LIMIT SHAPES OF STABLE AND RECURRENT CONFIGURATIONS OF A GENERALIZED BULGARIAN SOLITAIRE

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ABSTRACT. Bulgarian solitaire is played on n cards divided into several piles; a move consists of picking one card from each pile to form a new pile. This can be seen as a process on the set of integer partitions of n: If sorted configurations are represented by Young diagrams, a move in the solitaire consists of picking all cards in the bottom layer of the diagram and inserting the picked cards as a new column. Here we consider a generalization, L-solitaire, wherein a fixed set of layers L (that includes the bottom layer) are picked to form a new column.

L-solitaire has the property that if a stable configuration of n cards exists it is unique. Moreover, the Young diagram of a configuration is convex if and only if it is a stable (fixpoint) configuration of some L-solitaire. If the Young diagrams representing card configurations are scaled down to have unit area, the stable configurations corresponding to an infinite sequence of pick-layer sets $(L_1, L_2, \dots)$ may tend to a limit shape ϕ. We show that every convex ϕ with certain properties can arise as the limit shape of some sequence of L_n. We conjecture that recurrent configurations have the same limit shapes as stable configurations.

For the special case $L_n = \{1, 1 + \lfloor 1/q_n \rfloor, 1 + \lfloor 2/q_n \rfloor, \dots\}$, where the pick layers are approximately equidistant with average distance $1/q_n$ for some $q_n \in (0, 1]$, these limit shapes are linear (in case $nq_n^2 \to 0$), exponential (in case $nq_n^2 \to \infty$), or interpolating between these shapes (in case $nq_n^2 \to C > 0$).

1. INTRODUCTION

The game of Bulgarian solitaire is played with a deck of n identical cards divided arbitrarily into several piles. A move consists of picking a card from each pile and letting these cards form a new pile. This move is repeated over and over again. For information about the earlier history of the Bulgarian solitaire and a summary of subsequent research, see reviews by Hopkins [10] and Drensky [5].

Let $\mathcal{P}$ denote the set of integer partitions. An integer partition of n is a $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$ and $\sum_{i=1}^{\ell} \lambda_i = n$. For $i > \ell$ it will be convenient to define $\lambda_i = 0$. The sum of the parts of λ is denoted by $|\lambda| = n$, and the number of non-zero parts is denoted by $\ell = \ell(\lambda)$. If piles of cards are sorted in order of decreasing size, any configuration of n cards can be regarded as an integer partition of n. A geometric shape arises when a configuration λ is represented by a Young diagram of unit squares in the first quadrant of a coordinate system for the real plane, such that the ith column has height λ_i. A move of the Bulgarian solitaire then has

Date: August 31, 2020.

the geometric interpretation of picking the first (i.e., bottom) layer of the diagram and making it the new first column, left-shifting cards if needed so that the configuration remains sorted. See Figure 1 for an example.

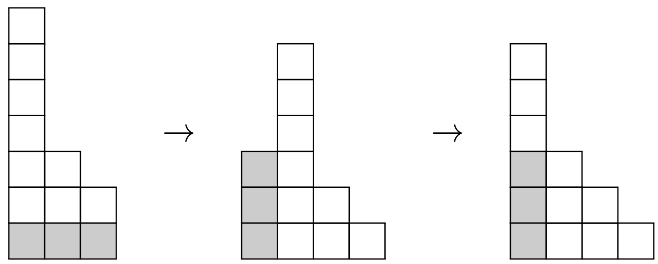

FIGURE 1. A move in Bulgarian solitaire from $\lambda = (7,3,2) \in \mathcal{P}(12)$: The bottom layer is picked to form a new pile with three cards, higher levels are then left-shifted.

In this paper we consider a generalization of Bulgarian solitaire, in which not only the bottom layer (layer number 1) is picked but also some other layers. This layer-based solitaire will be referred to as L-solitaire, where L is the set of layers to be picked. In terms of Young diagrams, a move of an L-solitaire on n cards consists of removing layers $L = \{h_1 = 1, h_2, h_3, \dots\} \subseteq \{1, 2, \dots, n\}$ of the Young diagram, counting from the bottom, to form a new column.[1] See Figure 2 for an example of a move in an L-solitaire. A set $L \ni 1$ of layer numbers may be referred to as a *pick-layer set*, and its elements as *pick layers*.

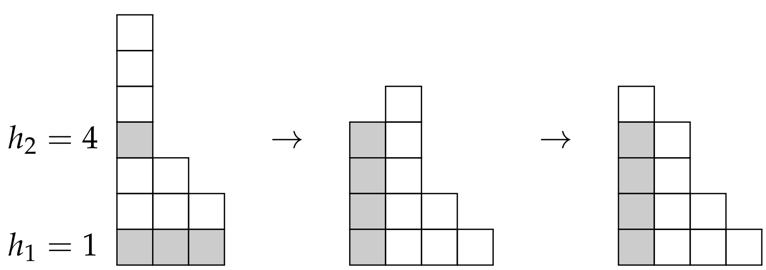

FIGURE 2. A move from the partition $\lambda = (7,3,2) \in \mathcal{P}(12)$ in the $\{1,4\}$-solitaire in which layers number $h_1 = 1$ and $h_2 = 4$ are picked to form a new pile with four cards.

[1] Of course, an even wider generalization would be to pick *any* layers (not necessarily including the bottom layer). However, if the bottom layer is not picked (i.e. if $h_1 > 1$), any layer $< h_1$ will never be picked and therefore none of its cards will ever "rotate", yielding a degenerate solitaire which eventually leads to a diagram with height $< h_1$ in which no card is picked.

1.1. **Outline of the paper.** For $L = \{1\}$ the L-solitaire reverts to the ordinary Bulgarian solitaire. In Section 2 we discuss how L-solitaire relates to an even more far-reaching generalization of Bulgarian solitaire by Olson [11].

In the remainder of the paper we shall be concerned with *stable* and *recurrent* configurations of the L-solitaire. These concepts can be defined as follows.

Definition 1. *For a given L-solitaire, let $f : \mathcal{P} \to \mathcal{P}$ denote the map defined by the rules for making a move in the solitaire. A configuration $\lambda \in \mathcal{P}$ is called* recurrent *with respect to this solitaire if there exists a positive integer k such that $f^k(\lambda) = \lambda$. A recurrent configuration that satisfies the stronger condition $f(\lambda) = \lambda$ is called* stable.

Bulgarian solitaire has the property that if a stable configuration exists for a given number of cards, it is unique [4]. In Section 3 we demonstrate that uniqueness of stable configurations holds for any L-solitaire.

In the Bulgarian solitaire, a stable configuration exists if and only if the total number n of cards is a triangular number, in which case the unique stable configuration is a staircase. In Section 4 we generalize this result by characterizing stable configurations of L-solitaires as convex, that is, satisfying the inequality $\lambda_i - \lambda_{i+1} \geq \lambda_{i+1} - \lambda_{i+2}$ for all $i \geq 1$.

In the Bulgarian solitaire, if n increases but the staircase shape is rescaled so that it always has the same area, the limit shape (as n tends to infinity) becomes a straight line segment of negative slope. In the more general case of L-solitaire we may let the pick-layer set L change with the number of cards. In Sections 5 and 6 we define limit shapes of stable configurations for an infinite sequence $\{L_n\}_{n=1}^{\infty}$ of pick-layer sets, and we show that any convex shape can be obtained as the limit shape of such a sequence.

By definition, the stable configurations constitute a subset of the recurrent configurations. Note that for any given n, the set of all configurations on n cards is finite. Regardless of choice of starting configuration, the process must therefore inevitably enter the set of recurrent configurations after a finite number of moves. In the Bulgarian solitaire, recurrent configurations are close to staircase shapes and therefore have the same linear limit shape as the stable configurations have [1, 3, 8, 9]. In Section 7 we conjecture that this equivalence between limit shapes of recurrent and stable configurations holds also for sequences of L-solitaires. In Sections 8 and 9 we prove the conjecture in the special case $L_n = \{1, 1 + \lfloor 1/q_n \rfloor, 1 + \lfloor 2/q_n \rfloor, \dots\}$, where the pick layers are approximately equidistant with average distance $1/q_n$ for $q_n \in (0, 1]$. The limit shapes of stable and recurrent configurations are then linear in case $q_n^2 n \to 0$, and exponential in case $q_n^2 n \to \infty$, as $n \to \infty$.

2. L-solitaire and σ-solitaire

Olson [11] recently introduced a generalization of Bulgarian solitaire, which we call *σ-solitaire*, in which the number of cards picked from a pile of size $h \geq 0$ is given by $\sigma(h)$, where $\sigma : \mathbb{N} \to \mathbb{N}$ can be any function such that $\sigma(h) \leq h$ for all $h \in \mathbb{N}$. Let us call σ the *pick function*. The ordinary Bulgarian solitaire is obtained for the constant function $\sigma(h) = 1$. Olson studied cycle lengths, proving a general upper bound on cycle lengths for any specification of σ.

Clearly, any L-solitaire is a σ-solitaire for some unique σ. Let us denote by σ_L the pick function that corresponds to a given pick-layer set L.

Observation 1. *The pick function corresponding to L is given by $\sigma_L(h) = |L \cap \{1, 2, \dots, h\}|$, the number of picked layers up to and including layer h.*

It is not true that every σ-solitaire is an L-solitaire. The properties that a pick function must have to correspond to a pick-layer set is that (1) from a pile with just a single card, you pick that card; (2) you never pick fewer cards from a larger pile than from a smaller pile; and (3) the number of unpicked cards are never fewer in the larger pile than in a smaller pile. Formally:

Theorem 1. *Let σ be a pick function. Then $\sigma = \sigma_L$ for some pick-layer set L if and only if*

(1) $\sigma(1) = 1$,
(2) $\sigma(h)$ is a weakly increasing function of h, and
(3) the "non-pick" function $\bar{\sigma}(h) := h - \sigma(h)$ is a weakly increasing function of h.

Proof. Let us first prove that any σ_L fulfills the three conditions. Condition 1 follows from the assumption that $1 \in L$. Condition 2 follows from the fact that a layer that is picked from a pile of size h is also picked from a pile of size greater than h. Condition 3 follows from the fact that a layer that is *not* picked from a pile of size h is also *not* picked from a pile of size greater than h.

Assuming that the three conditions are satisfied for some σ, we shall find a corresponding L. First note that conditions 2 and 3 together are equivalent to the condition that for all pile sizes $h > 0$ the difference $\Delta\sigma(h) := \sigma(h) - \sigma(h-1)$ equals either 1 or 0. By choosing the pick layer

$$L = \{h > 0 : \Delta\sigma(h) = 1\}$$

it is straightforward to see that we obtain $\sigma_L = \sigma$. □

The aim of the present paper is to show that several interesting properties of Bulgarian solitaire generalize to all L-solitaires, although they do not generalize to all σ-solitaires.

As an illustration, consider the following dominance preserving property. Say that $\lambda \leq \kappa$ if the configuration λ is dominated by configuration κ in the sense that $\lambda_i \leq \kappa_i$ holds for all i. If one move of σ-solitaire is played in parallel from two configurations λ and κ, let λ^{new} and κ^{new} denote the new configurations thereby reached. In the special case of ordinary Bulgarian solitaire, it is obvious that a dominance relation is always preserved, that is, $\lambda \leq \kappa$ implies $\lambda^{\text{new}} \leq \kappa^{\text{new}}$. This dominance preserving property does not hold for σ-solitaire in general. A simple counter-example is obtained by defining $\sigma(3) = 0$ and $\sigma(4) = 2$, and setting $\lambda = (3)$ and $\kappa = (4)$. We then obtain $\lambda^{\text{new}} = (3)$ and $\kappa^{\text{new}} = (2, 2)$.

Theorem 2. *The implication $\lambda \le \kappa \Rightarrow \lambda^{\mathrm{new}} \le \kappa^{\mathrm{new}}$ holds in σ-solitaire if both σ and $\bar{\sigma}$ are weakly increasing functions. In particular, the implication holds for any L-solitaire.*

Proof. If $\bar{\sigma}$ is weakly increasing, what remains of the old piles of λ will be dominated by what remains of the old piles of κ. If σ is weakly increasing, the new pile formed from λ will be dominated by the new pile formed from κ. This pilewise dominance clearly remains when the piles in each configuration are sorted by size. By Theorem 1, for any L-solitaire with pick-layer set L, σ_L has the property that both σ_L and $\bar{\sigma}_L$ are weakly increasing functions. □

3. Uniqueness of stable configurations

Uniqueness of stable configurations does not generally hold for the σ-solitaire; a simple counter-example is obtained by defining $\sigma(1) = 1$, $\sigma(2) = 1$, and $\sigma(3) = 3$, in which case both $(2, 1)$ and (3) are stable configurations of three cards. Note that this pick function σ violates condition 3 in Theorem 1, and therefore does not define an L-solitaire. Here we show that uniqueness of stable configurations holds for L-solitaires.

Lemma 1. *Let σ be a pick function such that $\bar{\sigma}$ is weakly increasing and $\sigma(h) > 0$ for any $h > 0$ (e.g., σ could be σ_L for any pick-layer set L). Then λ is a stable configuration of the σ-solitaire if and only if $\lambda_{i+1} = \bar{\sigma}(\lambda_i)$ for all $i \ge 1$.*

Proof. A move of the σ-solitaire decreases the size of any nonempty pile from λ_i to $\bar{\sigma}(\lambda_i)$ and then creates a new pile such that the sum of all pile sizes stays constant at n, the total number of cards. Because $\bar{\sigma}$ is assumed to be a weakly increasing function, the decreased piles will still satisfy $\bar{\sigma}(\lambda_i) \ge \bar{\sigma}(\lambda_{i+1})$ for all $i \ge 1$, that is, they will not need to be reordered. Therefore λ is a stable configuration if $\bar{\sigma}(\lambda_i) = \lambda_{i+1}$ for all $i \ge 1$, as the new pile will then automatically have size λ_1. Conversely, if λ is a stable configuration, then, since the size of any nonempty pile decreases, the new pile must have size λ_1, and the decreased piles must match the rest of the original piles, that is, $\bar{\sigma}(\lambda_i) = \lambda_{i+1}$ for all $i \ge 1$. □

Theorem 3 (Uniqueness of stable configurations)**.** *Let σ be a pick function such that $\bar{\sigma}$ is weakly increasing and $\sigma(h) > 0$ for any $h > 0$ (e.g., σ could be σ_L for any pick-layer set L). Then (a) for each possible size of the first pile, λ_1, there is a unique stable configuration of the σ-solitaire, which is given by $\lambda_{i+1} = \bar{\sigma}^i(\lambda_1)$ for all $i > 0$, and (b) there is at most one stable configuration of the σ-solitaire on any given total number n of cards.*

Proof. Part (a) of the theorem follows immediately by induction from Lemma 1. To prove part (b), let λ be a stable configuration with n cards and consider another stable configuration λ' with $\lambda_1 < \lambda'_1$. Using the assumption that $\bar{\sigma}$ is weakly increasing, it follows immediately by induction that $\lambda_i \le \lambda'_i$ for all $i \ge 1$, and consequently that the total number of cards in these two configurations are different. □

So, for a fixed L-solitaire and a fixed total number of cards, there is either exactly one stable configuration or none at all. Next we shall bound the difference in the total number of cards between consecutive stable configurations of a fixed L-solitaire.

Corollary 1. *Fix an L-solitaire and let λ and λ' be the stable configurations determined by first piles of size λ_1 and $\lambda'_1 = \lambda_1 + 1$, respectively. Then the difference in the total number of cards between λ' and λ is at most $\ell(\lambda) + 1$.*

Proof. As we noted in the proof of Theorem 1, the assumption that both σ_L and $\bar{\sigma}_L$ are weakly increasing functions implies that for any pile size h we have that $\sigma_L(h+1) - \sigma_L(h)$ equals either 1 or 0. Starting from the relation $\lambda'_1 = \lambda_1 + 1$, it follows immediately by induction that as long as $\sigma_L(\lambda_i + 1) - \sigma_L(\lambda_i) = 0$ we will also have $\lambda'_{i+1} = \lambda_{i+1} + 1$. The first time we instead have $\sigma_L(\lambda_i + 1) - \sigma_L(\lambda_i) = 1$, we will obtain $\lambda'_{i+1} = \lambda_{i+1}$, and from that point on the pile sizes will be identical in the two configurations. Thus, the difference in the total number of cards is equal to the number of piles that differed in size, which is at most the number of piles in the larger configuration λ'. Because each of its piles is at most one larger than the corresponding piles in the smaller configuration λ, it can have at most one pile more. Hence, the difference in the total number of cards is bounded by $\ell(\lambda) + 1$. $\square$

4. Convexity of stable configurations

We shall now characterize what stable configurations of L-solitaires look like. Define a configuration λ as *convex* if $\lambda_i - \lambda_{i+1} \geq \lambda_{i+1} - \lambda_{i+2}$ for all $i \geq 1$.

Lemma 2. *A configuration λ is convex if and only if it is a stable configuration of an L-solitaire for some pick-layer set L.*

Proof. First assume that λ is a stable configuration of an L-solitaire. Then Lemma 1 (together with Theorem 1) says that $\lambda_i - \lambda_{i+1} = \sigma_L(\lambda_i)$ for all $i \geq 1$. As σ_L is weakly increasing, this inequality implies that λ is convex.

To prove the converse, assume that λ is a convex configuration with ℓ nonzero piles. Then for each $i \geq 1$ we can choose a subset of $(\lambda_i - \lambda_{i+1}) - (\lambda_{i+1} - \lambda_{i+2})$ layers in the interval of layers $(\lambda_{i+1}, \lambda_i]$. Note that this means all layers in the interval $(0, \lambda_\ell]$ are chosen, in particular layer 1. The union of these subsets therefore constitutes a pick-layer set L. Moreover, for all $i \geq 1$ the corresponding pick function σ_L will (by Observation 1) satisfy $\sigma_L(\lambda_i) = \lambda_i - \lambda_{i+1}$, as the latter expression equals the number of picked layers up to layer λ_i. Thus, λ is a stable configuration of this L-solitaire. $\square$

5. The concept of limit shapes of stable and recurrent configurations

We shall now define what we mean by limit shapes of stable or recurrent configurations, given an infinite sequence $L_1, L_2, \ldots$ of pick-layer sets. We first need to define the limit shape of an infinite sequence of Young diagrams.

5.1. Downscaling of diagram-boundary functions.

For any partition λ, define its *diagram-boundary function* as the nonnegative, weakly decreasing and piecewise constant function $\partial\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by

$$\partial\lambda(x) = \lambda_{\lfloor x \rfloor + 1}.$$

To illustrate, Figure 3 depicts the function graph $y = \partial\lambda(x)$ for the partition $\lambda = (4,4,2,1,1)$.

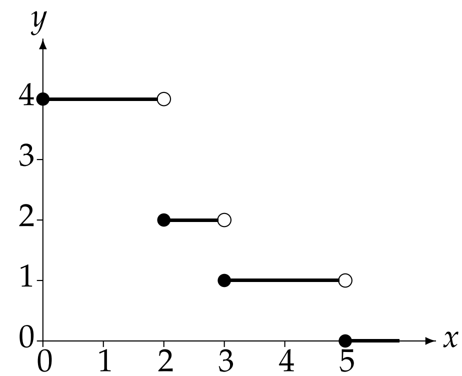

FIGURE 3. Function graph $y = \partial\lambda(x)$ for the partition $\lambda = (4,4,2,1,1) \in \mathcal{P}(12)$.

To achieve limiting behavior of such function graphs as $|\lambda|$ grows we need to rescale the diagrams. Following [7] and [14] we apply a scaling factor $a > 0$ such that all row lengths are multiplied by $1/a$ and all column heights are multiplied by $a/|\lambda|$, yielding a constant area of 1. Thus, given a partition λ and a scaling factor $a > 0$, we define the *a-downscaled* diagram-boundary function of λ as the nonnegative, real-valued, weakly decreasing and piecewise constant function $\partial^a\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by

$$\partial^a\lambda(x) = \frac{a}{|\lambda|}\partial\lambda(ax) = \frac{a}{|\lambda|}\lambda_{\lfloor ax\rfloor+1}. \tag{1}$$

5.2. Limit shapes of sequences of Young diagrams.

Definition 2. *Given an infinite sequence $\lambda^{(1)}, \lambda^{(2)}, \ldots$ of Young diagrams and a sequence of scaling factors $\{a_n\}_{n=1}^{\infty}$, we say that $\phi : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ is a* limit shape of $\{\lambda^{(n)}\}$ under the scaling $\{a_n\}$ *if the downscaled diagrams converge pointwise to ϕ, i.e.*

$$\partial^{a_n}\lambda^{(n)}(x) \to \phi(x) \text{ as } n \to \infty \tag{2}$$

for all $x > 0$.

Note that we do not require that $|\lambda^{(n)}| = n$. (However, in all our applications we will have $|\lambda^{(n)}|/n \to 1$ as $n \to \infty$.)

5.3. Limit shapes of recurrent configurations of L_n-solitaires. Consider a sequence of pick-layer sets $\{L_n\}_{n=1}^{\infty}$. By a *sequence of recurrent configurations* we mean a sequence $\{\rho^{(n)}\}_{n=1}^{\infty}$ of configurations such that, for any n, $|\rho^{(n)}| = n$ and $\rho^{(n)}$ is a recurrent configuration with respect to the L_n-solitaire.

Definition 3. *Given a sequence of pick-layer sets $\{L_n\}_{n=1}^{\infty}$ and a sequence of positive scaling factors $\{a_n\}_{n=1}^{\infty}$, we say that $\phi : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ is a* limit shape of recurrent configurations *of $\{L_n\}_{n=1}^{\infty}$ under the scaling sequence $\{a_n\}_{n=1}^{\infty}$ if ϕ is a limit shape of any sequence of recurrent configurations under this scaling sequence.*

5.4. **Limit shapes of stable configurations of L_n-solitaires.** Again, consider a sequence $\{L_n\}_{n=1}^{\infty}$ of pick-layer sets. For each value of n, consider the L_n-solitaire and, among the stable configurations with at most n cards, let $\zeta^{(n)}$ be the stable configuration with the largest number of cards. This is well-defined since there always exists a stable configuration with a single card and there is at most one stable configuration with any given number of cards according to Theorem 3.

Definition 4. *A* limit shape of stable configurations *of the sequence $\{L_n\}_{n=1}^{\infty}$ under the scaling sequence $\{a_n\}_{n=1}^{\infty}$ is a limit shape of the sequence $\{\zeta^{(n)}\}_{n=1}^{\infty}$ under this scaling sequence.*

In general, the stable configuration $\zeta^{(n)}$ has fewer than n cards, but never significantly fewer, as the following lemma asserts.

Lemma 3. $|\zeta^{(n)}|/n \to 1$ *as* $n \to \infty$.

Proof. By Theorem 3(a), for each n there is a unique stable configuration $\lambda^{(n)}$ with respect to the L_n-solitaire such that the size of the first pile is $\lambda_1^{(n)} = \zeta_1^{(n)} + 1$. According to Corollary 1 we have

$$|\lambda^{(n)}| - |\zeta^{(n)}| \leq \ell(\zeta^{(n)}) + 1. \tag{3}$$

Since $|\lambda^{(n)}| > |\zeta^{(n)}|$ it follows from the definition of $\zeta^{(n)}$ that $|\lambda^{(n)}| > n$, and combining this with the inequality (3) yields

$$|\zeta^{(n)}| \geq n - \ell(\zeta^{(n)}). \tag{4}$$

Since at least one card is removed from each non-zero pile in each move, it follows from Theorem 3(a) that the sequence of piles of the stable configuration $\zeta^{(n)}$ decreases by at least one card per pile. Thus, $|\zeta^{(n)}| \geq 1 + 2 + \cdots + \ell(\zeta^{(n)}) = \ell(\zeta^{(n)})\big(\ell(\zeta^{(n)}) + 1\big)/2 > \ell(\zeta^{(n)})^2/2$ and hence $\ell(\zeta^{(n)}) < \sqrt{2|\zeta^{(n)}|} \leq \sqrt{2n}$. Combining this with the inequality (4) yields $|\zeta^{(n)}| > n - \sqrt{2n}$ and the lemma follows. □

6. Characterization of limit shapes of stable configurations of L_n-solitaires

It is well known [4] that the Bulgarian solitaire has a stable configuration if and only if the total number of cards in the deck is a triangular number, $n = 1 + 2 + \cdots + k$ for some positive integer k, in which case the unique stable configuration has one pile of each integer size from k down to 1. Thus, the Young diagrams of stable configurations are staircase-shaped. After scaling by $a_n = \sqrt{n}$ the staircase has unit area. As n tends to infinity the downscaled staircases converge to a limit shape that is a line with slope -1, from $(0, \sqrt{2})$ to $(\sqrt{2}, 0)$.

When generalizing from ordinary Bulgarian solitaire to L-solitaire, the limit shapes that arise will not necessarily be linear. Indeed, in Theorem 4 we prove that essentially any convex shape can be obtained as the limit shape of a suitably chosen infinite sequence of pick-layer sets $\{L_n\}_{n=1}^{\infty}$.

It is a well known fact that a convex function on the real line has left and right derivatives everywhere and that these derivatives coincide at all but a finite or countably infinite number of points. We will also need two elementary lemmas on convex functions that we have not found in any textbook. The first one is due to Tsuji [13, Lemma 1] and a proof for the second one can be found in [2, Theorem 6].

Lemma 4 (Tsuji 1952). *Let $f_n : \mathbb{R}_{>0} \to \mathbb{R}$ be convex functions for $n = 1, 2, \dots$, and suppose there is a function $f : \mathbb{R}_{>0} \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for any $x > 0$. Then we also have pointwise convergence of derivatives wherever they are defined: For any $x > 0$ such that $f'(x)$ exists and $f'_n(x)$ exists for all n, we have $\lim_{n\to\infty} f'_n(x) = f'(x)$.*

Lemma 5. *The right derivative of a convex function is right continuous.*

Theorem 4. *Let $\phi : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ be a function and let $a_1, a_2, \dots \to \infty$ be any (positive) scaling factors such that a_n^2/n converges to some $c \geq 0$ as $n \to \infty$. Then the following are equivalent.*

(a) There is a sequence $\{L_n\}_{n=1}^{\infty}$ of pick-layer sets such that ϕ is a limit shape of stable configurations of $\{L_n\}_{n=1}^{\infty}$ under the scaling sequence $\{a_n\}_{n=1}^{\infty}$.

(b) ϕ is convex with $\int_0^\infty \phi(x)\,dx \leq 1$, and if $c > 0$ the right derivative $\phi'_R(x)$ is an integer multiple of c for any $x > 0$.

Proof. To prove that (a) implies (b), suppose ϕ is a limit shape of stable configurations of $\{L_n\}$ under the scaling $\{a_n\}$.

For each n, define a piecewise linear function $\phi_n : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ as the function whose graph joins the inner corners of the downscaled Young diagram of the stable configuration $\zeta^{(n)}$ using the scaling factor a_n. In other words,

$$\phi_n(x) = \frac{a_n}{|\zeta^{(n)}|}\big((1 - t(x))\zeta^{(n)}_{\lfloor a_n x\rfloor+1} + t(x)\zeta^{(n)}_{\lfloor a_n x\rfloor+2}\big),$$

where $t(x) := a_n x - \lfloor a_n x\rfloor$. By Lemma 2, $\zeta^{(n)}$ is convex, and therefore also ϕ_n. Since ϕ is weakly decreasing, its set D of discontinuity points is finite or countable. By the construction of ϕ_n it is clear that $\phi_n(x) \to \phi(x)$ for any x outside D. Thus, since each ϕ_n is convex, so is ϕ, and it follows that D is empty and that $\phi_n(x) \to \phi(x)$ everywhere. By Fatou's lemma

$$\int_0^\infty \phi(x)\,dx \leq \liminf_{n\to\infty} \int_0^\infty \phi_n(x)\,dx \leq 1.$$

Now suppose $c > 0$. It is a well known fact that a convex function is differentiable almost everywhere[2], so by Lemma 4, for almost every $x > 0$ we have $\phi'_n(x) \to \phi'(x)$ and hence, by Lemma 3,

$$\frac{\phi'_n(x)}{a_n^2/|\zeta^{(n)}|} \to \phi'(x)/c$$

[2]From here on, we will use the term "almost everywhere" as a synonym for "everywhere except on a finite or countably infinite set"

as $n \to \infty$. But for the right derivative we have that

$$\frac{(\phi_n)'_R(x)}{a_n^2/|\zeta^{(n)}|} = \zeta^{(n)}_{\lfloor a_n x\rfloor+2} - \zeta^{(n)}_{\lfloor a_n x\rfloor+1},$$

which is an integer, so it follows that $\phi'(x)/c$ is an integer almost everywhere. From Lemma 5 it follows that $\phi'_R(x)/c$ is an integer everywhere.

For the other direction, suppose (b) holds true and that $c = 0$. Since ϕ is convex and $\int_0^\infty \phi(x)\,dx$ is bounded, we know that ϕ is weakly decreasing and that $\lim_{x\to\infty}\phi(x) = 0$. It also follows that ϕ has a nonpositive and weakly increasing right derivative ϕ'_R.

Let $s_1, s_2, \ldots$ be a sequence of positive real numbers such that $s_n \to \infty$ but $s_n a_n^2/n \to 0$ as $n \to \infty$, and such that $s_n a_n$ is an integer for any n.

Define a convex partition $\lambda^{(n)}$ by letting

$$\lambda^{(n)}_k = \sum_{i=k+1}^{s_n a_n} \Big\lfloor -\frac{n}{a_n^2}\phi'_R(i/a_n)\Big\rfloor \tag{5}$$

for $k = 1, 2, \ldots$. Since ϕ'_R is a weakly increasing function, the sum above can be estimated by integrals:

$$\lambda^{(n)}_k \le -\frac{n}{a_n}\int_{k/a_n}^{\infty} \phi'_R(x)\,dx,$$
$$\lambda^{(n)}_k \ge -\frac{n}{a_n}\int_{(k+1)/a_n}^{s_n} \phi'_R(x)\,dx - s_n a_n,$$

where the last term $s_n a_n$ stems from the floor function in (5). By the fundamental theorem of calculus, the integrals can be expressed in terms of values of ϕ, and we obtain

$$\lambda^{(n)}_k \le \frac{n}{a_n}\phi(k/a_n), \tag{6}$$

$$\lambda^{(n)}_k \ge \frac{n}{a_n}\big(\phi((k+1)/a_n) - \phi(s_n)\big) - s_n a_n. \tag{7}$$

From the first of these inequalities, and from the fact that ϕ is weakly decreasing, it follows that

$$\sum_{k=1}^{\infty}\lambda^{(n)}_k \le a_n \int_0^\infty \frac{n}{a_n}\phi(x)\,dx \le n,$$

where the last inequality uses the assumption that $\int_0^\infty \phi(x)\,dx \le 1$.

Now, let $\mu^{(n)}_k = \lambda^{(n)}_k$ for $k = 2, 3, \ldots$ but choose $\mu^{(n)}_1$ so that $\mu^{(n)}_1 + \mu^{(n)}_2 + \cdots = n$. Since $\lambda^{(n)}$ is convex, clearly $\mu^{(n)}$ is too, so by Lemma 2, $\mu^{(n)}$ is a stable configuration of the L_n-solitaire for some pick-layer set L_n.

By (6) and the facts that ϕ is continuous and $a_n \to \infty$, it follows that, for any $x > 0$,

$$\frac{a_n}{n}\lambda_{\lfloor a_n x\rfloor+1} \le \phi((\lfloor a_n x\rfloor + 1)/a_n) \to \phi(x). \tag{8}$$

Similarly, by (7) and the facts that $\phi(s_n) \to 0$ and $s_n a_n^2/n \to 0$ we obtain for any $x > 0$ that

$$\frac{a_n}{n}\lambda_{\lfloor a_n x\rfloor+1} \geq \phi((\lfloor a_n x + 1\rfloor + 1)/a_n) - \phi(s_n) - \frac{s_n a_n^2}{n} \to \phi(x). \tag{9}$$

From (8) and (9) it follows that $\frac{a_n}{n}\lambda_{\lfloor a_n x\rfloor+1} \to \phi(x)$ and hence $\frac{a_n}{n}\mu_{\lfloor a_n x\rfloor+1} \to \phi(x)$ for any $x > 0$, establishing that ϕ is the desired limit shape.

Now suppose (b) holds true and $c > 0$. Define a convex partition $\lambda^{(n)}$ by letting

$$\lambda_k^{(n)} = -\frac{1}{c}\sum_{i=k+1}^{\infty} \phi_R'(i/\sqrt{cn})$$

for $k = 1, 2, \ldots$.

The remaining reasoning is completely analogous to the previous case. Since ϕ_R' is weakly increasing, we have

$$\lambda_k^{(n)} \leq -\frac{\sqrt{cn}}{c}\int_{k/\sqrt{cn}}^{\infty} \phi_R'(x)\,dx = \sqrt{\frac{n}{c}}\,\phi(k/\sqrt{cn}),$$
$$\lambda_k^{(n)} \geq -\frac{\sqrt{cn}}{c}\int_{(k+1)/\sqrt{cn}}^{\infty} \phi_R'(x)\,dx = \sqrt{\frac{n}{c}}\,\phi((k+1)/\sqrt{cn}).$$

From the first of these inequalities it follows that

$$\sum_{k=1}^{\infty} \lambda_k^{(n)} \leq \sqrt{cn}\int_0^{\infty} \sqrt{\frac{n}{c}}\,\phi(x)\,dx \leq n.$$

Now, as before, let $\mu_k^{(n)} = \lambda_k^{(n)}$ for $k = 2, 3, \ldots$ but choose $\mu_1^{(n)}$ so that $\mu_1^{(n)} + \mu_2^{(n)} + \cdots = n$. Again, $\mu^{(n)}$ is convex, so by Lemma 2 it is a stable configuration for some pick-layer set L_n. Finally, since $a_n^2/n \to c$ as $n \to \infty$, and since ϕ is continuous, for any $x > 0$ we have

$$\phi(x) \leftarrow \frac{a_n}{\sqrt{cn}}\phi\left(\frac{\lfloor a_n x + 1\rfloor + 1}{\sqrt{cn}}\right) \leq \frac{a_n}{n}\lambda_{\lfloor a_n x\rfloor+1} \leq \frac{a_n}{\sqrt{cn}}\phi\left(\frac{\lfloor a_n x\rfloor + 1}{\sqrt{cn}}\right) \to \phi(x).$$

Thus, $\frac{a_n}{n}\mu_{\lfloor a_n x\rfloor+1} \to \phi(x)$ for any $x > 0$, and ϕ is the desired limit shape. □

Note that a downscaled Young diagram will have unit area. The reason for the inequality $\int_0^\infty \phi(x)\,dx \leq 1$ in Theorem 4 is that the largest pile (or a few of the largest piles) may be arbitrarily large without affecting the limit shape ϕ. By Definition 2, the limit shape does not include $x = 0$, which allows for $\lim_{x\to 0+}\phi(x)$ to be infinite.

7. A conjecture on limit shapes of recurrent configurations of L-solitaires

The ordinary Bulgarian solitaire has the property that when a stable configuration exists (i.e., when the total number of cards is a triangular number), it will eventually be reached from any starting configuration. This property does not hold for L-solitaires in general. A counter-example is given by the $\{1,4\}$-solitaire on $n = 11$ cards, which allows both a stable configuration $(5,3,2,1)$ and a non-trivial cycle

$$(6,2,2,1) \mapsto (5,4,1,1) \mapsto (6,3,2) \mapsto (4,4,2,1) \mapsto (6,2,2,1).$$

However, it is worth noting that the pile sizes in these recurrent configurations never deviate by more than one card from the corresponding pile sizes in the stable configuration.

This is akin to the ordinary Bulgarian solitaire in the case when no stable configuration exists. The game will then eventually reach a cycle of recurrent configurations that are close to staircase-shaped (namely, they can be constructed by starting with some staircase configuration $(k, k-1, \ldots, 1)$ and adding at most one card to each pile, and possibly adding one more pile of size 1) [1, 3, 8, 9]. As n grows to infinity and the diagram is rescaled to unit area using scaling factor $a_n = \sqrt{n}$, the deviation of recurrent configurations from the perfect staircase tends to zero. Thus, the limit shape of recurrent configurations of the ordinary Bulgarian solitaire exists and is the same as the limit shape for stable configurations (namely, a line segment with negative slope). We believe that the same holds true for L-solitaire in general:

Conjecture 1. *If ϕ is a limit shape of the stable configurations for the sequence of pick-layer sets $\{L_n\}_{n=1}^{\infty}$ under the scaling sequence $\{a_n\}_{n=1}^{\infty}$, then ϕ is also a limit shape of recurrent configurations under the same scaling.*

For L-solitaire in general we leave this conjecture as an open problem. Below we shall prove the conjecture for a special class of L-solitaires for which we can determine the exact form of limit shapes.

8. Limit shapes of stable configurations of q-proportion solitaire

Choose a $q \in (0,1]$ and consider the L-solitaire defined by a pick-layer set with the distance between adjacent pick layers being approximately $1/q$:

$$L = \{1 + \lfloor i/q \rfloor : i = 0,1,2,\ldots\}.$$

Using Observation 1, it follows that the corresponding pick function is

$$\sigma_L(h) = \lceil qh \rceil.$$

This pick function means that from each pile we pick a number of cards given by the proportion q of the pile size, rounded upward to the closest integer. We refer to this special case of L-solitaire as *q-proportion solitaire*.

We may let the choice of q depend on the total number of cards n, in which we write q_n. Note that for $q_n \leq 1/n$ only one card is picked in any pile. Thus by choosing $q_n \leq 1/n$ we obtain ordinary Bulgarian solitaire.

Thanks to Lemma 1, all stable configurations of a q-proportion solitaire are determined by first choosing the size of the largest part and then repeatedly applying the function $\bar{\sigma}(h) = h - \lceil qh \rceil$ to that part to obtain the remaining parts of the configuration. From this description of the stable configurations we will be able to determine their limit shapes.

8.1. **Three regimes of limit shapes.** There will be three different regimes of limit shapes defined by the asymptotic behavior of nq_n^2, as described in the following theorem and illustrated in Figure 4.

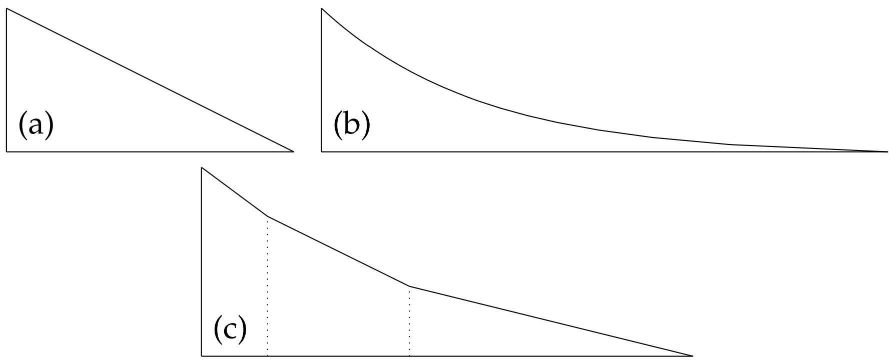

FIGURE 4. The three cases of limit shapes in Theorem 5: (a) triangular, (b) exponential, and (c) interpolating with Z linear sections, here illustrated for $Z = 3$.

Theorem 5. *There are three cases for limit shapes of stable configurations of q-proportion Bulgarian solitaire, depending on the asymptotic behavior of nq_n^2 as n tends to infinity:*

(a) *In case $nq_n^2 \to 0$, the scaling sequence $a_n = \sqrt{n/2}$ yields the linear limit shape $\phi(x) = \max\{0, 1 - \frac{x}{2}\}$.*

(b) *In case $nq_n^2 \to \infty$ and $q_n \to 0$, the scaling sequence $a_n = 1/q_n$ yields the exponential limit shape $\phi(x) = e^{-x}$.*

(c) *Interpolating between the two previous cases is the case $nq_n^2 \to C > 0$. Define $z > 0$ by the equation*

$$2C = \frac{z^2 + \lceil z \rceil^2}{\lceil z \rceil} - \sum_{i=0}^{\lceil z \rceil - 1} \frac{1}{\lceil z \rceil - i}$$

and set $W_0 = \frac{z}{C} \frac{1+z-\lceil z \rceil}{\lceil z \rceil}$ and $W_k = \frac{z}{C} \frac{1}{\lceil z \rceil - k}$ for $1 \le k \le \lceil z \rceil - 1$. Then there is a limit shape under the scaling $a_n = nq_n/z$. This shape approximates the exponential shape using $Z := \lceil z \rceil$ linear segments such that the first segment has width W_0 and every subsequent segment, numbered $k = 1, 2, \ldots, Z - 1$, has width W_k. The slope of the kth segments is $-\frac{C(Z-k)}{z^2}$ for all $k = 0, 1, \ldots, Z - 1$.

Proof. Let us treat one regime at a time.

(a) In case $nq_n^2 \to 0$ the effect of rounding turns out to dominate in a move from the stable configuration. Specifically, for all sufficiently large n we have $q_n \lceil \sqrt{2n} \rceil < 1$ and hence $\lceil q_n h \rceil = 1$ for all $0 < h \le \lceil \sqrt{2n} \rceil$. Consider such a large n, and let λ be a stable configuration of the q_n-proportion solitaire such that $\lambda_1 \leq \lceil\sqrt{2n}\rceil$. Then λ is a staircase configuration with $|\lambda| = \lambda_1(\lambda_1+1)/2$ cards. If $\lambda_1 = \lceil\sqrt{2n}\rceil$, we would have $|\lambda| > n$, so $\zeta_1^{(n)} < \lceil\sqrt{2n}\rceil$. Hence $\zeta^{(n)}$ is a staircase configuration and we obtain the same linear limit shape as in the case of Bulgarian solitaire.

(b) In case $nq_n^2 \to \infty$ and $q_n \to 0$ the effect of rounding turns out to be negligible in a move from the stable configuration $\zeta^{(n)}$. By repeated application of Lemma 1, we see that, for any k,

$$\zeta_1^{(n)}(1-q_n)^{k-1} - (k-1) \leq \zeta_k^{(n)} \leq \zeta_1^{(n)}(1-q_n)^{k-1}, \tag{10}$$

where the term $-(k-1)$ on the left-hand side is the contribution from rounding downwards in each move.

Let s_n be a sequence of positive real numbers such that $s_n \to \infty$ but $s^2 / nq_n^2 \to 0$ as $n \to \infty$, and such that s_n/q_n is an integer for any n. By summing the inequalities (10), we can estimate the total number of cards $|\zeta^{(n)}|$:

$$\sum_{k=1}^{s_n/q_n} \left(\zeta_1^{(n)}(1-q_n)^{k-1} - (k-1)\right) \leq \sum_{k=1}^{s_n/q_n} \zeta_k^{(n)} \leq |\zeta^{(n)}| \leq \sum_{k=1}^{\infty} \zeta_1^{(n)}(1-q_n)^{k-1}.$$

This can be written as

$$\zeta_1^{(n)} \frac{1-(1-q_n)^{s_n/q_n}}{q_n} - \binom{s_n/q_n}{2} \leq |\zeta^{(n)}| \leq \zeta_1^{(n)}/q_n,$$

and it follows that $|\zeta^{(n)}| = (1-o(1))\zeta_1^{(n)}/q_n$, and thus, by Lemma 3, that

$$\zeta_1^{(n)} = (1+o(1))nq_n. \tag{11}$$

Now, for any fixed $x > 0$, it follows from (10) and (11) that

$$\begin{aligned}\partial^{a_n}\zeta^{(n)}(x) = \frac{1}{nq_n}\zeta^{(n)}_{\lfloor x/q_n\rfloor+1} &= \frac{1}{nq_n}\left(\zeta_1^{(n)}(1-q_n)^{\lfloor x/q_n\rfloor} - O(1/q_n)\right)\\ &= e^{-x}(1+o(1)) - O(1/nq_n^2),\end{aligned}$$

which tends to e^{-x} since $nq_n^2 \to \infty$.

(c) For the remaining case, the crucial observation is that the rate by which a pile melts away depends on how the pile size relates to multiples of $1/q_n$. Any pile size can be expressed in the form y/q_n for some $y > 0$. From a pile of that size, a move will take away the amount $\lceil y\rceil$. Thus, a pile starting at a size of z/q_n will initially melt away at a slope of $Z = \lceil z\rceil$ per move for $B_0 = \left\lceil \frac{1+z-Z}{Zq_n}\right\rceil$ moves, i.e. until the pile size reaches the threshold $(Z-1)/q_n$. At this point the slope decreases to $Z-1$ per move for B_1 (possible zero) moves until the pile size reaches the next threshold, $(Z-2)/q_n$, etc. This pattern ends with a section of slope 1 per move for B_{Z-1} moves. See Figure 5. By Lemma 1 this sequence of pile sizes constitutes a stable configuration $\lambda^{(n)}$.

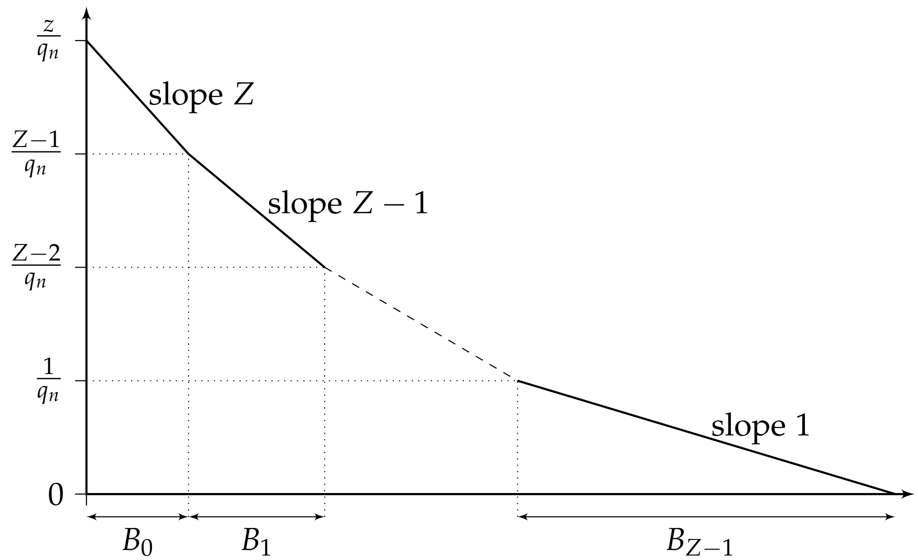

FIGURE 5. The stable configuration $\lambda^{(n)}$ in case (c) of the proof of Theorem 5.

For a moment, fix $k \in \{1, 2, \ldots, Z-1\}$ and consider only the k-th segment of $\lambda^{(n)}$, that is, the piles in $\lambda^{(n)}$ of sizes between $(Z-k-1)/q_n$ (exclusive) and $(Z-k)/q_n$ (inclusive). The number of such piles, B_k, is approximately $\frac{1}{q_n(Z-k)}$, and it is easy to see that the error in that approximation is at most 1, so

$$B_k = \frac{1+o(1)}{q_n(Z-k)}. \tag{12}$$

The average size among those piles, A_k, is approximately $\frac{1}{2}\left(\frac{Z-k}{q_n} + \frac{Z-k-1}{q_n}\right)$, and it is easy to see that

$$\left| A_k - \frac{Z-k-\frac{1}{2}}{q_n} \right| \leq \frac{Z-k}{2},$$

and hence

$$A_k = (1+o(1))\frac{Z-k-\frac{1}{2}}{q_n}. \tag{13}$$

By combining (12) and (13), we can estimate the total number of cards in the k-th segment of $\lambda^{(n)}$:

$$A_k B_k = \frac{1+o(1)}{2q_n^2}\left(2 - \frac{1}{Z-k}\right).$$

The average number of cards A_0 in the first B_0 piles of $\lambda^{(n)}$ is approximately $\frac{1}{2}\left(\frac{z}{q_n} + \frac{Z-1}{q_n}\right)$, and it is easy to see that

$$\left| A_0 - \frac{z+Z-1}{2q_n} \right| \leq \frac{Z}{2}$$

and hence

$$A_0 = (1+o(1))\frac{z+Z-1}{2q_n}.$$

It follows that the total number of cards in those piles is

$$A_0 B_0 = A_0 \left\lceil \frac{1+z-Z}{Zq_n} \right\rceil = (1+o(1))\frac{z^2-(Z-1)^2}{2q_n^2 Z}.$$

The total number of cards in $\lambda^{(n)}$ is thus

$$|\lambda^{(n)}| = \sum_{k=0}^{Z-1} A_k B_k = \frac{1+o(1)}{2q_n^2}\left(\frac{z^2-(Z-1)^2}{Z} + \sum_{k=1}^{Z-1}\left(2-\frac{1}{Z-k}\right)\right) \tag{14}$$

$$= \frac{1+o(1)}{2q_n^2}\left(\frac{z^2+Z^2}{Z} - \sum_{k=0}^{Z-1}\frac{1}{Z-k}\right) \tag{15}$$

$$= \frac{1+o(1)}{2q_n^2}\psi(z), \tag{16}$$

where we define the real function ψ on $\mathbb{R}_{>0}$ by

$$\psi(y) = \frac{y^2+\lceil y\rceil^2}{\lceil y\rceil} - \sum_{i=0}^{\lceil y\rceil-1}\frac{1}{\lceil y\rceil - i}.$$

Since $q_n^2 n \to C$, it follows from (16) that $|\lambda^{(n)}|/n \to \psi(q_n\lambda_1^{(n)})/2C$. By Lemma 3, we know that $|\zeta^{(n)}|/n \to 1$, so it follows that $\psi(q_n\zeta_1^{(n)}) \to 2C$. It is easy to check that ψ is continuous and strictly increasing and that $\lim_{y\to 0}\psi(y) = 0$ and $\lim_{y\to\infty}\psi(y) = \infty$, so ψ has a continuous inverse ψ^{-1} defined on $\mathbb{R}_{>0}$. It follows that $q_n\zeta_1^{(n)} \to \psi^{-1}(2C)$, and from now on we let $z = \psi^{-1}(2C)$ in accordance with the definition in the theorem.

Let W_k be the length of the k-th section, $0 \le k \le Z-1$, after downscaling $\zeta^{(n)}$. Then

$$W_0 = \frac{z}{nq_n}\left\lceil \frac{1+z-Z}{q_n Z}\right\rceil \to \frac{z(1+z-Z)}{CZ} \text{ and}$$

$$W_k = \frac{z}{nq_n}B_k = \frac{z}{nq_n}\cdot\frac{1+o(1)}{q_n(Z-k)} \to \frac{z}{C(Z-k)},\ 1 \le k \le Z-1$$

as $n \to \infty$. The proposed slopes of the sections follow immediately. Analogously to the proof in case (a), it follows that the above describes the limit shape. □

9. Limit shapes of recurrent configurations of q_n-proportion solitaire

Although we have not been able to prove Conjecture 1 in full generality, we can prove the conjecture in the two main regimes of q-proportion solitaire.

Lemma 6. *After n moves of q_n-proportion solitaire on n cards there are at most $2\sqrt{n}$ nonempty piles and the largest pile has size $nq_n + O(\sqrt{n})$.*

Proof. Every nonempty pile decreases by at least one card in each move. As all pile sizes are bounded by n, all original piles must have died out after n moves. Moreover, because there are n cards in total there are always at most $\sqrt{n}$ piles of size greater than $\sqrt{n}$. After $\sqrt{n}$ moves all other piles will have died and $\sqrt{n}$ new piles will have been created, hence there will then be at most $2\sqrt{n}$ nonempty piles. From then on, when new piles are formed they will have size $nq_n + O(\sqrt{n})$, where the latter term is the contribution from the number of picked cards in each pile being rounded upwards to the closest integer. $\square$

9.1. **The limit shape of recurrent configurations in the case $q_n^2 n \to 0$.** In case $q_n^2 n \to 0$, Lemma 6 implies that after n moves the largest pile size is $O(\sqrt{n})$ (since $nq = \sqrt{n(q_n^2 n)} = o(\sqrt{n})$). Then the number of picked cards in each pile is bounded by $\lceil q_n O(\sqrt{n}) \rceil$. This number equals 1 for sufficiently large n. From then on the solitaire is therefore equivalent to Bulgarian solitaire. As the recurrent configurations of Bulgarian solitaire are known to converge to a linear limit shape under appropriate choice of scaling, it follows that the recurrent configurations of q_n-proportion solitaire do too in this case.

9.2. **The limit shape of recurrent configurations in the case $q_n^2 n \to \infty$.** Finally, we shall prove that in the case $q_n^2 n \to \infty$ and $q_n \to 0$, the recurrent configurations of q_n-proportion solitaire have an exponential limit shape under the scaling $a_n = 1/q_n$. We do this by showing that regardless of which configuration we start at we must eventually reach configurations that are close to the exponential shape. Our proof works with piles sorted by time of creation rather than by size. Thus, as mathematical objects the configurations are then compositions rather than partitions. However, as we prove in [6, Lemma 2], if a sequence of compositions has a decreasing limit shape then the same limit shape is obtained by the corresponding partitions.

Lemma 6 implies that after n moves the largest pile is always of size $nq_n + O(\sqrt{n}) = nq_n(1 + o(1))$ and that after an additional $2\sqrt{n}$ moves all nonempty piles will be stemming from piles of that size. At this point, let α_k denote the current size of the pile that was created k moves ago ($k = 1, 2, \dots$) and has since been decreased $k - 1$ times. Thus $\alpha_k = (1 - q_n)^{k-1} nq_n(1 + o(1)) - O(k)$ where the latter term is the contribution from rounding downward in each move.

After downscaling with $a_n = 1/q_n$:

$$\begin{aligned}
\partial^{a_n}\alpha(x) = \frac{1}{q_n n}\alpha_{\lfloor a_n x \rfloor + 1} &= \frac{(1 - q_n)^{x/q_n} q_n n(1 + o(1)) - O(1/q_n)}{q_n n} \\
&= (1 + o(1))(1 - q_n)^{x/q_n} - O\Big(\frac{1}{q_n^2 n}\Big) \\
&= (1 - q_n)^{x/q_n} - o(1),
\end{aligned}$$

where we used $q_n^2 n \to \infty$ as $n \to \infty$ in the last step. Since $q_n \to 0$ as $n \to \infty$, we have $(1 - q_n)^{x/q_n} \to e^{-x}$ and thus $\partial^{a_n}\alpha(x) \to e^{-x}$ for any $x > 0$.

Finally, thanks to the abovementioned result from [6, Lemma 2], the same limit shape is obtained when the piles of the weak compositions are reordered to form partitions. (Note that in our earlier work [6] we require *uniform* convergence to the limit shape, but by Dini's theorem this distinction does not matter in this case, since the limit shape is continuous and the downscaled Young diagrams are bounded.)

10. Discussion

In this paper we have introduced L-solitaire, a generalization of Bulgarian solitaire, and proved that some well-known results for the Bulgarian solitaire generalize nicely to L-solitaire. Our main focus was limit shape results for stable and recurrent configurations. For a subclass of L-solitaires, called q-proportion solitaire, we found explicit limit shapes.

One may also consider limit shapes of random versions of Bulgarian solitaire. Popov [12] studied the limit shape of the configurations drawn from the stationary distribution of a random version of Bulgarian solitaire, in which a card is picked from a pile only with probability p (and independently of other piles). He found that also this random version yields a linear limit shape, in the sense that the probability of deviations larger than some $\varepsilon > 0$ tends to zero as n tends to infinity. See our other paper [6] for related work on random versions of q-proportion solitaire.

References

[1] E. Akin and M. Davis, *Bulgarian solitaire*, The American Mathematical Monthly **92** (1985), no. 4, pp. 237–250.

[2] H. Álvarez, *On the characterization of convex functions*, Revista de la Union Matematica Argentina **48** (2007), no. 1, 1–6.

[3] H.J. Bentz, *Proof of the Bulgarian solitaire conjectures*, Ars Combin. **23** (1987), 151–170.

[4] J. Brandt, *Cycles of partitions*, Proc. Amer. Math. Soc. **85** (1982), no. 3, pp. 483–486.

[5] V. Drensky, *The Bulgarian solitaire and the mathematics around it*, `arXiv:1503.00885v1 [math.CO]` (2015).

[6] K. Eriksson, M. Jonsson, and J. Sjöstrand, *An exponential limit shape of random q-proportion Bulgarian solitaire*, Integers **18** (2018), A58.

[7] K. Eriksson and J. Sjöstrand, *Limiting shapes of birth-and-death processes on Young diagrams*, Adv. in Appl. Math. **48** (2012), no. 4, 575–602.

[8] G. Etienne, *Tableaux de Young et solitaire bulgare*, J. Combin. Theory Ser. A **58** (1991), no. 2, 181–197.

[9] J.R. Griggs and C.C. Ho, *The cycling of partitions and compositions under repeated shifts*, Adv. in Appl. Math. **21** (1998), no. 2, 205–227.

[10] B. Hopkins, *30 years of Bulgarian solitaire*, College Math. J. **43** (2012), no. 2, 136–141.

[11] J.S. Olson, *Variants of Bulgarian solitaire*, Integers **16** (2016), A8.

[12] S. Popov, *Random Bulgarian solitaire*, Random Structures Algorithms **27** (2005), no. 3, 310–330.

[13] M. Tsuji, *On F.Riesz' fundamental theorem on subharmonic functions*, Tohôku Mathematical Journal, second series **4** (1952), 131–140.

[14] A. Vershik, *Statistical mechanics of combinatorial partitions, and their limit shapes*, Funct. Anal. Appl. **30** (1996), no. 2, 90–105.

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