THE BLOCK ENERGY OF A GRAPH

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ABSTRACT. In this paper, we introduce the concept block matrix (B-matrix) of a graph G, and obtain some coefficients of the characteristic polynomial $\phi(G,\ \mu)$ of the B-matrix of G. The block energy $E_B(G)$ is established. Further upper and lower bounds for $E_B(G)$ are obtained. In addition, we define a uni-block graph. Some properties and new bounds for the block energy of the uni-block graph are presented.

Keywords: Block of a graph, Block matrix (B-matrix), block eigenvalues, block energy of a graph.
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1. **Introduction**

In this paper, all graphs are assumed to be finite connected simple graphs. A graph $G = (V, E)$ is a simple graph, that is, having no loops, no multiple and directed edges. As usual, we denote n to be the order and m to be the size of the graph G. For a vertex $v \in V$, the open neighborhood of v in a graph G, denoted $N(v)$, is the set of all vertices that are adjacent to v and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The degree of a vertex v in G is $d(v) = |N(v)|$. A graph G is said to be k-regular graph if $d(v) = k$ for every $v \in V(G)$. The distance $d(u, v)$ between any two vertices u and v in a graph G is the length of the shortest path connecting them. A vertex v of a graph G is a cut vertex of G if the graph $G - v$ consists of a greater number of components than G. A block of a graph G is a maximal connected subgraph with no cut vertex - (A subgraph with as many edges as possible and no cut vertex). The complement of a graph G is a graph $\overline{G}$ has $V(G)$ as its vertex set, but two vertices adjacent in $\overline{G}$ if and only if they are not adjacent in G. All the definitions and terminologies about the graph in this paragraph available in [8].

The concept energy of a graph introduced by I. Gutman [6], in (1978). Let G be a graph with n vertices and m edges and let $A(G) = (a_{ij})$ be the adjacency matrix of G, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_iv_j \in E, \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of a matrix $A(G)$, assumed in non-increasing order, are the eigenvalues of the graph G [11]. Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_s$, for $s \leq n$ be the distinct eigenvalues of G with multiplicities $m_1, m_2, ..., m_s$, respectively. The multiset of eigenvalues of $A(G)$ is called the spectrum of G and denoted by

$$Sp(G) = \begin{bmatrix} \lambda_1 & \lambda_2 & ... & \lambda_s \\ m_1 & m_2 & ... & m_s \end{bmatrix}$$

As A is real symmetric with zero trace, the eigenvalues of G are real with sum equal to zero [14]. The energy $E(G)$ of a graph G is defined to be the sum of the absolute values of the eigenvalues of G [6], i.e.,

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Adiga et al. [1], introduced the concept of color energy of a graph $E_c(G)$ and computed the color energy $E_\chi(G)$ of few families of graphs with minimum number of colors. It depends on the underlying graph and colors on its vertices. They established an upper bound and a lower bound for color energy. Also, they introduced the concept of complement of a colored graph and computed energies of complement of colored graphs of few families of graphs.

Sharada et al. [17], introduced the Laplacian sum-eccentricity matrix $LS_e(G)$ of a graph G. They obtained the Laplacian sum-eccentricity energy $LS_eE(G)$ of a graph G. Upper bounds for $LS_eE(G)$ are established. They defined the Laplacian sum-eccentricity equienergetic graph, and discussed some graphs which are Laplacian sum-eccentricity equienergetic.

Sowaity et al. [18], introduced the eccentricity extended matrix $A_{eex}(G)$, so that its (i,j)-entry is equal to $\frac{1}{2}(\frac{e_i}{e_j} + \frac{e_j}{e_i})$ for $v_iv_j \in E$ and 0 otherwise. Some properties of the eccentricity extended spectral radius are obtained. The eccentricity extended energy $E_{eex}(G)$ of G is defined. Upper and lower bounds for $E_{eex}(G)$ are established.
For more details on the mathematical aspects of the theory of graph energy we refer to [3, 5, 11, 12, 19] and the references therein.

Motivated by these works we introduce the concept of block matrix (B-matrix) of a graph G, and obtain some coefficients of the characteristic polynomial of the B-matrix of G. The block energy $E_B(G)$ is established. Upper and lower bounds for $E_B(G)$ are obtained. We define a uni-block graph. Some properties and new bounds for the block energy of the uni-block graph are presented.

2. **The B-matrix of graphs**

If a graph G contains t blocks, $B_1, B_2, ,..., B_t$, then we call B_r the r^{th} block of G. If two vertices v_i and v_j lies in the same block, we call $v_iv_j \in B_r$.

Definition 2.1. *Let G be a graph with n vertices. Then the block matrix (B-matrix) of a graph G denoted by $B(G)$, is defined as $B(G) = (b_{ij})$, where*

$$b_{ij} = \begin{cases} 2, & \text{if } v_iv_j \in E \text{ and } v_iv_j \in B_r, \\ 1, & \text{if } v_iv_j \notin E \text{ and } v_iv_j \in B_r, \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $B(G)$ is defined by

$$\phi(G, \mu) = \det(\mu I - B(G)),$$

where I is the unit matrix of order n. The eigenvalues of $B(G)$ are the roots of the characteristic polynomial $\phi(G, \mu)$.
Since $B(G)$ is real symmetric with zero trace, it follows that its eigenvalues must be real with sum equal to zero, i.e., $trace(B(G)) = 0$. We label the eigenvalues $\mu_1, \mu_2, ..., \mu_n$ in a non-increasing manner $\mu_1 \geq \mu_2 \geq ... \geq \mu_n$. The block energy of a graph G is denoted by $E_B(G)$ and is defined as the summation of the absolute values of the eigenvalues

$$E_B(G) = \sum_{i=1}^{n} |\mu_i|.$$

The following examples explain the concept.

Example 2.2. *Let G_1 be the graph as in Figure 1.*

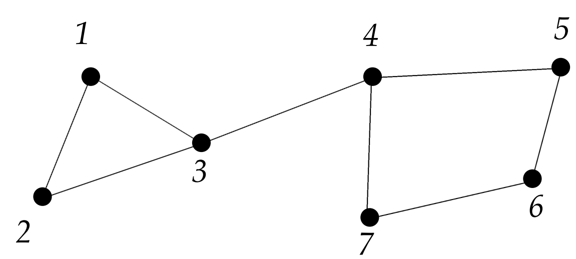

Figure 1: Graph G_1 with 3 blocks

Then the B-matrix of G_1 is

$$B(G_1) = \begin{bmatrix} 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 2 & 1 & 2 & 0 \end{bmatrix}$$

The characteristic polynomial of $B(G_1)$ is

$$\begin{aligned} \phi(G_1, \mu) &= |\mu I_n - B(G_1)| \\ &= \mu^7 - 34\mu^5 - 48\mu^4 + 253\mu^3 + 704\mu^2 + 548\mu + 112. \end{aligned}$$

The block eigenvalues of G_1 are
$\mu_1 = 5.386$, $\mu_2 = 3.889$, $\mu_3 = -0.3202$, $\mu_4 = -1$, $\mu_5 = -2$, $\mu_6 = -2.2582$, $\mu_7 = -3.6966$.
Therefore the block energy of G_1 is

$$E_B(G_1) = 18.55.$$

Example 2.3. *Let G_2 be the K_4 graph.*

Then the B-matrix of G_2 is

$$B(G_2) = \begin{bmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{bmatrix}$$

The characteristic polynomial of $B(G_2)$ is

$$\begin{aligned} \phi(G_2, \mu) &= \begin{vmatrix} \mu & -2 & -2 & -2 \\ -2 & \mu & -2 & -2 \\ -2 & -2 & \mu & -2 \\ -2 & -2 & -2 & \mu \end{vmatrix} \\ &= \mu^4 - 24\mu^2 - 64\mu - 48 \\ &= (\mu + 2)^3(\mu - 6). \end{aligned}$$

The block eigenvalues of G_2 are $\mu_1 = 6$, $\mu_2 = -2$, $\mu_3 = -2$, $\mu_4 = -2$.
Therefore the block energy of G_2 is $E_B(G_2) = 12$.

3. **Block energy for some standard graphs**

In this section we present and derive the block energy $E_B(G)$, for some well-known graphs. We need the following Lemma to proof our main result.

Lemma 3.1. [11] *The energy of the path P_n, $n \geq 2$, is given by*

$$E(P_n) = \begin{cases} \dfrac{2}{sin(\frac{\pi}{2(n+1)})} - 2, & if\ \ n \equiv 0(mod2), \\ \dfrac{2cos(\frac{\pi}{2(n+1)})}{sin(\frac{\pi}{2(n+1)})} - 2, & if\ \ n \equiv 1(mod2). \end{cases} \tag{3.1}$$

Theorem 3.2. *(1) The block eigenvalues of a complete graph K_n are -2 and $2(n-1)$, with multiplicities $(n-1)$ and 1 respectively, and the block energy for K_n is*

$$E_B(K_n) = 4(n-1) = 2E(K_n).$$

(2) The block energy for The star $K_{1,n-1}$, is

$$E_B(k_{1,n-1}) = 4\sqrt{n-1}.$$

(3) The block energy of a path P_n, $n \geq 3$ is given by

$$E_B(P_n) = \begin{cases} \dfrac{4}{sin(\frac{\pi}{2(n+1)})} - 4, & if\ n \equiv 0(mod2), \\ \dfrac{4cos(\frac{\pi}{2(n+1)})}{sin(\frac{\pi}{2(n+1)})} - 4, & if\ n \equiv 1(mod2). \end{cases} \tag{3.2}$$

Proof. The proof of parts (1) and (2) are similar to the proof of Theorem 4.2 in [19] and Theorem 2.6 in [16], respectively.
To show (3), we will start the proof by comparing the matrices $B(P_n)$ and $A(P_n)$. For the path P_n, we have

$$B(P_n) = \begin{bmatrix} 0 & 2 & 0 & 0 & \dots & 0 & 0 \\ 2 & 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 2 \\ 0 & 0 & 0 & 0 & \dots & 2 & 0 \end{bmatrix} = 2A(P_n).$$

Hence, $\mu_i = 2\lambda_i$, for $1 \leq i \leq n$. Therefore $E_B(P_n) = 2E(P_n)$. Using Lemma 3.1, we get the wanted result. □

4. Bounds for the block energy

We now give the explicit expression for the coefficient c_i of μ^{n-i} $(i = 0, 1, 2,\ and\ n)$ in the characteristic polynomial of the $B(G)$.

Theorem 4.1. *Let G be a graph with n vertices, $t \geq 1$, blocks and let*

$$\phi(G, \mu) = c_0\mu^n + c_1\mu^{n-1} + c_2\mu^{n-2} + ... + c_n,$$

be the characteristic polynomial of the B-matrix of G. Then

(1) $c_0 = 1.$
(2) $c_1 = 0.$
(3) $c_2 = -(4m + \sum_{r=1}^{t} \overline{m}_r),$
where $\overline{m}_r =$ *number of edges in the complement of* B_r.

(4) *for* $n \geq 2$ *we have* $c_n = (-1)^n \, det(B(G))$.

Proof. We prove only the equality in part (3), the proofs of equalities in parts (1), (2) and (4) are similar to the proof of Theorem 3.1 in [19].
3. Since

$$c_2 = \begin{vmatrix} 0 & b_{ij} \\ b_{ji} & 0 \end{vmatrix} = \sum_{1 \leq i < j \leq n} [0 - (b_{ij} b_{ji})] = - \sum_{1 \leq i < j \leq n} b_{ij}^2$$

and since

$$b_{ij} = \begin{cases} 2, & \text{if } v_i v_j \in E \text{ and } v_i v_j \in B_r, \\ 1, & \text{if } v_i v_j \notin E \text{ and } v_i v_j \in B_r, \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$c_2 = -(4m + \sum_{r=1}^{t} \overline{m}_r).$$

□

Example 4.2. *For the graph* G_1 *in Figure 1, the coefficient* c_2 *of* μ^5 *in the characteristic polynomial of* $B(G_1)$ *is equal to*

$$\begin{aligned} c_2 &= -(4m + \sum_{r=1}^{t} \overline{m}_r) \\ &= -[4 \times 8 + 0 + 0 + 2] \\ &= -34. \end{aligned}$$

Theorem 4.3. *Let* $\mu_1, \mu_2, ..., \mu_n$, *be the block eigenvalues of graph* G. *Then*

$$\sum_{i=1}^{n} \mu_i^2 = 8m + 2 \sum_{r=1}^{t} \overline{m}_r.$$

Proof. The summation of squares of the eigenvalues of $B(G)$ is just the trace of $B^2(G)$, i.e. $\sum_{i=1}^{n} \mu_i^2 = trace(B^2(G))$. Hence

$$\begin{aligned}\sum_{i=1}^{n} \mu_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}b_{ji} \\ &= \sum_{i\neq j} b_{ij}b_{ji} + \sum_{i=1}^{n} b_{ii}^2 \\ &= 2\sum_{1\leq i<j\leq n} b_{ij}^2 + 0 \\ &= 2(4m + \sum_{r=1}^{t} \overline{m}_r) \\ &= 8m + 2\sum_{r=1}^{t} \overline{m}_r.\end{aligned}$$

□

Corollary 4.4. *If $\mu_1,\ \mu_2, ..., \mu_n$, are the block eigenvalues of a graph G, then*

$$\sum_{i=1}^{n} \mu_i^2 = -2c_2.$$

Example 4.5. *If $G = K_n$, $n \geq 1$, then $c_2 = -2n(n-1)$.*

Proof. Since $c_2 = -(4m + \sum_{r=1}^{t} \overline{m}_r)$, for any graph G, and since for the complete graph K_n, $m = \frac{n(n-1)}{2}$, and $\sum_{r=1}^{t} \overline{m}_r = 0$, it follows that

$$c_2 = -2n(n-1).$$

□

Example 4.6. *In the graph $G_2 = K_4$, the coefficient c_2 of μ^2 in the characteristic polynomial of $B(G_2)$ is $-2(4)(3) = -24$.*

Example 4.7. *For the complete graph K_n, we have*

$$\sum_{i=1}^{n} \lambda_i^2 = 4n(n-1).$$

Theorem 4.8. *Let G be a graph with n vertices, m edges and $t \geq 1$, blocks. If $L = \prod_{i=1}^{n} \mu_i$. Then*

$$\sqrt{2(4m + \sum_{r=1}^{t} \overline{m}_r) + n(n-1)L^{\frac{2}{n}}} \leq E_B(G) \leq \sqrt{n(4m + \sum_{r=1}^{t} \overline{m}_r)}.$$

Proof. We have

$$\begin{aligned} E_B^2(G) &= (\sum_{i=1}^{n} |\mu_i|)^2 \\ &= \sum_{i=1}^{n} |\mu_i|^2 + \sum_{i \neq j} |\mu_i||\mu_j|. \end{aligned}$$

Employing the inequality between the Arithmetic mean, Geometric mean and bring in mind Theorem 4.3 we obtain

$$E_B(G) \geq \sqrt{2(4m + \sum_{r=1}^{t} \overline{m}_r) + n(n-1)L^{\frac{2}{n}}}.$$

On the other hand, using the Cauchy Schwartz inequality

$$\begin{aligned} \sum_{i=1}^{n} |\mu_i| &\leq \sqrt{n(\sum_{i=1}^{n} \mu_i^2)} \\ &= \sqrt{n(4m + \sum_{r=1}^{t} \overline{m}_r)}. \end{aligned}$$

Hence

$$E_B(G) \leq \sqrt{n(4m + \sum_{r=1}^{t} \overline{m}_r)}.$$

□

5. The uni-block graph

For $t \geq 1$, if $B_1,\ B_2,\ ,..., B_t$ are the blocks of a graph G, then we define G_b, to be the union of the blocks of G, i.e. $G_b = \bigcup_{r=1}^{t} B_r$, and n_b, to be the number of vertices in G_b, i.e. $n_b = |V(G_b)|$.

Definition 5.1. *A graph G with n vertices and m edges is a uni-block graph if G has no blocks other than itself (or $G_b \cong G$).*

Remark 5.2. *For any graph G, we have*

(1) $n_b = n + t - 1$.
(2) $n_b = n$ *if and only if G is a uni-block graph.*
(3) $|E(G_b)| = |E(G)|$.

In the following results, we present the relationship between the B-matrix $B(G)$ and the adjacency matrix of a graph $A(G)$.

Proposition 5.3. *Let G be a uni-block graph with n vertices and m edges. Then*

$$\begin{aligned} B(G) &= A(G) + A(K_n) \\ &= 2A(G) + A(\overline{G}). \end{aligned}$$

Lemma 5.4. *Let G be a graph with n vertices and let H be a proper subgraph of G. Then*

$$E_B(H) \leq E_B(G).$$

Theorem 5.5. *For a graph G with n vertices and t blocks. Then*

$$\sum_{r=1}^{t} E_B(B_r) \leq tE_B(G).$$

Proof. We know that each block of G is a subgraph with a smaller number of vertices. Using Lemma 5.4, we get

$$E_B(B_r) \leq E_B(G),\ r = 1, 2, ..., t.$$

hence

$$\sum_{r=1}^{t} E_B(B_r) \leq tE_B(G).$$

$\square$

The following fundamental results will be used to prove our main results.

Lemma 5.6. *(Weyl's inequality)*[13] *Let A and B be hermitian $n \times n$ matrices, if $1 \leq i \leq n$ and $\lambda(A),\ \lambda(B),\ \lambda(A+B)$ are the eigenvalues of $A,\ B,\ A+B$ respectively. Then*

$$\lambda_i(A) + \lambda_n(B) \leq \lambda_i(A+B) \leq \lambda_i(A) + \lambda_1(B).$$

Lemma 5.7. [9] *Let G be a connected graph of order n and size m. Then*

$$\lambda_1 \leq \sqrt{2m - n + 1}$$

with equality holding if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$.

Lemma 5.8. [3] *The graph G is bipartite if and only if its eigenvalues are symmetric with respect to the origin.*

Lemma 5.9. [15] *Let A and B be two real square matrices of order n. Let $C = A + B$. Then*

$$E(C) \leq E(A) + E(B).$$

From Proposition 5.3, and bring in mind that $E(G) \leq 2m$, [11], since $E(K_n) = 2(n-1)$, we get the following result.

Theorem 5.10. *Let G be a uni-block graph with n vertices and m edges. Then*

$$E_B(G) \leq 2(m + n - 1),$$

with equality holds if and only if $G \cong K_2$.

Theorem 5.11. *Let G be a uni-block graph with n vertices. Then*

$$E_B(G) \leq \frac{5n}{2} + \frac{n\sqrt{n}}{2} - 2.$$

Proof. Using Proposition 5.3, we have

$$B(G) = A(G) + A(K_n).$$

Using Lemma 5.9, we get

$$\begin{aligned} E_B(G) &\leq E(G) + E(K_n) \\ &= E(G) + 2(n-1). \end{aligned}$$

Bringing in mind that $E(G) \leq \frac{n}{2}(\sqrt{n+1})$, [11], we get

$$\begin{aligned} E_B(G) &\leq \frac{n}{2}(1+\sqrt{n}) + 2(n-1) \\ &= \frac{5n}{2} + \frac{n\sqrt{n}}{2} - 2. \end{aligned}$$

□

Theorem 5.12. *Let G be a bipartite uni-block graph with n vertices and m edges. Then*

$$E_B(G) \leq 2(n-1) + n\sqrt{2m-n+1}.$$

Proof. We will start the proof from Proposition 5.3, which gives

$$B(G) = A(G) + A(K_n).$$

Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$, be the eigenvalues of the adjacency matrix $A(G)$. Using Lemma 5.6, we get

$$\lambda_i(A(K_n)) + \lambda_n(A(G)) \leq \mu_i \leq \lambda_i(A(K_n)) + \lambda_1(A(G)).$$

For the bipartite graph, using Lemma 5.8, we have $\lambda_1(A(G)) = -\lambda_n(A(G))$, which implies

$$\lambda_i(A(K_n)) - \lambda_1(A(G)) \leq \mu_i \leq \lambda_i(A(K_n)) + \lambda_1(A(G)).$$

Hence

$$|\mu_i| \leq max\{|\lambda_i(A(K_n)) - \lambda_1(A(G))|,\ |\lambda_i(A(K_n)) + \lambda_1(A(G))|\}.$$

But $\lambda_i(A(K_n)) = -1,\ n-1$, with multiplicities $n-1,\ 1$, respectively. So

$$\max\{|\lambda_i(A(K_n)) - \lambda_1(A(G))|,\ |\lambda_i(A(K_n)) + \lambda_1(A(G))|\} = \begin{cases} 1+\lambda_1(A(G)), & \text{if } \lambda_i(A(K_n)) = -1, \\ n-1+\lambda_1(A(G)), & \text{if } \lambda_1(A(K_n)) = n-1. \end{cases}$$

Hence

$$\begin{aligned}\sum_{i=1}^{n}|\mu_i| &= \sum_{i=1}^{n-1}|\mu_i| + |\mu_n| \\ &\leq \sum_{i=1}^{n-1}[(1+\lambda_1(A(G))] + n - 1 + \lambda_1(A(G)) \\ &= n-1+(n-1)\lambda_1(A(G)) + n - 1 + \lambda_1(A(G)) \\ &= 2(n-1) + n\lambda_1(A(G)).\end{aligned}$$

Hence

$$E_B(G) \leq 2(n-1) + n\lambda_1(A(G)).$$

Using Lemma 5.7, we get

$$E_B(G) \leq 2(n-1) + n\sqrt{2m-n+1}.$$

□

Theorem 5.13. *Let G be a bipartite uni-block graph of order n and size m. Then*

$$E_B(G) \leq n(n-1+\sqrt{2m-n+1}).$$

Proof. Let G be a bipartite uni-block graph with n vertices and m edges, if we exchange the matrices $A(G)$ and $A(K_n)$ in Lemma 5.6, we get

$$\lambda_i(A(G)) + \lambda_n(A(K_n)) \leq \mu_i \leq \lambda_i(A(G)) + \lambda_1(A(K_n)).$$

Using the eigenvalues $\lambda_i = n-1,\ -1$,of the complete graph, we get

$$\lambda_i(A(G)) - 1 \leq \mu_i \leq \lambda_i(A(G)) + n - 1.$$

Using Lemma 5.8, we get

$$-\lambda_1(A(G)) - 1 \leq \mu_i \leq \lambda_1(A(G)) + n - 1.$$

Hence

$$\begin{aligned}|\mu_i| &\leq max\{|-\lambda_1(A(G)) - 1|,\ |\lambda_1(A(G)) + n - 1|\} \\ &= max\{\lambda_1(A(G)) + 1,\ \lambda_1(A(G)) + n - 1\}.\end{aligned}$$

For $n \geq 2$, we get

$$\begin{aligned}|\mu_i| &\leq \lambda_1(A(G)) + n - 1 \\ &\leq \sqrt{2m-n+1} + n - 1,\end{aligned}$$

by Lemma 5.7.
So

$$\sum_{i=1}^{n}|\mu_i| \leq n(n-1+\sqrt{2m-n+1}).$$

Hence

$$E_B(G) \leq n(n-1+\sqrt{2m-n+1}).$$

□

Clearly if we compare the bounds in Theorem 5.12, and Theorem 5.13, we can easily get that the bound in Theorem 5.12 is more efficient and smaller than that in Theorem 5.13.

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