DIOPHANTINE EQUATIONS FOR ANALYTIC FUNCTIONS

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ABSTRACT. We consider analogs of several classical diophantine equations, such as Fermat's last theorem and Catalan's conjecture, for certain classes of analytic functions. We give simple direct proofs avoiding use of deep theorems in complex analysis. As a byproduct of our results, we obtain new proofs for the corresponding results over polynomials.

1. INTRODUCTION

The polynomial analog of Fermat's last theorem and Catalan's conjecture have simple short proofs based on Mason's theorem (see [2, Theorems 4.3.3 & 4.3.5]). Recall that, Mason's theorem asserts that if p, q and r are polynomials such that $p + q = r$, then $\deg(r) \leq \deg(rad(pqr)) - 1$. So the number of roots of r (the degree) is controlled by $\deg(rad(pqr))$ (the number of distinct roots of p, q, r together). A simple analog of Mason's theorem for analytic functions that "controls" the roots is unsustainable as we can have equations such as $e^g + e^h = k$, where g and h are linearly independent and so we have a case where each term in the LHS does not vanish while k has infinitely many zeroes (a simple consequence of little Picard theorem). In spite of this, as we see in this paper, we still can have analogs of Fermat's last theorem and Catalan's conjecture for certain families of analytic functions. Since our method of proof avoids Mason's theorem as a byproduct we obtain new proof for polynomial case. We conclude with a study of Waring type problems for analytic functions. The analog problem for polynomials is discussed in [2, Theorem 4.3.6] and for finite fields in [1].

We need the following lemma in our arguments throughout the paper.

Lemma 1.1. *Suppose that f is an entire function that does not take the value 0, then there exists an entire function $\psi(z)$, such that $f(z) = e^{\psi(z)}$.*

Proof. Let $h(z) := \int_0^z \frac{f'(\tau)}{f(\tau)} d\tau$ and $g(z) := \frac{e^{h(z)}}{f(z)}$. Then it can be seen that h is an entire function and $h'(z) = \frac{f'(z)}{f(z)}$. We have that

$$g'(z) = \frac{e^{h(z)}(h'(z)f(z) - f'(z))}{f^2(z)} = 0.$$

Therefore g is a constant function which implies $f(z) = ce^{h(z)} = e^{h(z)+\ln c}$. $\square$

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2. Main Results

We first study Fermat's last theorem for analytic functions. The following lemma is needed for our proof.

Lemma 2.1. *Suppose that h is an entire function such that for a natural number m, $h^m(z) = \prod_{j=1}^{n} f_j(z)$, where f_j's are all entire functions with no common zeros. Then, there exist entire functions $h_1, h_2, \dots, h_n$ such that $f_j(z) = h_j^m(z)$, $j \in \overline{1,n}$.*

Proof. Let f_{j_0}, $1 \leq j_0 \leq n$, be given. Suppose that $a_1, a_2, a_3, \dots$ is the sequence of (distinct) zeros of f_{j_0} of multiplicity $m_1, m_2, m_3, \dots$, respectively. Then since $h^m(z) = \prod_{j=1}^{n} f_j(z)$ each m_k is divisible by m, therefore there exist integers $m_1', m_2', m_3', \dots$ such that $m_k = m \cdot m_k'$ ($k \in \mathbb{N}$). By Weierstrass factorization theorem, we can find an entire function g_j, with zeros $a_1, a_2, a_3, \dots$ of multiplicity $m_1', m_2', m_3', \dots$. Let $\tilde{f}_j(z) := \frac{f_j(z)}{g_j(z)^m}$. Since $\tilde{f}_j(z)$ is an entire function with no zeros from lemma 1.1 there exist an entire function ϕ_j such that $\tilde{f}_j(z) = e^{\phi_j(z)}$. Now take $h_j(z) := g_j(z) e^{\frac{\phi_j(z)}{m}}$ and observe that $f_j(z) = h_j^m(z)$. □

Theorem 2.2 (Fermat's last theorem for entire functions)**.** *Suppose, $n \geq 3$ and f, g and h are entire functions that do not have common zeros. Moreover, assume that at least one of them has finitely many zeros. If $f^n(z) + g^n(z) = h^n(z)$, then $f(z) = a \cdot e^{\phi(z)}$, $g(z) = b \cdot e^{\phi(z)}$, $h(z) = c \cdot e^{\phi(z)}$, for an entire function $\phi(z)$ and complex numbers a, b, and c with $a^n + b^n = c^n$.*

Proof. We distinguish two cases:

Case 1. Assume that $h(z)$ does not have zeros. Divide both sides by $g^n(z)$ to get

$$\left(\frac{f(z)}{g(z)}\right)^n + 1 = \frac{h^n(z)}{g^n(z)}.$$

The right hand side is a meromorphic function, that does not take the value 0. This implies, that the left hand side does not take zero as well. In other words, the function $\frac{f(z)}{g(z)}$ omits all values of the form $\sqrt[n]{-1}$. Since $n \geq 3$, little Picard theorem implies that $\dfrac{f(z)}{g(z)}$ is constant which in turn leads to a solution of desirable form for our equation. If $f(z)$ or $g(z)$ does not have zeros, a similar argument shows that the result holds.

Case 2. Assume that each of the functions f, g and h has at least one zero. Out of all triples of functions $(f(z), g(z), h(z))$ satisfying $f^n(z) + g^n(z) = h^n(z)$ we choose the one, for which $h(z)$ has the least number of zeros. We now rewrite our equation in the form

$$h^n(z) = \prod_{j=1}^{n} (f(z) - \varepsilon_j g(z)),$$

where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are $n-$th roots of unity. We note that all terms on the right hand side are entire functions with no common zeros. By Lemma 2.1, there exist entire functions $h_1, h_2, \ldots, h_n$ such that $\prod_{j=1}^{n} h_j(z) = h(z)$ and $f(z) - \varepsilon_j g(z) = h_j^n(z)$, $j \in \overline{1,n}$. Since $h(z)$ has at leas one complex zero, at least one of the $h_j(z)$ has a zero. We can assume that this function is $h_1(z)$. We now write an analogous system of equations for the functions $h_1(z)$, $h_2(z)$ and $h_3(z)$

$$\begin{cases} f(z) - \varepsilon_1 g(z) = h_1^n(z) \\ f(z) - \varepsilon_2 g(z) = h_2(z)^n \\ f(z) - \varepsilon_3 g(z) = h_3^n(z) \end{cases}.$$

From these equations we obtain

$$g(z) = \frac{h_1^n(z) - h_2^n(z)}{\varepsilon_2 - \varepsilon_1} = \frac{h_2^n(z) - h_3^n(z)}{\varepsilon_3 - \varepsilon_2}.$$

Multiplying through by the denominators and collecting the corresponding terms,

$$(\varepsilon_3 - \varepsilon_2)h_1^n(z) + (\varepsilon_1 - \varepsilon_3)h_2^n(z) + (\varepsilon_2 - \varepsilon_1)h_3^n(z) = 0.$$

So, if we let $\tilde{h_1}(z) = \sqrt[n]{\varepsilon_3 - \varepsilon_2}h_1(z)$, $\tilde{h_2}(z) = \sqrt[n]{\varepsilon_1 - \varepsilon_3}h_2(z)$ and $\tilde{h_3}(z) = \sqrt[n]{\varepsilon_1 - \varepsilon_2}h_3(z)$, we have

$$\tilde{h_1}^n(z) + \tilde{h_2}^n(z) = \tilde{h_3}^n(z). \tag{1}$$

If at least one of the functions $h_2(z)$ and $h_3(z)$ does not have a zero, say $h_3(z)$, then from case 1, it follows that there exists entire $\phi(z)$ and complex numbers $a_1, a_2, a_3 \in \mathbb{C}$ such that $\tilde{h_j}(z) = a_j e^{\phi(z)}$, $j = 1, 2, 3$. This contradicts the assumption that $h_1(z)$ has at least one zero. Thus, both functions $h_2(z)$ and $h_3(z)$ have complex zeros. WLOG, we may assume that among $h_1(z)$, $h_2(z)$, $h_3(z)$ the function $h_3(z)$ has least number of zeros. Then, the number of zeros of $\tilde{h_3}(z)$ has to be smaller than the number of zeros of $h(z)$. But then since $h(z) = \prod_{j=1}^{n} h_j(z)$ and the functions $h_2(z)$ and $h_3(z)$ have at least one zero each, this contradicts the assumption that $h(z)$ has the least number of zeros among all triples of functions $(f(z), g(z), h(z))$ that satisfy Fermat equation. □

Corollary 2.3 (Fermat's last theorem for polynomials)**.** *Suppose, $n \geq 3$ and p, q and r are nonzero and relatively prime polynomials satisfying $p^n + q^n = r^n$, then p, q and r are constant.*

The following lemma studies the case $n = 2$ in Fermat's equation.

Lemma 2.4. *Suppose that f, g and ϕ are entire functions and P is a polynomial. Then the equation $f^2 + g^2 = P(z)e^{\phi(z)}$ has infinitely many solutions in the class of analytic functions that can be completely parametrized as follows:*

$$\begin{cases} f(z) = 1/2e^{\phi(z)} \left(P_1(z)e^{\psi(z)} + P_2(z)e^{-\psi(z)} \right) \\ g(z) = -i/2e^{\phi(z)} \left(P_1(z)e^{\psi(z)} - P_2(z)e^{-\psi(z)} \right) \end{cases},$$

where $P_1(z)$ and $P_2(z)$ are any polynomials such that $P_1(z)P_2(z) = P(z)$ and ψ is an entire function.

Proof. We factorize $f^2(z) + g^2(z)$ to get

$$P(z)e^{\phi(z)} = f^2(z) + g^2(z) = (f(z) + ig(z))\,(f(z) - ig(z))\,.$$

Let $f_1 := \dfrac{f}{e^{\phi(z)}}$ and $g_1 := \dfrac{f}{e^{\phi(z)}}$. Then we have that

$$P(z) = (f_1(z) + ig_1(z))\,(f_1(z) - ig_1(z))\,.$$

Since the left hand side has only finite number of roots we have that $f_1 + ig_1 = e^{\phi_1}P_1(z)$ and $f_1(z) - ig_1(z) = e^{\phi_2(z)}P_2(z)$. Now since $(f_1 + ig_1)\,(f_1 - ig_1) = P(z)$ we must have that $P_1(z)P_2(z) = P(z)$ and $\phi_1(z) = -\phi_2(z)$. Consider the following system of equations

$$\begin{cases} f_1(z) + ig_1(z) = e^{\phi_1(z)}P_1(z) \\ f_1 - ig_1 = e^{-\phi_1(z)}P_2(z) \end{cases} \tag{2}$$

and observe that

$$\begin{cases} f_1(z) = 1/2\left(P_1(z)e^{\phi_1(z)} + P_2(z)e^{-\phi_1(z)}\right) \\ g_1(z) = -i/2\left(P_1(z)e^{\phi_1(z)} - P_2(z)e^{-\phi_1(z)}\right) \end{cases}$$

and so the result follows. □

The following proposition is an auxiliary result in the proof of Catalan's conjecture for rational functions.

Proposition 2.5. *Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions over $\mathbb{C}$. Assume that $f(z)$ has finitely many zeros. If $m \geq n \geq 2$, then*

$$f^m(z) - g^n(z) = 1 \tag{3}$$

implies that $m = n = 2$.

Proof. Suppose $m > 2$. Since $f(z)$ and $g(z)$ are meromorphic functions, we can write $f(z) = \frac{f_1(z)}{f_2(z)}$, $g(z) = \frac{g_1(z)}{g_2(z)}$, where $f_j(z), g_j(z)$, $j = 1, 2$, are entire functions, $f_1(z)$ and $f_2(z)$ do not have common zeros, and $g_1(z)$ and $g_2(z)$ do not have common zeros. We now substitute these expression into (3) and multiply by $f_2^m(z)g_2^n(z)$ to get

$$f_1^m(z)g_2^n(z) - g_1^n(z)f_2^m(z) = f_2^m(z)g_2^n(z). \tag{4}$$

Since $f_1(z)$ and $f_2(z)$ do not have common zeros and $g_1(z)$ and $g_2(z)$ do not have common zeros, we deduce that $f_2(z)$ and $g_2(z)$ have the same zeros, possibly with different multiplicities. Let a be the common root, of multiplicity k for $f_2(z)$ and multiplicity l for $g_2(z)$. Then equation (4) implies that $mk = nl$. By Weierstrass factorization theorem, we construct the function $h(z)$ such that $f_2(z) = h^n(z)\tilde{f_2}(z)$, $g_2(z) = h^m(z)\tilde{g_2}(z)$, and $\tilde{f_2}(z)$ and $\tilde{g_2}(z)$ do not have zeros. By Lemma 1.1, there exist entire functions $\phi(z)$ and $\psi(z)$ such that $\tilde{f_2}(z) = e^{\phi(z)}$, $\tilde{g_2}(z) = e^{\psi(z)}$, and so $f_2(z) = h^n(z)e^{\phi(z)}$, $g_2(z) = h^m(z)e^{\psi(z)}$.

We now plug in the expressions for $f_2(z)$ and $g_2(z)$ into (4) and cancel out $h^{nm}(z)$ to obtian:

$$f_1^m(z)e^{n\psi(z)} - g_1^n(z)e^{m\phi(z)} = h^{nm}(z)e^{n\psi(z)+m\phi(z)}.$$

Let $\tilde{f_1}(z) = f_1(z)e^{\frac{n\psi(z)}{m}}$, $\tilde{g_1}(z) = g_1(z)e^{\frac{m\phi(z)}{n}}$ and $\tilde{h}(z) = h(z)e^{\frac{n\psi(z)+m\phi(z)}{nm}}$. In terms of these functions the last equation can be rewritten as $\tilde{f_1}^m(z) = \tilde{g_1}^n(z) + \tilde{h}^{nm}(z)$ and so factorize as

$$\tilde{f_1}^m(z) = \prod_{j=1}^{n}(\tilde{g_1}(z) - \varepsilon_j\tilde{h}^m(z)),$$

where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are the $n-$th root of -1. Observe that different brackets in the right hand side do not have common zeros. An argument similar to that in the proof of Theorem 2.2, leads us to the Fermat's equation

$$\hat{h}^m(z) = \hat{h_1}^m(z) + \hat{h_2}^m(z),$$

where $\hat{h}(z) = \sqrt[m]{\varepsilon_2 - \varepsilon_1}\tilde{h}(z)$, $\hat{h_1}(z) = h_1(z)$, and $\hat{h}_2(z) = \sqrt[m]{-1}h_2(z)$. Note, that each $\hat{h_1}(z)$ and $\hat{h_2}(z)$ have finite number of zeros, because $\prod_{j=1}^{n} h_j(z) = \tilde{f_1}(z)$, and $\tilde{f_1}(z)$ and $f_1(z)$ have finitely many zeros. Since $m > 2$, we can apply Theorem 2.2 and conclude that both $\hat{h}(z)$ and $\hat{h_1}(z)$ do not have zeros. Consequently, $f(z)$ and $g(z)$ have to be both entire functions, and since $f_1(z)$ does not have zeros, $f(z)$ also does not have zeros. Lemma 1.1, now impies that there exists $\theta(z)$, such that $f(z) = e^{\theta(z)}$. Plugging into (3),we get the equation $e^{m\theta(z)} - g^n(z) = 1$, that can be rewritten as

$$\left(e^{\frac{m\theta(z)}{n}}\right)^n - g^n(z) = 1.$$

If $n > 2$, Theorem 2.2, implies that the solutions are constant. If $n = 2$, Lemma 2.4 implies an existence of entire $\vartheta(z)$,

$$e^{\frac{m\theta(z)}{n}} = \frac{e^{\vartheta(z)} + e^{-\vartheta(z)}}{2}.$$

The last equality is impossible, since the left hand side omits value 0, while the right hand side takes this value infinitely often. □

Corollary 2.6 (Catalan's equation for rational functions)**.** *If $f(z)$ and $g(z)$ are non constant rational functions and $m, n \geq 2$, then the Catalan's equation*

$$f^m(z) - g^n(z) = 1$$

has solutions only for $m = n = 2$.

We now proceed with studying Waring type problems for analytic functions. Classical Waring problem for polynomials looked over additive decomposition of polynomials as sums of n-th powers of polynomials.

Theorem 2.7. *Let $P[(z)] \in \mathbb{C}[z]$ be a polynomial of degree n. Then for any entire function f there exist entire functions $f_1(z), f_2(z), ...f_k(z)$ and a natural number $k = k(P)$ such that*

$$f(z) = \sum_{i=1}^{k} P(f_i(z)).$$

Proof. For each polynomial $Q(z) \in \mathbb{C}[z]$ of degree $m \geq 0$, define $\Delta_1(Q(z)) := Q(z+1) - Q(z)$. Observe that $\Delta_1(Q(z))$ is a polynomial of degree $m-1$ that is Δ_1 reduces the degree of Q by exactly one degree. You can now define the operator $\Delta_n : \mathbb{C}[z] \to \mathbb{C}[z]$ recursively where $\Delta_n(Q(z)) := \Delta_1(\Delta_{n-1}(Q(z)))$. Observe that for each polynomial $Q(z)$

$$\deg(\Delta_n(Q(z))) = \begin{cases} m-n & \text{if } m \geq n \\ 0 & \text{otherwise} \end{cases}$$

Suppose that $P[(z)] \in \mathbb{C}[z]$ is a polynomial of degree n. Then by the above observation $\Delta_{n-1}(P(z)) = a_0 z + b_0$ is a polynomial of degree 1. Now for any entire function f the desired representation is obtained by taking $z = \frac{f(z)-b_0}{a_0}$ in $\Delta_{n-1}(P(z)) = a_0 z + b_0$. □

Remark 2.8. Lemma 2.7 is also true for multivariable polynomials $P(z_1, z_2, ..., z_n)$. To see this note that by taking $(\alpha_1, \alpha_2, ..., \alpha_n)$ such that $P(\alpha_1, \alpha_2, ..., \alpha_n) \neq 0$ and performing the substitution $z_1 := \alpha_1 z$, $z_2 := \alpha_2 z$,...$z_n := \alpha_n z$ we can derive a one variable non-zero polynomial. Now we can apply theorem 2.7 to get the entire functions $f_1(z), f_2(z), ...f_k(z)$. We have that

$$f(z) = \sum_{i=1}^{k} P(\alpha_1 f_i(z), \alpha_2 f_i(z), ..., \alpha_n f_i(z)).$$

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