SEVERAL EXPLICIT FORMULAE OF SUMS AND HYPER-SUMS OF POWERS OF INTEGERS

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Abstract. In this paper, we present several explicit formulas of the sums and hyper-sums of the powers of the first $(n+1)$-terms of a general arithmetic sequence in terms of Stirling numbers and generalized Bernoulli polynomials.

1. Introduction

The problem of finding formulas for sums of powers of integers has attracted the attention of many mathematicians and has been developed in several different directions. For a recent treatment and references, see [2, 3, 8, 10, 12, 13, 14]. This paper is concerned both with sums $S_{p,(a,d)}(n)$ and hyper-sums $S_{p,(a,d)}^{(r)}(n)$ of the p-th powers of the first $(n+1)$-terms of a general arithmetic sequence. Let

$$\begin{aligned} S_{p,(a,d)}(n) &= a^p + (a+d)^p + \cdots + (a+nd)^p \\ &= \sum_{k=0}^{n} (a+kd)^p , \end{aligned}$$

be the power sum of arithmetic progression where n, p are non-negative integers and a and d are complex numbers with $d \neq 0$. For the most studied case $a = 0$ and $d = 1$,

$$S_{p,(0,1)}(n) = \begin{cases} n+1 & (p=0) \\ 1^p + 2^p + 3^p + \cdots + n^p & (p>0) \end{cases},$$

there have been a considerable number of results are known.

The basic properties for the $S_{p,(a,d)}(n)$ can be obtained from the following generating function [9]

$$\sum_{p\geq 0} S_{p,(a,d)}(n) \frac{z^p}{p!} = \sum_{k=0}^{n} e^{(a+kd)z}, \tag{1}$$

and we can easily verify that [13]

$$S_{p,(a,d)}(n) = \frac{d^p}{p+1}\left(B_{p+1}\left(n+\frac{a}{d}+1\right) - B_{p+1}\left(\frac{a}{d}\right)\right), \tag{2}$$

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where $B_n(x)$ denotes the classical Bernoulli polynomials, defined by the following generating function

$$\sum_{n\geq 0} B_n(x) \frac{z^n}{n!} = \frac{ze^{xz}}{e^z - 1}.$$

Recall that the weighted Stirling numbers $\mathcal{S}_n^k(x)$ of the second kind are defined by (see [6, 7])

$$\mathcal{S}_n^k(x) = \frac{1}{k!}\Delta^k x^n \tag{3}$$

$$= \frac{1}{k!}\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x+j)^n, \tag{4}$$

where Δ denotes the forward difference operator. The exponential generating function of $\mathcal{S}_n^k(x)$ is given by

$$\sum_{n=k}^{\infty} \mathcal{S}_n^k(x) \frac{z^n}{n!} = \frac{1}{k!} e^{xz} (e^z - 1)^k, \tag{5}$$

and $\mathcal{S}_n^k(x)$ satisfies the following recurrence relation

$$\mathcal{S}_{n+1}^k(x) = \mathcal{S}_n^{k-1}(x) + (x+k)\mathcal{S}_n^k(x), \qquad (1 \leq k \leq n).$$

In particular, we have for non-negative integer r

$$\mathcal{S}_n^k(0) = \left\{ {n \atop k} \right\} \text{ and } \mathcal{S}_n^k(r) = \left\{ {n+r \atop k+r} \right\}_r,$$

where $\left\{ {n \atop k} \right\}_r$ denotes the r-Stirling numbers of the second kind [5]. These numbers count the number of partitions of a set of n objects into exactly k non-empty disjoint subsets, such that the first r elements are in distinct subsets.

For any positive integer m, the r-Whitney numbers of the second kind $W_{m,r}(n,k)$ are the coefficients in the expansion

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k) x(x+1)\cdots(x+k-1),$$

and are given by generating function

$$\sum_{n\geq k} W_{m,r}(n,k) \frac{z^n}{n!} = \frac{1}{m^k k!} e^{rz} (e^{mz} - 1)^k.$$

Clearly, we have

$$W_{1,0}(n,k) = \left\{ {n \atop k} \right\}, \quad W_{1,r}(n,k) = \left\{ {n+r \atop k+r} \right\}_r$$

and

$$W_{m,r}(n,k) = m^{n-k} \mathcal{S}_n^k\left(\frac{r}{m}\right).$$

For more details on r-Whitney numbers, see for instance [15].

2. The sums of powers of integers $S_{p,(a,d)}(n)$

In the next theorem we present an explicit formula for $S_{p,(a,d)}(n)$.

Theorem 2.1. *For all integers $n, p \geq 0$ and a,d complex numbers with $d \neq 0$, we have*

$$S_{p,(a,d)}(n) = d^p \sum_{k=0}^{p} k! \binom{n+1}{k+1} \mathcal{S}_p^k \left(\frac{a}{d} \right).$$

Proof. It follows from (5) that

$$\begin{aligned} \sum_{p \geq 0} \left(d^p \sum_{k=0}^{p} k! \binom{n+1}{k+1} \mathcal{S}_p^k \left(\frac{a}{d} \right) \right) \frac{z^p}{p!} &= \sum_{k \geq 0} k! \binom{n+1}{k+1} \sum_{p \geq 0} \mathcal{S}_p^k \left(\frac{a}{d} \right) \frac{(dz)^p}{p!} \\ &= e^{az} \sum_{k \geq 0} \binom{n+1}{k+1} \left(e^{dz} - 1 \right)^k \\ &= e^{az} \frac{e^{(n+1)dz} - 1}{e^{dz} - 1} \\ &= \sum_{k=0}^{n} e^{(a+kd)z} \end{aligned}$$

and the proof is complete. □

The following corollary immediately follows from Theorem 2.1.

Corollary 2.2. *If we assume that d divides a, then we have for $p > 0$*

$$S_{p,(a,d)}(n) = d^p \sum_{k=0}^{p} k! \binom{n+1}{k+1} \left\{ \begin{matrix} p + \frac{a}{d} \\ k + \frac{a}{d} \end{matrix} \right\}_{\frac{a}{d}}.$$

The next corollary contains an explicit formula for $S_{p,(a,d)}(n)$ expressed in terms of the r-Whitney numbers of the second kind $W_{m,r}(n,k)$.

Corollary 2.3. *For all integers $n, p \geq 0$ and a,d complex numbers with $d \neq 0$, we have*

$$S_{p,(a,d)}(n) = \sum_{k=0}^{p} k! d^k \binom{n+1}{k+1} W_{d,a}(p,k).$$

An explicit formula for $S_{p,(a,d)}(n)$ involving Bernoulli polynomials is given by the following theorem.

Theorem 2.4. *For all integers $n, p \geq 0$ and a,d complex numbers with $d \neq 0$, we have*

$$S_{p,(a,d)}(n) = \frac{d^p}{p+1} \sum_{k=0}^{p} \binom{p+1}{k} (n+1)^{p+1-k} B_k \left(\frac{a}{d} \right),$$

Proof. It follows from [16] that

$$B_n(x) = \sum_{k=0}^{n} (-1)^k \frac{k!}{k+1} \mathcal{S}_n^k(x). \tag{6}$$

Thus (2) becomes

$$S_{p,(a,d)}(n) = \frac{d^p}{p+1} \sum_{k=0}^{p+1} (-1)^k \frac{k!}{k+1} \left(\mathcal{S}_{p+1}^k \left(n + \frac{a}{d} + 1 \right) - \mathcal{S}_{p+1}^k \left(n + \frac{a}{d} \right) \right).$$

Now, from (4), we get

$$\begin{aligned} S_{p,(a,d)}(n) &= \frac{d^p}{p+1} \sum_{k=0}^{p+1} \frac{(-1)^k k!}{k+1} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k!} \binom{k}{j} \sum_{s=0}^{p} \binom{p+1}{s} \left(\frac{a}{d} + j \right)^s (n+1)^{p+1-s} \\ &= \frac{d^p}{p+1} \sum_{k=0}^{p+1} \frac{(-1)^k k!}{k+1} \sum_{s=0}^{p} \binom{p+1}{s} (n+1)^{p+1-s} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k!} \binom{k}{j} \left(\frac{a}{d} + j \right)^s \\ &= \frac{d^p}{p+1} \sum_{k=0}^{p+1} (-1)^k \frac{k!}{k+1} \sum_{s=0}^{p} \binom{p+1}{s} (n+1)^{p+1-s} \mathcal{S}_s^k \left(\frac{a}{d} \right) \\ &= \frac{d^p}{p+1} \sum_{s=0}^{p} \binom{p+1}{s} (n+1)^{p+1-s} \sum_{k=0}^{p+1} (-1)^k \frac{k!}{k+1} \mathcal{S}_s^k \left(\frac{a}{d} \right). \end{aligned}$$

Using again (6), we get the desired result. ☐

3. The hyper-sums of powers of integers $S_{p,(a,d)}^{(r)}(n)$

The hyper-sums of powers of integers $S_{p,(a,d)}^{(r)}(n)$ $(p \geq 0)$ (or the r-fold summation of pth powers) are defined recursively as

$$\begin{cases} S_{p,(a,d)}^{(0)}(n) = \sum_{k=0}^{n} (a+kd)^p = S_{p,(a,d)}(n) \\ S_{p,(a,d)}^{(r)}(n) = \sum_{k=0}^{n} S_{p,(a,d)}^{(r-1)}(k) \end{cases}.$$

In this section, we generalize the results obtained recently by the same authors in [11]. An explicit formula for $S_{p,(a,d)}^{(r)}(n)$ is given in the following theorem.

Theorem 3.1. *The hyper-sums of powers of integers $S_{p,(a,d)}^{(r)}(n)$ are given by*

$$S_{p,(a,d)}^{(r)}(n) = \sum_{k=0}^{n} \binom{n+r-k}{r} (a+kd)^p.$$

Proof. We can verify this fact easily by induction on r using the well-known identity

$$\sum_{k=i}^{n} \binom{k-i+r-1}{r-1} = \binom{n+r-i}{r}.$$

□

Next, we derive the generating functions (ordinary, exponential, and double) for $S_{p,(a,d)}^{(r)}(n)$ using Theorem 3.1.

Theorem 3.2. *The ordinary generating function of* $S_{p,(a,d)}^{(r)}(n)$ *is given by*

$$\sum_{r\geq 0} S_{p,(a,d)}^{(r)}(n)\, z^r = \frac{1}{(1-z)^{n+1}} \sum_{k=0}^{n} (1-z)^k (a+kd)^p . \tag{7}$$

Proof. Using Theorem 3.1 and the identity

$$\sum_{k\geq 0} \binom{n+k-i}{k} z^r = (1-z)^{i-n-1} ,$$

the claim follows. □

Theorem 3.3. *The exponential generating function of the hyper-sums of powers of integers* $S_{p,(a,d)}^{(r)}(n)$ *is given by*

$$\sum_{p\geq 0} S_{p,(a,d)}^{(r)}(n)\, \frac{z^p}{p!} = \sum_{k=0}^{n} \binom{n+r-k}{r} e^{(a+kd)z}. \tag{8}$$

Proof. We have

$$\begin{aligned}
\sum_{p\geq 0} S_{p,(a,d)}^{(r)}(n)\, \frac{z^p}{p!} &= \sum_{p\geq 0} \left(\sum_{k=0}^{n} \binom{n+r-k}{r} (a+kd)^p \right) \frac{z^p}{p!} \\
&= \sum_{k=0}^{n} \binom{n+r-k}{r} \sum_{p\geq 0} \frac{((a+kd)\, z)^p}{p!} \\
&= \sum_{k=0}^{n} \binom{n+r-k}{r} e^{(a+kd)z}.
\end{aligned}$$

□

Theorem 3.4. *The double generating function of* $S_{p,(a,d)}^{(r)}(n)$ *is given by*

$$\sum_{r\geq 0}\sum_{p\geq 0} S_{p,(a,d)}^{(r)}(n)\, \frac{z^p}{p!} t^r = \frac{e^{az} - (1-t)^{n+1} e^{(a+(n+1)d)z}}{(1-t)^{n+1} \left(1-(1-t)\, e^{dz}\right)}.$$

Proof. From (8) and (7), we obtain

$$\begin{aligned}\sum_{r\geq 0}\sum_{p\geq 0} S^{(r)}_{p,(a,d)}(n)\frac{z^p}{p!}t^r &= \sum_{k=0}^{n}\left(\sum_{r\geq 0}\binom{n+r-k}{r}t^r\right)e^{(a+kd)z}\\ &= \frac{e^{az}}{(1-t)^{n+1}}\sum_{k=0}^{n}\left((1-t)\,e^{dz}\right)^k\\ &= \frac{e^{az}}{(1-t)^{n+1}}\left(\frac{1-(1-t)^{n+1}e^{(n+1)dz}}{1-(1-t)\,e^{dz}}\right)\\ &= \frac{e^{az}-(1-t)^{n+1}e^{(a+(n+1)d)z}}{(1-t)^{n+1}\left(1-(1-t)\,e^{dz}\right)}.\end{aligned}$$

□

Our next goal is to give the exponential generating function in terms of Gaussian hypergeometric functions. The Gaussian hypergeometric function ${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};z\right)$ is defined by

$$\sum_{n\geq 0}\frac{(a)^{\overline{n}}(b)^{\overline{n}}}{(c)^{\overline{n}}}\frac{z^n}{n!},$$

and $(x)^{\overline{n}}$ denotes the Pochhammer symbol defined by

$$(x)^{\overline{0}} = 1 \text{ and } (x)^{\overline{n}} = x(x+1)\cdots(x+n-1).$$

Theorem 3.5. *The exponential generating function of the hyper-sums of powers of integers* $S^{(r)}_{p,(a,d)}(n)$ *is*

$$\sum_{p\geq 0} S^{(r)}_{p,(a,d)}(n)\frac{z^p}{p!} = \binom{n+r+1}{r+1}e^{az}\,{}_2F_1\left(\begin{matrix}1,-n\\r+2\end{matrix};1-e^{dz}\right). \tag{9}$$

Proof. From (8), we have

$$\begin{aligned}
\sum_{p\geq 0} S_{p,(a,d)}^{(r)}(n)\frac{z^p}{p!} &= e^{az}\sum_{k=0}^{n}\binom{k+r}{r}e^{d(n-k)z} \\
&= \frac{(n+r+1)!e^{az}}{n!r!}\sum_{k=0}^{n}\binom{n}{k}\frac{(n-k)!\,(k+r)!}{(n+r+1)!}e^{d(n-k)z} \\
&= \binom{n+r+1}{r+1}(r+1)\,e^{az}\sum_{k=0}^{n}\binom{n}{k}e^{d(n-k)z}\int_0^1 (1-x)^{r+k}\,x^{n-k}dx \\
&= \binom{n+r+1}{r+1}(r+1)\,e^{az}\int_0^1 (1-x)^r\left(\sum_{k=0}^{n}\binom{n}{k}\left(xe^{dz}\right)^{(n-k)}(1-x)^k\right)dx \\
&= \binom{n+r+1}{r+1}e^{az}\,(r+1)\int_0^1 (1-x)^r\left(1-x+xe^{dz}\right)^n dx.
\end{aligned}$$

It is well-known that the the Gaussian hypergeometric function ${}_2F_1\left(\begin{matrix}1,-n\\ r+2\end{matrix};1-e^{dz}\right)$ has an integral representation given by

$${}_2F_1\left(\begin{matrix}1,-n\\ r+2\end{matrix};1-e^{dz}\right)=(r+1)\int_0^1 (1-x)^r\left(1-x+xe^{dz}\right)^n dx,$$

this fact implies (9) and the theorem is proven. □

Now, according to the well-known formula, for $n\geq 0$ and $m>1$

$${}_2F_1\left(\begin{matrix}-n,1\\ m\end{matrix};z\right)=\frac{n!\,(z-1)^{m-2}}{(m)^{\overline{n}}\,z^{m-1}}\left(\sum_{k=0}^{m-2}\frac{(n+1)^{\overline{k}}}{k!}\left(\frac{z}{z-1}\right)^k-(1-z)^{n+1}\right).$$

we can rewrite the exponential generating function of the hyper-sums of powers of integers $S_p^{(r)}(n)$ as

Theorem 3.6.

$$\sum_{p\geq 0} S_{p,(a,d)}^{(r)}\frac{z^p}{p!}=\frac{e^{(a+d(r+(n+1)))z}}{\left(e^{dz}-1\right)^{r+1}}-\sum_{k=0}^{r}\binom{n+k}{k}\frac{e^{(a+(r-k)d)z}}{\left(e^{dz}-1\right)^{r-k+1}}. \tag{10}$$

The next result gives an explicit formula for $S_{p,(a,d)}^{(r)}(n)$ involving the generalized Bernoulli polynomials. Recall that the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of degree n in x are defined by the exponential generating function

$$\sum_{n\geq 0} B_n^{(\alpha)}(x)\frac{z^n}{n!}=\left(\frac{z}{e^z-1}\right)^{\alpha}e^{xz} \tag{11}$$

for arbitrary parameter α. In particular, $B_n^{(1)}(x) := B_n(x)$ denotes the classical Bernoulli polynomials with $B_1(0) = -\frac{1}{2}$. For a recent treatment see [1, 4, 17].

Theorem 3.7. *For all $n, p, r \geq 0$, we have*

$$S_{p,(a,d)}^{(r)}(n) = \frac{p!d^p}{(p+r+1)!} B_{p+r+1}^{(r+1)}\left(\frac{a}{d} + (r + (n+1))\right) - p!d^p \sum_{k=0}^{r} \binom{n+k}{k} \frac{1}{(p+r+1-k)!} B_{p+r+1-k}^{(r-k+1)}\left(\frac{a}{d} + (r-k)\right).$$

Proof. By (10) and (11), we have

$$\sum_{p\geq 0} S_{p,(a,d)}^{(r)}(n) \frac{z^p}{p!} = \sum_{p\geq 0} d^{p-r-1} B_p^{(r+1)}\left(\frac{a}{d} + (r + (n+1))\right) \frac{z^{p-r-1}}{p!} - \sum_{k=0}^{r} \binom{n+k}{k} \sum_{p\geq 0} d^{p-r+k-1} B_p^{(r-k+1)}\left(\frac{a}{d} + (r-k)\right) \frac{z^{p-r+k-1}}{p!}.$$

By shifting indices, we find

$$\sum_{p\geq 0} S_{p,(a,d)}^{(r)}(n) \frac{z^p}{p!} = \sum_{p\geq 0} \frac{z^p}{p!} \left(\frac{p!d^p}{(p+r+1)!} B_{p+r+1}^{(r+1)}\left(\frac{a}{d} + (r + (n+1))\right) - p!d^p \sum_{k=0}^{r} \binom{n+k}{k} \frac{1}{(p+r+1-k)!} B_{p+r+1-k}^{(r-k+1)}\left(\frac{a}{d} + (r-k)\right)\right).$$

Comparing the coefficients of $\frac{z^p}{p!}$, we get the result. □

When $r = 0$, Theorem 3.7 reduces to (2).

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