THE POLYGONAL CYLINDER AND ITS HOSOYA POLYNOMIAL

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Abstract. We introduce a polygonal cylinder $C_{m,n}$, using the Cartesian product of paths P_m and P_n and using topological identification of vertices and edges of two opposite sides of $P_m \times P_n$, and give its Hosoya polynomial, which, depending on odd and even m, is covered in seven separate cases.

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1. Introduction

The Hosoya (or Wiener) polynomial was introduced by Hosoya in 1988 to count the number of paths of different lengths in G [7]. The most interesting application of the Hosoya polynomial is that almost all distance-based graph invariants, which are used to predict physical, chemical and pharmacological properties of organic molecules, can be recovered from it.

Hosoya polynomial has been computed for several classes of graphs. In 2002 Diudea computed the Hosoya polynomial of several classes of toroidal nets and recovered their Wiener indices [2]. In 2011 Ali found the Hosoya polynomial of concatenated pentagonal rings [1]. In 2012 Kishori gave a recursive method for calculating the Hosoya polynomial of Hanoi graphs, and computed some of their distance-based invariants [8]. In 2013 Farahani computed the Hosoya polynomial of polycyclic aromatic hydrocarbons [3]. To learn more about Hosoya polynomial see [4, 5, 6, 9, 10, 11, 12, 13, 14].

This paper is organized as follows: The basic definitions are given in Section 2, main results are presented in Section 3, and conclusive remarks are given in Section 4.

2. Preliminary Notes

A *graph* G is a pair (V, E), where V is the set of vertices and E the set of edges. The edge e between two vertices u and v is denoted by $e = (u, v)$. A *path* from a vertex v to a vertex w is a sequence of vertices and edges that starts from v and stops at w. The number of edges in a path is the *length* of that path. The *distance* between two vertices u and v, denoted by $d(u, v)$, is the length of the shortest path between them. The *diameter* of G, denoted by $d(G)$, is the longest distance in G. A graph is said to be *connected* if there is a path between any two of its vertices.

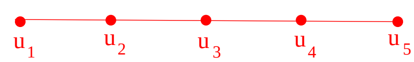
Definition 2.1. A function I which assigns to every connected graph G a unique number $I(G)$ is called a *graph invariant*. Instead of the function I it is custom to say the number $I(G)$ as the invariant.

Definition 2.2. [7] The Hosoya polynomial of a connected graph G is defined as

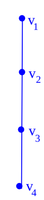
$$H(G,x) = \sum_{\{v,u\}\in V} x^{d(u,v)} = \sum_{k=1}^{d(G)} d(G,k)x^k.$$

where $d(u,v)$ is the distance between u and v and $d(G,k)$ is the number of pairs of vertices of G laying at distance k from each other.

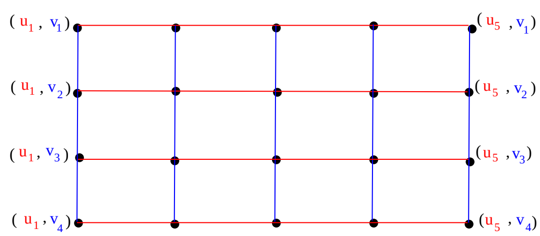
Definition 2.3. Consider the Cartesian product $P_m \times P_n$ of paths $P_m, m \geq 4$, and $P_n, n \geq 2$, with vertices $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, u_n$, respectively. Identify the vertices (u_1, v_1), $(u_1, v_2), \ldots, (u_1, v_n)$ with the vertices (u_m, v_1), (u_m, v_2), $\ldots, (u_m, v_n)$, respectively, and identify the edge $\big((u_1, v_i), (u_1, v_{i+1})\big)$ with the edge $\big((u_m, v_i), (u_m, v_{i+1})\big)$, where $1 \leq i \leq n-1$. What we receive is the polygonal $C_{m,n}$; we may call it (m-1)-gonal cylinder. You can see $C_{5,4}$ along with its grid form in the figure:



P_5

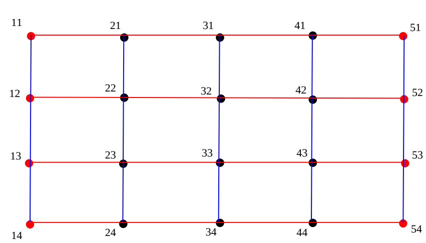


P_4



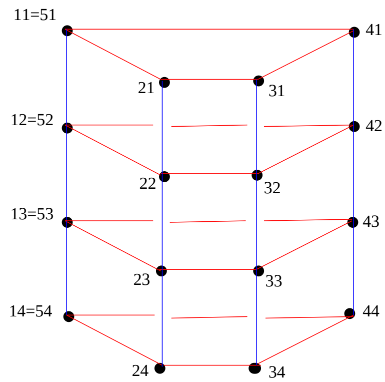
$P_5 \times P_4$

For brevity we shall use the symbol $v_{i,j}$ (v_{ij} or simply ij) to represent the vertex (u_i, v_j) of $C_{m,n}$. In the following you can see the grid form of $C_{5,4}$ along with simple labels.



Grid form of $C_{5,4}$

The polygonal cylinder obtained from $P_5 \times P_4$ is:



$C_{5,4}$

3. Main Results

Here we give the Hosoya polynomial of the polygonal cylinder and give closed formulas of all seven possible cases depending on odd and even m.

Theorem 3.1. *Let $m > 2n+1$ be odd, and $n \geq 3$. Then the Hosoya polynomial of the polygonal cylinder $C_{m,n}$ is*

$$\begin{aligned} H(C_{m,n}) &= \sum_{k=1}^{n-1} c_k x^k + \sum_{k=0}^{\frac{m-1}{2}-1-n} c_{n+k} x^{n+k} + c_{\frac{m-1}{2}} x^{\frac{m-1}{2}} \\ &\quad + \sum_{k=1}^{n-1} c_{\frac{m-1}{2}+k} x^{\frac{m-1}{2}+k}, \end{aligned}$$

where $c_k = (m-1)(2kn-k^2), c_{n+k} = (m-1)n^2, c_{\frac{m-1}{2}} = (m-1)(n^2 - \frac{n}{2})$, and $c_{\frac{m-1}{2}+k} = (m-1)(n-k)^2$.

Proof. We prove it using the distance matrix D corresponding to the polygonal cylinder $C_{m,n}$, which is symmetric and have order $(m-1)n \times (m-1)n$. Each row of D represents the distances from a vertex v_{ij} to the vertices $v_{1,1}, v_{1,2}, \ldots, v_{1,n}, v_{2,1}, v_{2,2}, \ldots, v_{2,n}, \ldots, v_{m-1,1}, v_{m-1,2}, \ldots, v_{m-1,n}$, respectively. Since we need distinct paths, we shall consider only its upper triangular part. For this we represent the upper-triangular part by submatrices. There are $\frac{m+1}{2}$ distinct submatrices $A_0, A_1, A_2, A_3, \ldots, A_{\frac{m-3}{2}}$, and $A_{\frac{m-1}{2}}$. All these submatrices are symmetric, each having order $n \times n$. Each A_i appears $m-1$ times except $A_{\frac{m-1}{2}}$, which appears $\frac{m-1}{2}$ times. A_0 appears only on the main diagonal of D. $A_i, 1 \leq i \leq \frac{m-3}{2}$, appears $m-(i+1)$ times in *ith* secondary diagonal and i times in $(m-(i-1))th$ secondary diagonal. $A_{\frac{m-1}{2}}$ appears only in $\frac{m-1}{2}$th secondary diagonal.

Thus, the general form of the distance matrix D is:

$$\begin{pmatrix}
A_0 & A_1 & A_2 & \cdots & A_{\frac{m-3}{2}} & A_{\frac{m-1}{2}} & A_{\frac{m-3}{2}} & A_{\frac{m-5}{2}} & \cdots & A_2 & A_1 \\
 & A_0 & A_1 & \cdots & A_{\frac{m-5}{2}} & A_{\frac{m-3}{2}} & A_{\frac{m-1}{2}} & A_{\frac{m-3}{2}} & \cdots & A_3 & A_2 \\
 & & A_0 & \cdots & A_{\frac{m-7}{2}} & A_{\frac{m-5}{2}} & A_{\frac{m-3}{2}} & A_{\frac{m-1}{2}} & \cdots & A_4 & A_3 \\
 & & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & A_0 & A_1 & A_2 & A_3 & \cdots & A_{\frac{m-3}{2}} & A_{\frac{m-1}{2}} \\
 & & & & & A_0 & A_1 & A_2 & \cdots & A_{\frac{m-5}{2}} & A_{\frac{m-3}{2}} \\
 & & & & & \ddots & & \ddots & \vdots & \vdots & \vdots \\
 & & & & & & & A_0 & A_1 & A_2 & A_3 \\
 & & & & & & & & A_0 & A_1 & A_2 \\
 & & & & & & & & & A_0 & A_1 \\
 & & & & & & & & & & A_0
\end{pmatrix}$$

Now we give the entries of the submatrices. Since A_0 lies on the main diagonal of D, only its upper triangular part contributes towards counting the distinct paths. So, A_0 is

$$A_0 = \begin{pmatrix}
0 & 1 & 2 & 3 & \cdots & n-2 & n-1 \\
 & 0 & 1 & 2 & \cdots & n-3 & n-2 \\
 & & 0 & 1 & \cdots & n-4 & n-3 \\
 & & & \ddots & \vdots & \vdots & \vdots \\
 & & & & 0 & 1 & 2 \\
 & & & & & 0 & 1 \\
 & & & & & & 0
\end{pmatrix}.$$

However, although all the entries of $A_i, 1 \leq i \leq \frac{m-1}{2}$, contribute towards counting the distinct paths, we give only entries of its upper triangular part as it is symmetric.

$$A_i = \begin{pmatrix}
i & i+1 & i+2 & i+3 & \cdots & i+(n-2) & i+(n-1) \\
 & i & i+1 & i+2 & \cdots & i+(n-3) & i+(n-2) \\
 & & i & i+1 & \cdots & i+(n-4) & i+(n-3) \\
 & & & \ddots & \vdots & \vdots & \vdots \\
 & & & & i & i+1 & i+2 \\
 & & & & & i & i+1 \\
 & & & & & & i
\end{pmatrix}$$

Now we give $c_is, 1 \leq i \leq \frac{m-1}{2} + n - 1$, which is the number of paths of length i.

$c_k =$ (no. of k in A_0) $\times$ (no. of A_0s) + (no. of k in A_1) $\times$ (no. of A_1s) + (no. of k in A_2) $\times$ (no. of A_2s) $+ \cdots +$ (no. of k in A_{k-1}) $\times$ (no. of $A_{k-1}s$) + (no. of k in A_k) $\times$ (no. of A_ks)
$= (n-k)(m-1) + 2(n-(k-1))(m-1) + 2(n-(k-2))(m-1) + \cdots + 2(n-1)(m-1) + n(m-1)$
$= (m-1)(n-k+2(n-(k-1)+2(n-(k-2)) + 2(n-(k-3)) + \cdots + 2(n-1) + n)$
$= (m-1)(n-k+2n-2(k-1)+2n-2(k-2)+2n-2(k-3)+\cdots+2n-2(1)+$

$n)(m-1)(2kn-2(1+2+3+\cdots+k-1)-k)$
$=(m-1)(2kn-(k-1)(k)-k)=(m-1)(2kn-k^2)$

Now we go for c_{n+k}:

$c_{n+k}=$ (no. of $n+k$ in A_{k+1}) $\times$ (no. of A_{k+1}) + (no. of $n+k$ in A_{k+2}) $\times$ (no. of A_{k+2}) + (no. of $n+k$ in A_{k+3}) $\times$ (no. of A_{k+3}) $+\cdots+$ (no. of $n+k$ in A_{k+n-1}) $\times$ (no. of A_{k+n-1}) + (no. of $n+k$ in A_{k+n}) $\times$ (no. of A_{k+n})
$=2(m-1)+2(2)(m-1)+2(3)(m-1)+\cdots+2(n-1)(m-1)+n(m-1)$
$=(m-1)[2(1+2+3+\cdots+n-1)+n]=(m-1)[(n-1)n+n]=(m-1)n^2$

$c_{\frac{m-1}{2}}=$ (no. of $\frac{m-1}{2}$ in $A_{\frac{m-1}{2}+1-n}$) $\times$ (no. of $A_{\frac{m-1}{2}+1-n}$) + (no. of $\frac{m-1}{2}$ in $A_{\frac{m-1}{2}+2-n}$) $\times$ (no. of $A_{\frac{m-1}{2}+2-n}$) + (no. of $\frac{m-1}{2}$ in $A_{\frac{m-1}{2}+3-n}$) $\times$ (no. of $A_{\frac{m-1}{2}+3-n}$) + (no. of $\frac{m-1}{2}$ in $A_{\frac{m-1}{2}+4-n}$) $\times$ (no. of $A_{\frac{m-1}{2}+4-n}$) $+\cdots+$ (no. of $\frac{m-1}{2}$ in $A_{\frac{m-1}{2}-1}$) $\times$ (no. of $A_{\frac{m-1}{2}-1}$) + (no. of $\frac{m-1}{2}$ in $A_{\frac{m-1}{2}}$) $\times$ (no. of $A_{\frac{m-1}{2}}$)
$=2(m-1)+2(2)(m-1)+2(3)(m-1)+\cdots+2(n-1)(m-1)+n(\frac{m-1}{2})$
$=(m-1)(2(1+2+3+\ldots+(n-1))+\frac{n}{2}(m-1)=(m-1)[\frac{(2n-2)(n)+n}{2}]=(m-1)(n^2-\frac{n}{2})$

Finally, $c_{\frac{m-1}{2}+k}, 1\leq k\leq n-1$:

$c_{\frac{m-1}{2}+k}=$ (no. of $\frac{m-1}{2}+k$ in $A_{\frac{m-1}{2}-n+(k+1)}$) $\times$ (no. of $A_{\frac{m-1}{2}-n+(k+1)}$) + (no. of $\frac{m-1}{2}+k$ in $A_{\frac{m-1}{2}-n+(k+2)}$) $\times$ (no. of $A_{\frac{m-1}{2}-n+(k+2)}$) + (no. of $\frac{m-1}{2}+k$ in $A_{\frac{m-1}{2}-n+(k+3)}$)$\times$ (no. of $A_{\frac{m-1}{2}-n+(k+3)}$) $+\cdots+$ (no. of $\frac{m-1}{2}+k$ in $A_{\frac{m-1}{2}-1}$) $\times$ (no. of $A_{\frac{m-1}{2}-1}$) + (no. of $\frac{m-1}{2}+k$ in $A_{\frac{m-1}{2}}$) $\times$ (no. of $A_{\frac{m-1}{2}}$)
$=(2)(m-1)+2(2)(m-1)+2(3)(m-1)+\ldots+2(n-(k+1))+2(n-k)(\frac{m-1}{2})$
$=(m-1)(2(1+2+3+\cdots+n-(k+1)))+(n-k)(m-1)$
$=(m-1)[(n-(k+1))(n-k)+(n-k)]=(m-1)(n-k)^2$ □

Example. The Hossoya polynomial for $C_{11,3}$ is $H(C_{11,3}) = 50x + 80x^2 + 90x^3 + 90x^4 + 75x^5 + 40x^6 + 10x^7$. Here the distance matrix is

$$D = \begin{pmatrix} A_0 & A_1 & A_2 & A_3 & A_4 & A_5 & A_4 & A_3 & A_2 & A_1 \\ & A_0 & A_1 & A_2 & A_3 & A_4 & A_5 & A_4 & A_3 & A_2 \\ & & A_0 & A_1 & A_2 & A_3 & A_4 & A_5 & A_4 & A_3 \\ & & & A_0 & A_1 & A_2 & A_3 & A_4 & A_5 & A_4 \\ & & & & A_0 & A_1 & A_2 & A_3 & A_4 & A_5 \\ & & & & & A_0 & A_1 & A_2 & A_3 & A_4 \\ & & & & & & A_0 & A_1 & A_2 & A_3 \\ & & & & & & & A_0 & A_1 & A_2 \\ & & & & & & & & A_0 & A_1 \\ & & & & & & & & & A_0 \end{pmatrix},$$

$A_0 = \begin{pmatrix} 0 & 1 & 2 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$, and $A_i = \begin{pmatrix} i & i+1 & i+2 \\ i+1 & i & i+1 \\ i+2 & i+1 & i \end{pmatrix}$, $1 \leq i \leq 5$.

Theorem 3.2. *Let $m < 2n-1$ be odd, and $n \geq 3$. Then*

$$\begin{aligned} H(C_{m,n}) &= \sum_{k=1}^{\frac{m-1}{2}-1} c_k x^k + c_{\frac{m-1}{2}} x^{\frac{m-1}{2}} + \sum_{k=1}^{n-1-(\frac{m-1}{2})} c_{\frac{m-1}{2}+k} x^{\frac{m-1}{2}+k} \\ &\quad + \sum_{k=0}^{\frac{m-1}{2}-1} c_{n+k} x^{n+k}, \end{aligned}$$

where $c_k = (m-1)(2nk-k^2)$, $c_{\frac{m-1}{2}} = \frac{-1}{4}(m-1)[m^2-(4n+2)m+(6n+1)]$, $c_{\frac{m-1}{2}+k} = \frac{-1}{4}(m-1)[m^2-2(2n-2k+1)m+(4n-4k+1)]$, and $c_{n+k} = \frac{1}{4}(m-1)[m-2k-1]^2$.

Proof. For D and its submatrices we refer to Theorem 3.1. Now we give $c_i s, 1 \leq i \leq \frac{m-1}{2}+n-1$. The c_k is same as is given in Theorem 3.1; we need to find $c_{\frac{m-1}{2}}$, $c_{\frac{m-1}{2}+k}, 1 \leq k \leq n-1-\frac{m-1}{2}$, and $c_{n+k}, 0 \leq k \leq \frac{m-1}{2}-1$.

$$\begin{aligned} c_{\frac{m-1}{2}} &= (\text{no. of } \tfrac{m-1}{2} \text{ in } A_0) \times (\text{no. of } A_0 s) + (\text{no. of } \tfrac{m-1}{2} \text{ in } A_1) \times (\text{no. of } A_1 s) + \\ &(\text{no. of } \tfrac{m-1}{2} \text{in } A_2) \times (\text{no. of } A_2 s) + (\text{no. of } \tfrac{m-1}{2} \text{in } A_3) \times (\text{no. of } A_3 s) + \cdots \\ &+ (\text{no. of } \tfrac{m-1}{2} \text{ in } A_{\frac{m-1}{2}-1}) \times (\text{no. of } A_{\frac{m-1}{2}-1} s) + (\text{no. of } \tfrac{m-1}{2} \text{ in } A_{\frac{m-1}{2}}) \times (\text{no. of} A_{\frac{m-1}{2}} s) \\ &= (n-\tfrac{m-1}{2})(m-1) + 2(n-\tfrac{m-1}{2}+1)(m-1) + (n-\tfrac{m-1}{2}+2)(m-1) + \\ &(n-\tfrac{m-1}{2}+3)(m-1) + \cdots + (n-\tfrac{m-1}{2}+\tfrac{m-1}{2}-1)(m-1) + n(\tfrac{m-1}{2}) \\ &= (n-\tfrac{m-1}{2}) + 2[(\tfrac{1}{2}\tfrac{m-3}{2})(2n-\tfrac{m-1}{2})](m-1) + \tfrac{n}{2}(m-1) \\ &= [(\tfrac{1}{2}(3n-m+1) + (m-3)(n-\tfrac{m-1}{4}))](m-1) \\ &= \tfrac{-1}{4}(m-1)[m^2-(4n+2)m+(6n+1)]. \end{aligned}$$

Now, $c_{\frac{m-1}{2}+k}, 1 \leq k \leq n-1-\frac{m-1}{2}$:

$c_{\frac{m-1}{2}+k} =$ (no. of $\frac{m-1}{2}+k$ in A_0) $\times$ (no. of A_0s) + (no. of $\frac{m-1}{2}+k$ in A_1) $\times$ (no. of A_1s) + (no. of $\frac{m-1}{2}+k$ in A_2) $\times$ (no. of A_2s) + (no. of $\frac{m-1}{2}+k$ in A_3) $\times$ (no. of A_3s) $+\cdots+$ (no. of $\frac{m-1}{2}+k$ in $A_{\frac{m-1}{2}-1}$) $\times$ (no. of $A_{\frac{m-1}{2}-1}s$) + (no. of$\frac{m-1}{2}+k$ in $A_{\frac{m-1}{2}}$) $\times$ (no. of$A_{\frac{m-1}{2}}s$)
$= (n-\frac{m-1}{2}-k)(m-1)+2(n-\frac{m-1}{2}+1-k)(m-1)+(n-\frac{m-1}{2}+2-k)(m-1)+$
$(n-\frac{m-1}{2}+3-k)(m-1)+\cdots+(n-\frac{m-1}{2}+\frac{m-1}{2}-k-1)(m-1)+2(n-k)(\frac{m-1}{2})$
$= \frac{-1}{4}(m-1)[m^2-2(2n-2k+1)m+(4n-4k+1)]$.

Finally, $c_{n+k}, 0 \leq k \leq \frac{m-1}{2}-1$:

$c_{n+k} =$ (no. of $n+k$ in A_{k+1}) $\times$ (no. of A_{k+1}) + (no. of $n+k$ in A_{k+2}) $\times$ (no. of A_{k+2}) + (no. of $n+k$ in A_{k+3}) $\times$ (no. of A_{k+3}) $+\cdots+$ (no. of $n+k$ in $A_{\frac{m-1}{2}-1}$) $\times$ (no. of $A_{\frac{m-1}{2}-1}$) + (no. of $n+k$ in $A_{\frac{m-1}{2}}$) $\times$ (no. of $A_{\frac{m-1}{2}}$)
$= 2(m-1)+2(2)(m-1)+2(3)(m-1)+\cdots+2(\frac{m-1}{2}-k-1)(m-1)+$
$2(\frac{m-1}{2}-k)(\frac{m-1}{2}) = (\frac{m-1}{2}-k)^2$. □

Theorem 3.3. *Let $m = 2n-1$ and $n \geq 3$. Then*

$$H(C_{m,n}) = \sum_{k=1}^{\frac{m-1}{2}-1} c_k x^k + c_{\frac{m-1}{2}} x^{\frac{m-1}{2}} + \sum_{k=0}^{n-2} c_{\frac{m-1}{2}+k+1} x^{\frac{m-1}{2}+k+1},$$

where $c_k = (m-1)(2nk-k^2)$, $c_{\frac{m-1}{2}} = \frac{-1}{4}(m-1)[m^2-(4n+2)m+(6n+1)]$, *and* $c_{\frac{m-1}{2}+k+1} = \frac{1}{4}(m-1)[m-2k-1]^2$.

Proof. For c_k and $c_{\frac{m-1}{2}}$ see the previous proofs. Since $m = 2n-1$, $c_{\frac{m-1}{2}+k+1}$ becomes c_{n+k}, which is also proved in Theorem 3.2. □

Theorem 3.4. *Let $m = 2n+1$ and $n \geq 3$. Then the Hosoya polynomial of the polygonal cylinder $C_{m,n}$ is*

$$H(C_{m,n}) = \sum_{k=1}^{\frac{m-1}{2}-1} c_k x^k + c_{\frac{m-1}{2}} x^{\frac{m-1}{2}} + \sum_{k=0}^{n-2} c_{\frac{m-1}{2}+k+1} x^{\frac{m-1}{2}+k+1},$$

where $c_k = (m-1)(2nk-k^2), c_{\frac{m-1}{2}} = (m-1)[n^2-\frac{n}{2}]$, *and*
$c_{\frac{m-1}{2}+k+1} = \frac{1}{4}(m-1)[m-2k-1]^2$.

Proof. For c_k and $c_{\frac{m-1}{2}}$ see Theorem 3.1. Here $c_{\frac{m-1}{2}+k}$ becomes c_{n+k}, which is also proved in Theorem 3.2. □

Theorem 3.5. *Let $m > 2n$ be even, and $n \geq 3$. Then the Hosoya polynomial of the polygonal cylinder $C_{m,n}$ is*

$$H(C_{m,n}) = \sum_{k=1}^{n-1} c_k x^k + \sum_{k=0}^{\frac{m}{2}-n-1} c_{n+k} x^{n+k} + \sum_{k=0}^{n-2} c_{\frac{m}{2}+k} x^{\frac{m}{2}+k},$$

where $c_k = (m-1)(2kn-k^2), c_{n+k} = (m-1)n^2$, and $c_{\frac{m}{2}+k} = (m-1)[(n-k)^2-(n-k)]$.

Proof. The distance matrix D is

$$\begin{pmatrix}
A_0 & A_1 & A_2 & \cdots & A_{\frac{m}{2}-2} & A_{\frac{m}{2}-1} & A_{\frac{m}{2}-1} & A_{\frac{m}{2}-2} & A_{\frac{m}{2}-3} & \cdots & A_2 & A_1 \\
 & A_0 & A_1 & \cdots & A_{\frac{m}{2}-3} & A_{\frac{m}{2}-2} & A_{\frac{m}{2}-1} & A_{\frac{m}{2}-1} & A_{\frac{m}{2}-2} & \cdots & A_3 & A_2 \\
 & & A_0 & \cdots & A_{\frac{m}{2}-4} & A_{\frac{m}{2}-3} & A_{\frac{m}{2}-2} & A_{\frac{m}{2}-1} & A_{\frac{m}{2}-1} & \cdots & A_4 & A_3 \\
 & & & \ddots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
 & & & & A_0 & A_1 & A_2 & A_3 & A_4 & \cdots & A_{\frac{m}{2}-1} & A_{\frac{m}{2}-1} \\
 & & & & & A_0 & A_1 & A_2 & A_3 & \cdots & A_{\frac{m}{2}-2} & A_{\frac{m}{2}-1} \\
 & & & & & & A_0 & A_1 & A_2 & \cdots & A_{\frac{m}{2}-3} & A_{\frac{m}{2}-2} \\
 & & & & & & & \ddots & \cdots & \vdots & \vdots & \vdots \\
 & & & & & & & & A_0 & A_1 & A_2 & A_3 \\
 & & & & & & & & & A_0 & A_1 & A_2 \\
 & & & & & & & & & & A_0 & A_1 \\
 & & & & & & & & & & & A_0
\end{pmatrix}$$

Each submatrix $A_0, A_1, A_2, A_3, \ldots, A_{\frac{m}{2}-2}$, and $A_{\frac{m}{2}-1}$ appears $m-1$ times. A_0 appears only on the main diagonal of D. $A_i, 1 \leq i \leq \frac{m}{2}-1$, appears $m-(i+1)$ times in ith secondary diagonal and i times in $[m-(i-1)]th$ secondary diagonal. These submatrices are same as are in Theorem 3.1. The proofs of c_k and c_{k+n} are given in Theorem 3.1. We need only $c_{\frac{m}{2}+k}$.

$c_{\frac{m}{2}+k} =$ (no. of $\frac{m}{2}+k$ in $A_{\frac{m}{2}-n+(k+1)}$) $\times$ (no. of $A_{\frac{m}{2}-n+(k+1)}$) $+$ (no. of $\frac{m}{2}+k$ in $A_{\frac{m}{2}-n+(k+2)}$) $\times$ (no. of $A_{\frac{m}{2}-n+(k+2)}$) $+$ (no. of $\frac{m}{2}+k$ in $A_{\frac{m}{2}-n+(k+3)}$) $\times$ (no. of $A_{\frac{m}{2}-n+(k+3)}$) $+ \cdots +$ (no. of $\frac{m}{2}+k$ in $A_{\frac{m}{2}-2}$) $\times$ (no. of $A_{\frac{m}{2}-2}$) $+$ (no. of$\frac{m}{2}+k$ in $A_{\frac{m}{2}-1}$) $\times$ (no. of $A_{\frac{m}{2}-1}$)$c_{\frac{m}{2}+k}$
$= 2(m-1) + 2(2)(m-1) + 2(3)(m-1) + \cdots + 2(n-(k+2))(m-1) +$
$2(n-(k+1))(m-1)$
$= (m-1)[2(1+2+3+\cdots+(n-(k+2)) + (n-(k+1))]$
$= (m-1)[(n-(k+1))(n-k)] = (m-1)[(n-k)^2-(n-k)].$ □

Example. The Hossoya polynomial for $C_{10,3}$ is $H(C_{10,3}) = 45x + 72x^2 + 81x^3 + 54x^4 + 18x^6$. Its distance matrix is

$$D = \begin{pmatrix} A_0 & A_1 & A_2 & A_3 & A_4 & A_4 & A_3 & A_2 & A_1 \\ & A_0 & A_1 & A_2 & A_3 & A_4 & A_4 & A_3 & A_2 \\ & & A_0 & A_1 & A_2 & A_3 & A_4 & A_4 & A_3 \\ & & & A_0 & A_1 & A_2 & A_3 & A_4 & A_4 \\ & & & & A_0 & A_1 & A_2 & A_3 & A_4 \\ & & & & & A_0 & A_1 & A_2 & A_3 \\ & & & & & & A_0 & A_1 & A_2 \\ & & & & & & & A_0 & A_1 \\ & & & & & & & & A_0 \end{pmatrix},$$

where $A_0 = \begin{pmatrix} 0 & 1 & 2 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$ and $A_i = \begin{pmatrix} i & i+1 & i+2 \\ i+1 & i & i+1 \\ i+2 & i+1 & i \end{pmatrix}$, $1 \leq i \leq 4$.

Theorem 3.6. *Let $m < 2n$ be even, and $n \geq 3$. Then the Hosoya polynomial of the polygonal cylinder $C_{m,n}$ is*

$$H(C_{m,n}) \;=\; \sum_{k=1}^{\frac{m}{2}-1} c_k x^k + \sum_{k=0}^{n-\frac{m}{2}-1} c_{\frac{m}{2}+k} x^{\frac{m}{2}+k} + \sum_{k=0}^{\frac{m}{2}-2} c_{n+k} x^{n+k},$$

where $c_k = (m-1)(2kn-k^2)$, $c_{\frac{m}{2}+k} = \frac{-1}{4}(m-1)[m^2 - 4(n-k)m - (4k-4n)]$, and $c_{n+k} = \frac{1}{4}(m-1)[m^2 - 2(2k+1)m + 4k(k+1)]$.

Proof. Everything is same as is in Theorem 3.5. We need only $c_{\frac{m}{2}+k}$ and c_{n+k}.

$c_{\frac{m}{2}+k} =$ (no. of $\frac{m}{2}+k$ in A_0) $\times$ (no. of A_0s) $+$ (no. of $\frac{m}{2}+k$ in A_1) $\times$ (no. of A_1s) $+$ (no. of $\frac{m}{2}+k$ in A_2) $\times$ (no. of A_2s) $+\cdots+$ (no. of $\frac{m}{2}+k$ in $A_{\frac{m}{2}-1}$) $\times$ (no. of $A_{\frac{m}{2}-1}s$)
$= (n-k-\frac{m}{2})(m-1) + 2(n-\frac{m}{2}-(k-1))(m-1) + 2(n-\frac{m}{2}-(k-2))(m-1) + \cdots + 2(n-\frac{m}{2}-(k-\frac{m}{2}+1)(m-1) = (m-1)[n-k-\frac{m}{2} + (m-2)(n-\frac{m}{4}-k)]$
$= \frac{-1}{4}(m-1)[m^2-4(n-k)m-(4k-4n)]$

$c_{n+k} =$ (no. of $n+k$ in A_{k+1}) $\times$ (no. of A_{k+1}) $+$ (no. of $n+k$ in A_{k+2}) $\times$ (no. of A_{k+2}) $+$ (no. of $n+k$ in A_{k+3}) $\times$ (no. of A_{k+3}) $+\cdots+$
(no. of $n+k$ in$A_{\frac{m}{2}-1}$) $\times$ (no. of $A_{\frac{m}{2}-1}$) $= 2(m-1) + 2(2)(m-1) + 2(3)(m-1) + \cdots + 2(\frac{m}{2}-k-1)(\frac{m-1}{2}) = \frac{1}{4}(m-1)[m^2-2(2k+1)m+4k(k+1)]$.

□

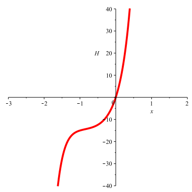
Theorem 3.7. *Let $m = 2n$ and $n \geq 3$. Then*

$$H(C_{m,n}) \quad = \quad \sum_{k=1}^{\frac{m}{2}-1} c_k x^k + \sum_{k=0}^{n-2} c_{\frac{m}{2}+k} x^{\frac{m}{2}+k},$$

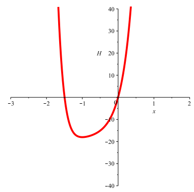
where $c_k = (m-1)(2kn - k^2)$, and $c_{\frac{m}{2}+k} = \frac{1}{4}(m-1)[m^2 - 2(2k+1)m + 4k(k+1)]$.

Proof. c_k is given in Theorem 3.1. Here $c_{\frac{m}{2}+k}$ becomes c_{n+k}, which is proved in Theorem 3.6. □

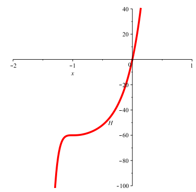
Remark 3.8. It is observed that if the Hosoya polynomial of the polygonal cylinder has an inflection point then it does not has any extrema, and if it has an extrema then it does not has any inflection point; you may see the situation in the following figures.



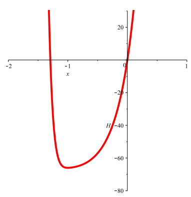
$H(C_{11,3})$



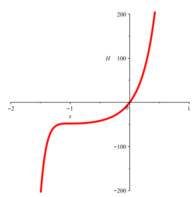
$H(C_{13,3})$



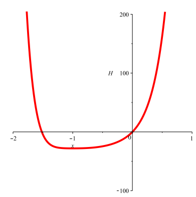
$H(C_{21,6})$



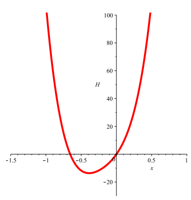
$H(C_{23,6})$



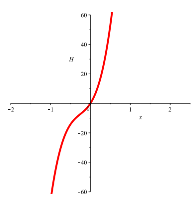
$H(C_{13,8})$



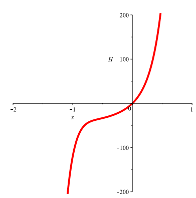
$H(C_{9,7})$



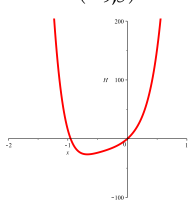
$H(C_{9,5})$



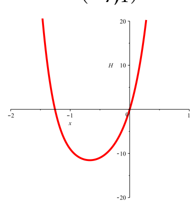
$H(C_{7,4})$



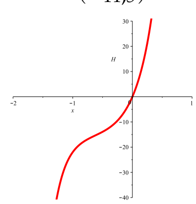
$H(C_{11,5})$



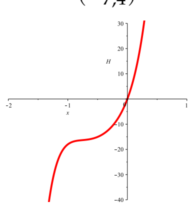
$H(C_{7,4})$



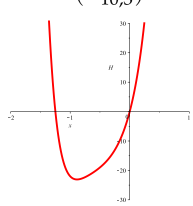
$H(C_{10,3})$



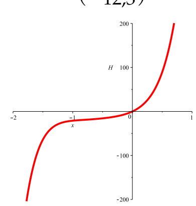
$H(C_{12,3})$



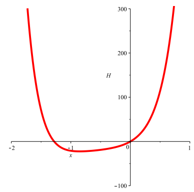
$H(C_{10,4})$



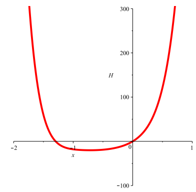
$H(C_{12,4})$



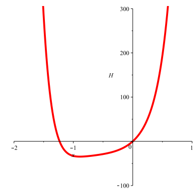
$H(C_{8,5})$



$H(C_{8,6})$



$H(C_{10,5})$



$H(C_{12,6})$

4. Conclusions

In this paper we introduced a polygonal cylinder $C_{m,n}$, using the Cartesian product of paths P_m, P_n and using topological identification of vertices and edges of two opposite sides of $P_m \times P_n$. The parameter m made the base while the parameter n made the length of $C_{m,n}$. Secondly, we gave general closed form of the Hosoya polynomial of $C_{m,n}$, which, depending on odd and even m, is covered in seven separate cases. We also gave two examples, one for odd m and one for even m. Moreover, we figured out that if the polynomial has an inflection point then it does not has any extrema, and if a polynomial has an extrema then it does not has any inflection point.

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