500813865 + 361331474368068126n - 11350183857053970928n^2 \\ & +122495341798811565472n^3 - 638721081045422287360n^4 \\ & +1803655225141389082624n^5 - 2819074826980059676672n^6 \\ & +2283478369362404835328n^7 - 739092735370828185600n^8.\end{aligned}$$

(b) $\ln 2 = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+2}{8n+1}} \frac{16^{-n}}{77915597760}$ *where*

$$\begin{aligned}\Lambda_8(n) = \ & -357525247275 - 55976773238526n + 522639413428528n^2 \\ & +8592761566691008n^3 - 154592820720488192n^4 \\ & +780841645872429568n^5 - 1768420698754174976n^6 \\ & +1681579429371412480n^7 - 367430322192384000n^8.\end{aligned}$$

(c) $$\ln 2 = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+4}{8n+2}} \frac{16^{-n}}{29967537600} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = \ & -546631100505 - 19494570294198n - 88974674819064n^2 \\ & +1311649781557024n^3 - 15597250259445120n^4 \\ & -14554676302380032n^5 + 333996711591911424n^6 \\ & -1034546136876253184n^7 + 1060015059684556800n^8.\end{aligned}$$

(d) $$\ln 2 = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+6}{8n+3}} \frac{16^{-n}}{4086482400} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = \ & -134737150275 - 4243678495170n - 64630341455828n^2 \\ & -394959291933976n^3 + 275373394844800n^4 \\ & +2765514190248320n^5 - 27442792015127552n^6 \\ & -31838806120259584n^7 + 164668133174476800n^8.\end{aligned}$$

(e) $$\ln 2 = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+8}{8n+4}} \frac{16^{-n}}{1816214400} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = \ & -56752842825 - 640065206919 0n - 95450001638087n^2 \\ & -322624861897856n^3 - 297914935097360n^4 \\ & -6260373176960000n^5 - 20308669690308608n^6 \\ & +26141400367169536n^7 + 90666260835532800n^8.\end{aligned}$$

(f) $$\ln 2 = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+10}{8n+5}} \frac{16^{-n}}{43243200} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = \ & 14317013655 - 348697629294n - 7663918858583n^2 \\ & -44503449922328n^3 - 67863393566800n^4 \\ & +31022674998144n^5 - 431224455140352n^6 \\ & -2012612682145792n^7 - 1932252404121600n^8.\end{aligned}$$

(g) $$\ln 2 = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+12}{8n+6}} \frac{16^{-n}}{389188800} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = \ & 81581571225 - 13161624487674n - 255231896215045n^2 \\ & -1767327940259936n^3 - 4400859325233520n^4 \\ & +3067988714040064n^5 + 33879809082429440n^6 \\ & +56125117560733696n^7 + 29824842935500800n^8.\end{aligned}$$

(h) $$\ln 2 = \sum_{n=0}^{\infty} \frac{\Lambda_8(n)}{\binom{16n+14}{8n+7}} \frac{16^{-n}}{227026800} \qquad \textit{where}$$

$$\begin{aligned}\Lambda_8(n) = \ & 898798560660 - 20673643870839n - 496510232712647n^2 \\ & -4452375890189213n^3 - 21120196275799880n^4 \\ & -56688653329948976n^5 - 85960734474651008n^6 \\ & -68451297460585472n^7 - 22202202377625600n^8.\end{aligned}$$

Finally, we point out that most formulae derived in this paper for π and logarithm functions involving the binomial coefficients $\binom{2mn+2\gamma}{mn+\gamma}$ with $\gamma \not\equiv 0 \pmod{m}$ have not appeared previously in literature.

References

[1] G. Almkvist *et al.*, *Some new formulas for π*, Exper. Math. 12 (2003), 441–456.

[2] R. Apéry, *Irrationalitié de $\zeta(2)$ et $\zeta(3)$*, Journees Arithmétiques de Luminy: Astérisque 61 (1979), 11–13.

[3] J. M. Borwein and R. Girgensohn, *Evaluations of binomial series,* Aequationes Math. 70 (2005), 25–36.

[4] W. Chu and D. Y. Zheng, *Infinite series with harmonic numbers and central binomial coefficients,* Int. J. Number Theory 5:3 (2009), 429–448.

[5] L. Comtet, *Advanced Combinatorics,* Reidel, Boston, Massachusetts, 1974.

[6] C. Elsner, *On recurrence formulae for sums involving Binomial Coefficients,* Fibonacci Quart. 43:1 (2005), 31–45.

[7] D. H. Lehmer, *Interesting series involving the central binomial coefficient,* Amer. Math. Monthly 92 (1985), 449–457.

[8] H. Muzaffar, *Some interesting seires arising from the power series expansion of $(\sin^{-1} x)^q$*, Int. J. Math. Math. Sci. 14 (2005), 2329–2336.

[9] A. J. van der Poorten, *A proof that Euler missed,* Math. Intelligencer 1 (1979), 195–203.

[10] D. Y. Zheng, *Multisection method and further formulae for π,* Indian J. Pure Appl. Math. 39:2 (2008), 137–155.

[11] I. J. Zucker, *On the series $\sum_{k=1}^{\infty}\binom{2k}{k}^{-1}k^{-n}$*, J. Number Theory 20:1 (1985), 92–102.

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