EDGE HUB NUMBER IN GRAPHS

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Abstract. Let G be a graph, a subset $S \subseteq E(G)$ is called an edge hub set of G if every pair of edges $e, f \in E(G) \setminus S$ are connected by a path where all internal edges are from S. The minimum cardinality of an edge hub set is called edge hub number of G, and is denoted by $h_e(G)$. If G is a disconnected graph then any edge hub set must contain all of the edges in all but one of the components, as well as an edge hub set in the remaining component. In this paper the edge hub number for several classes of graphs is computed, bounds in terms of other graph parameters are also determined.

1. Introduction

By a graph $G = (V, E)$ we mean a finite and undirected graph without loops and multiple edges. A graph G with p vertices and q edges is called a (p, q) graph, the number p is referred to as the order of a graph G and q is referred to as the size of a graph G. In general, the degree of a vertex v in a graph G denoted by $deg(v)$ is the number of edges of G incident with v, where $\Delta(G) = \max\{deg(u), u \in V(G)\}$ [4]. The degree of an edge uv is defined to be $deg(u) + deg(v) - 2$. Also $\Delta'(G)$ denotes the maximum degree among the edges of G. Given any vertex $v \in V(G)$, the graph obtained from G by removing the vertex v and all of its incident edges is denoted by $G - v$. In a tree, a leaf is a vertex of degree one, a leaf edge is an edge incident to a leaf. We refer to [4] for terminology and notations not defined here.

Introduced by Walsh [15], a hub set in a graph G is a set H of vertices in G such that any two vertices outside H are connected by a path whose internal vertices lie in H. The hub number of G, denoted $h(G)$, is the minimum size of a hub set in G. A connected set in G is a vertex set F such that the subgraph of G induced by F (denoted $G[F]$) is connected. The connected hub number of G, denoted $h_c(G)$, is the minimum size of a connected hub set in G. For any set $C \subseteq E$, the edge-induced subgraph $G[C]$ is the maximal subgraph of G with edge set C, thus two edges of C are adjacent in $G[C]$ if and only if they are adjacent in G.

Let G be a graph, let $e = (u, v)$ and $f = (u_1, v_1)$ be edges in G. A path between e and f is a path between one end vertex from e and another end vertex from f such that $d(e, f) = min\{d(u, u_1), d(u, v_1), d(v, u_1), d(v, v_1)\}$. Internal edges of a path between two edges e and f are all the edges of the path except e and f [12]. A subset $S \subseteq E(G)$ is called an edge hub set of G if every pair of edges $e, f \in E \setminus S$ are connected by a path

2010 *Mathematics Subject Classification.* Primary 05C40; Secondary 05C99.
Key words and phrases. Edge hub number, Hub number, Domination number, Line graph.

where all internal edges are from S. The minimum cardinality of an edge hub set is called edge hub number of G, and is denoted by $h_e(G)$ [12]. For more details on the hub studies we refer to [8, 9, 11, 13, 14]

A set D of vertices in a graph G is called dominating set of G if every vertex in $V \setminus D$ is adjacent to some vertex in D, the minimum cardinality of a dominating set in G is called the domination number $\gamma(G)$ of a graph G [5].

A set B of edges in a graph G is called an edge dominating set of G if every edge in $E \setminus B$ is adjacent to some edge in B, the minimum cardinality of an edge dominating set in G is called the edge domination number $\gamma'(G)$ of a graph G [5]. The chromatic number $\chi(G)$ of a graph G, is the minimum number of colors required to assign to the vertices of G, in such a way that no two adjacent vertices of G receive the same color. The edge chromatic number $\chi'(G)$, gives the minimum number of colors with which graph's edges can be colored, such that no two adjacent edges of G receive the same color [4].

A double star $S_{n,m}$ is the tree obtained from two disjoint stars $K_{1,n-1}$ and $K_{1,m-1}$, by connecting their centers [3]. The line graph $L(G)$ of G, has the edges of G as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G [4]. The following results will be useful in the proof of our results.

Theorem 1.1. [15] *If G is a connected graph, then $h(G) \leqslant p - \Delta(G)$, and the inequality is sharp.*

Theorem 1.2. [2] *For any graph G, $\chi(G) \leqslant \Delta(G) + 1$.*

Theorem 1.3. [6] *If G is a (p, q) graph without isolated vertices, then*

$$\frac{q}{\Delta'(G) + 1} \leqslant \gamma'(G).$$

2. Edge hub number

It is clear that $h_e(G)$ is well-defined for any graph G with at least one edge, since $E(G)$ is an edge hub set.

An edge hub set $S \subseteq E(G)$ is called a connected edge hub set, if the subgraph $G[S]$ is connected. The minimum cardinality of a connected edge hub set of G, is called a connected edge hub number and is denoted by $h_{ce}(G)$.

Observation 2.1. *We have,*

(1) *for any graph G, $h_e(G) \leqslant h_{ce}(G)$.*
(2) *for any graph G, $h_e(G) = h(L(G))$.*

Proposition 2.1.

(1) For any path P_p with $p \geqslant 3$, $h_e(P_p) = p - 3$.
(2) For any cycle C_p, $p \geqslant 4$, $h_e(C_p) = p - 3$.
(3) For the star $K_{1,p-1}$, $h_e(K_{1,p-1}) = 0$.
(4) For the double star $S_{n,m}$, $h_e(S_{n,m}) = 1$.

(5) For the wheel graph $W_{1,p-1}$ with $p \geqslant 5$,

$$h_e(W_{1,p-1}) = \begin{cases} \frac{p}{2}, \text{ if } p \text{ is even;} \\ \frac{p-1}{2}, \text{ if } p \text{ is odd.} \end{cases}$$

It is clear from the Proposition 2.1, that the relation between hub number and edge hub number is not the same for every graph. It is easy to see that edge hub number is less than the hub number for paths, $h_e(P_p) = p-3 < h(P_p) = p-2$. But for wheels edge hub number is greater than the hub number. On the other hand, the two parameters are equal for cycles, for the cycle with p vertices, $h_e(C_p) = h(C_p) = p-3$.

Let G_1 and G_2 be two graphs, now we think, is the edge hub number a suitable measure of accessibility? In other words, does the edge hub number discriminate between G_1 and G_2? There are many examples of graphs which propose that $h_e(G)$ is a suitable measure of accessibility which is able to discriminate between graphs. For example, consider the graphs G_1, G_2 and G_3 in Figure 1.

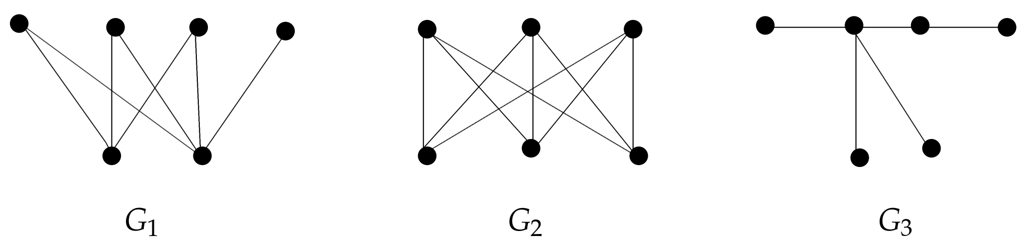

Figure 1: G_1, G_2, and G_3.

We have $h(G_1) = h(G_2) = h(G_3) = 2$, the hub number does not discriminate between graphs G_1, G_2 and G_3. But $h_e(G_1) = 2$, $h_e(G_2) = 3$ and $h_e(G_3) = 1$, this means $h_e(G_1) \neq h_e(G_2) \neq h_e(G_3)$, so the edge hub number discriminates between graphs G_1, G_2 and G_3.

Observation 2.2. *For any graph G, let G' be a subgraph of G. Then $h_e(G')$ need not be less than or equal to $h_e(G)$, for example, for the graphs G, G' shown in Figure 2, we have $h_e(G) = 1$, while $h_e(G') = 2$.*

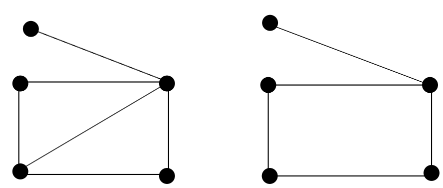

Figure 2: G, G'.

Theorem 2.1. *For any connected graph G, $h_e(G - v) \leqslant h_e(G)$.*

Proof. Let H_e be a minimum edge hub set of a graph G. Based on the cardinality of a minimum edge hub set, we discuss the following cases:

Case 1: $|H_e| = 0$. This means that all edges in G are adjacent. So $h_e(G) = 0$, and if we remove any vertex v from G, it follows that $G - v$ is either a connected graph of order $p - 1$ with all edges adjacent, or $G - v$ is totally disconnected graph, and in all cases $h_e(G - v) = 0$.
Case 2: $|H_e| = 1$. The following cases are considered. Suppose that e is any edge of G such that $e \in H_e$.
Subcase 2.1: v is a vertex incident with e. Removal of v from G, results in a graph $G - v$ such that either all its edges are adjacent, or disconnected graph consisting of two parts, G_1 and G_2 such that G_1 is totaly disconnected and G_2 is a graph with all edges adjacent. Thus $h_e(G - v) = 0$.
Subcase 2.2: v is a vertex not incident with e. If we remove it from G, the resulting graph $G - v$ is either a graph in which all edges are adjacent, so $h_e(G - v) = 0$, or $G - v$ is a graph of order $p - 1$ with $h_e(G - v) = 1$. Then $h_e(G - v) \leqslant h_e(G)$.
Case 3: $|H_e| \geqslant 2$. Let $e \in H_e$, if v is a vertex not incident with e, and we delete v from G, then H_i is still an edge hub set of G. Thus, $h_e(G - v) \leqslant h_e(G)$. If v is an end vertex of e, removal of v from G, produces a graph $G - v$ such that $h_e(G - v) < h_e(G)$.
Then, from Case 1, Case 2 and Case 3, we get the result. □

Theorem 2.2. *Let G be a connected graph and $v \in V(G)$, such that $h_e(G - v) < h_e(G)$. If S_1 is a minimum edge hub set of $G - v$ and e is an edge of G incident with v, then $S_1 \cup \{e\}$ is a minimum edge hub set of G.*

Proof. Suppose that G is connected graph, let f, g be any two edges of G, we discuss the following cases:
Case 1: $f, g \in E(G - v)$. Since S_1 is a minimum edge hub set of $G - v$, then there is a path between f and g with all internal edges from S_1. This implies that in G there is an $S_1 \cup \{e\}$ path between f and g. So, $S_1 \cup \{e\}$ is a minimum edge hub set of G.
Case 2: v is an end vertex of f and g. Then in G the edge e incident with v is adjacent to f and g. Note that, $h_e(G - v) < h_e(G)$ so e must be in any minimum edge hub set of G, hence $(S_1 \cup \{e\})$–path between f and g is there in G.
Case 3: $f \in E(G - v)$, and v is an end vertex of g, where $g \neq e$. Clearly g is adjacent to e. Since $h_e(G - v) < h_e(G)$, the edge e incident with v must be in any minimum edge hub set of G. Now S_1 is a minimum edge hub set of $G - v$, and e is adjacent to g and to one edge of S_1. Therefore, there is an $S_1 \cup \{e\}$ path between f and g, so that $S = S_1 \cup \{e\}$ is a minimum edge hub set of G. □

Proposition 2.2. *For any connected graph G, $h_e(G) = 0$ if and only if $G \cong K_{1,p-1}$, or $G \cong C_3$.*

Proof. Suppose that $h_e(G) = 0$, from definition of edge hub number, all edges in G are adjacent, hence $G \cong K_{1,p-1}$ or C_3. The converse is obvious. □

Theorem 2.3. *For any graph G, $h_e(G) \leqslant q - \Delta'(G)$, and the inequality is sharp for any path $P_p, p \geqslant 4$.*

Proof. Let G be a graph of size q, clearly that $\Delta'(G) = \Delta(L(G))$, also $h_e(G) = h(L(G))$, then by Theorem 1.1, $h(L(G)) \leqslant |V(L(G))| - \Delta(L(G))$. Since $|V(L(G))| = q$, the result is true. For any path P_p, $p \geqslant 4$, by Proposition 2.1, $h_e(P_p) = p - 3$. Note that, $\Delta'(P_p) = 2$ and $q = p - 1$. Hence the assertion follows. □

Proposition 2.3. *For any connected graph G, $h_e(G) = 1$ if and only if $G \cong S_{n,m}$, C_4, K_4, $K_4 - e$ where $e \in E(K_4)$, $K_{1,p-1} + e$ where e is an edge, $p \geqslant 3$, and the graphs shown in Figure 3.*

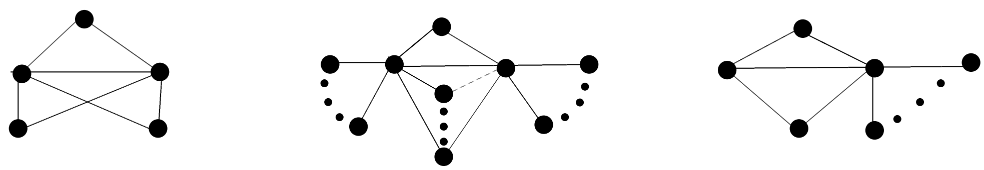

Figure 3.

Theorem 2.4. *Let G be any graph. Then $\chi'(G) + h_e(G) \leqslant q + 1$.*

Proof. Let G be any graph, by Theorem 1.2, it is clear that $\chi(L(G)) \leqslant \Delta(L(G)) + 1$, then $\chi'(G) \leqslant \Delta'(G) + 1$. By Theorem 2.3, it follows that $\chi'(G) + h_e(G) \leqslant q + 1$. □

Theorem 2.5. *For any graph G, $\gamma'(G) \leqslant h_e(G) + 1$.*

Proof. Let S be an edge hub set of G. Suppose that A is the set of edges which are not adjacent to any edge in S (nor in S themselves). The only S–paths between edges in A must therefore be trivial. Since we know that there are S–paths between all pairs of edges in $E \setminus S$, this implies that $G[A]$ is a graph with all edges are adjacent. Therefore, $S \cup A$ must be an edge dominating set for any edge $\{a\} \in A$. □

Theorem 2.6. *For any (p, q) graph G, $\frac{q}{\Delta'(G)+1} - 1 \leqslant h_e(G)$.*

Proof. By Theorem 1.3 and Theorem 2.5, we get the result. □

Proposition 2.4. *For any graph G with $p \geqslant 3$, $h_e(G) \leqslant p - 3$.*

Proof. Suppose that T is a spanning tree of G, and if u is pendant vertex of T, then the $p - 3$ edges of T other than those incident with u form an edge hub set of G, hence the result. □

Corollary 2.1.

For any graph G with $p \geqslant 3$, we have,

(1) $0 \leqslant h_e(G) + h_e(\overline{G}) \leqslant 2p - 6$.
(2) $0 \leqslant h_e(G) \cdot h_e(\overline{G}) \leqslant p^2 - 6p + 9$, and equality in upper bound holds if G is isomorphic to C_5, C_6, C_7 and lower bound for $G \cong K_{1,3}$.

Proposition 2.5. *For any connected graph G, $h_e(G) \leqslant 3q - 2p$, the equality is attained if G is a path P_p, $p \geqslant 3$.*

Proof. By Proposition 2.4, and since $p - 1 \leqslant q$ we have $h_e(G) \leqslant p - 3 = 3(p - 1) - 2p \leqslant 3q - 2p$. □

Theorem 2.7. *For any nonempty edge hub set S of a connected graph $G = (V, E)$, $|E \setminus S| \leqslant 1 + \sum_{s \in S} deg_e(s)$.*

Proof. Suppose that S is an edge hub set of a connected graph $G = (V, E)$. Since $\gamma'(G) \leqslant h_e(G) + 1$ for any $e \in E(G) \setminus S$ is adjacent to at least one edge of S in G. Therefore, each edge in $(E \setminus S) \setminus \{e\}$ contributes at least one to the sum of the degrees of the edges of S. Hence

$$|E \setminus S| \leqslant 1 + \sum_{s \in S} deg_e(s).$$

□

Theorem 2.8. *For any tree T with $p \geqslant 3$ vertices and l leaves,*

$$h_e(T) = h_{ce}(T) = p - l - 1.$$

Proof. In a tree, if the edge e is incident with a leaf, we say that e is a leaf edge. Let M be a set of all non-leaf edges in T, since the unique path between any two leaf edges never pass through another leaf edge, then M forms an edge hub set of T and $|M| = p - (l + 1)$. Note that any proper subset of M cannot be an edge hub set, since every edge is a bridge, and therefore separates a pair of leaf edges. Now, we must show that we can find a minimum edge hub set which is not containing any leaf edge.

Suppose that F is a minimum edge hub set which contains a leaf edge g. Since the set of non-leaf edges in T form an edge hub set, if F is minimum it must exclude at least one edge of degree 2 or greater, choose such an edge f, so that the path between g and f in T has all intermediate edges in F. Then the set $F' = (F \setminus \{g\}) + \{f\}$ is also an edge hub set, since any path through F between f and any other edge can now be extended to a path through F' between g and that edge.

This means, we removed a leaf edge and added a non leaf edge. We can repeat this process to find a minimum edge hub set containing only non leaf edges, and the set M is the only such set. Since M is the set of all non-leaf edges in T, M must be minimum and $h_e(G) = p - l - 1$. Also clearly $T[M]$ is connected. Therefore, $h_{ce} = p - l - 1$. □

Theorem 2.9. *Let $G = (p, q)$ be a connected non star graph. Then $h_e(G) \geqslant p - q$, and the inequality is sharp for C_3 and double star.*

Proof. Let G be a connected non star graph. The following cases are considered.
Case 1: $p \leqslant q$. Since $h_e(G) \geqslant 0$ the result is trivial.
Case 2: $p > q$. Clearly G is a tree because G is connected, and since G is not star then there are at least two non adjacent edges. This means that $h_e(G) \geqslant 1$. Since for any tree $p - q = 1$, then $h_e(G) \geqslant p - q$. □

Theorem 2.10. *Let G be a graph, and u, v are non-adjacent vertices of G. Then $h_e(G + uv) > h_e(G)$, if and only if $u, v \notin V(G[S])$ for every minimum edge hub set S of G.*

Proof. Assume that $h_e(G + uv) > h_e(G)$. Let S be any minimum edge hub set of G, then S cannot be an edge hub set of $(G + uv)$. Therefore, u cannot be an end vertex of any

edge of S and similarly v cannot be an end vertex of any edge in S. Then $u, v \notin V(G[S])$. Conversely, let $u, v \notin V(G[S])$. The following cases are considered.
Case 1: $h_e(G + uv) = h_e(G)$. Let S be any minimum edge hub set of $(G + uv)$, since $u, v \notin V(G[S])$ then $uv \notin S$. Therefore, any $f \in S$ then $f \in G$. Thus S is a minimum edge hub set of G but S is also minimum edge hub set of $(G + uv)$, this means that there is an edge e in S such that either u is an end vertex of e or v is an end vertex of e. This implies that $u \in V(G[S])$ or $v \in V(G[S])$, which is a contradiction. Then $h_e(G + uv) \neq h_e(G)$.
Case 2: $h_e(G + uv) < h_e(G)$. Assume that S is any minimum edge hub set of $(G + uv)$, then the edge $uv \in S$. Let a be any vertex adjacent to u and b be any vertex adjacent to v. Now $S' = (S - uv) + (ua, vb)$. Then clearly S' is a minimum edge hub set of G, thus $u \in V(G[S])$ and $v \in V(G[S])$. This is a contradiction. Then $h_e(G + uv) \not< h_e(G)$. Therefore, $h_e(G + uv) > h_e(G)$. □

3. Acknowledgements

The authors would like to thank the referee and the editor for their valuable comments which helped to improve the manuscript.

The Second Author is thanks to UGC for financial assistance under No. F.510/12/$DRS - II$/2018($SAP - I$).

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