HUB EDGE-INTEGRITY OF GRAPHS

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Abstract. The hub-integrity of a graph is given by the minimum of $|S| + m(G - S)$, where S is a hub set and $m(G - S)$ is the maximum order of the components of $G - S$. In this paper, the concept of hub edge-integrity is introduced as a new measure of the stability of a graph G and it is defined as $HEI(G) = \min\{|S| + m(G - S)\}$, where S is an edge hub set and $m(G - S)$ is the order of a maximum component of $G - S$. Furthermore, an $HEI-$ set of G is any set S for which this minimum is attained. Several properties and bounds on the HEI are presented, and the relationship between HEI and other parameters is investigated. The HEI of some classes of graphs is also computed.

1. Introduction

The integrity of a graph measures the reliability of a communication network. If the network is modeled by a graph, then the integrity measures how easy it is to cut the graph (or the network) into several small pieces by deleting as few vertices as possible. Formally, the integrity of a graph G with vertex set V is defined as $I(G) = \min\{|S| + m(G - S) : S \subseteq V(G)\}$, where $m(G - S)$ denotes the order of the largest component. In the most significant variation of integrity, edges rather than vertices are destroyed. Formally, the edge-integrity of a graph G is defined as $I'(G) = \min\{|S| + m(G - S) : S \subseteq E(G)\}$, where $m(G - S)$ denotes the order of the largest component. Both types of integrity were introduced by Barefoot, Entringer and Swart [3]. For more about integrity and edge-integrity one can see [1, 2]. Suppose that $H \subseteq V(G)$ and let $x, y \in V(G)$. An H-path between x and y is a path where all intermediate vertices are from H. (This includes the degenerate cases where the path consists of the single edge xy or a single vertex x if $x = y$, call such an H-path trivial). A set $H \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) - H$, there is an H-path in G between x and y [20]. Hub-integrity was introduced by Sultan, Veena and Ali [12] as an alternative measure of the vulnerability of graphs to disruption caused by the removal of vertices. It is defined as $HI(G) = \min\{|S| + m(G - S), S\ is\ a\ hub\ set\ of\ G\}$, where $m(G - S)$ is the order of a maximum component of $G - S$. For more details see [13, 14, 16].

By a graph $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges, with vertex set $V(G) = \{v_1, v_2, ..., v_p\}$. We use p to denote the number of vertices and q to denote the number of edges of a graph G. We refer to [4, 7] for terminology and notations not defined here. In general, the degree of a vertex v in a graph G denoted by $deg(v)$ is the number of edges of G incident with v. The maximum (minimum) degree among the vertices of G is denoted by $\Delta(G), (\delta(G))$. A

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vertex of degree one is called a pendant and its neighbor is called a steam. A steam x of G is called a strong steam if x is adjacent to at least $deg(x)-1$ pendants in G. An edge of a graph G is said to be pendant if one of its vertices is a pendant vertex. A bridge is an edge removing which increases number of disconnected components. The minimum number of edges in an edge cover of G (i.e., the edge cover number) is denoted as $\alpha_1(G)$ and the maximum number of edges in an independent set of edges of G (i.e., the edge independence number) by $\beta_1(G)$. The symbols $\alpha(G)$, $\kappa(G)$, $\lambda(G)$, $\chi(G)$ and $\beta(G)$ denote the vertex cover number, the connectivity, the edge-connectivity, the chromatic number and the independence number of G, respectively. A set $S \subseteq V(G)$ is called a dominating set of G if each vertex of $V-S$ is adjacent to at least one vertex of S. The domination number of a graph G, denoted as $\gamma(G)$ is the minimum cardinality of a dominating set in G [8].

The complement $\overline{G}$ of a graph G has $V(G)$ as its vertex set, two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G [7].

The line graph $L(G)$ of G has the edges of G as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G [7]. $\lceil x \rceil$ is the smallest integer greater than or equal to x. $\lfloor x \rfloor$ is the greatest integer less than or equal to x. The degree of an edge $(u,v) \in E(G)$ is defined to be $deg(u)+deg(v)-2$, $\Delta'(G)$ denotes the maximum degree among the edges of G. The double star graph $S_{n,m}$ is the graph constructed from $K_{1,n-1}$ and $K_{1,m-1}$ by joining their centers v_0 and u_0. That is, $V(S_{n,m}) = V(K_{1,n-1}) \cup V(K_{1,m-1})$ and $E(S_{n,m}) = \{v_0u_0, v_0v_i, u_0u_j : 1 \leq i \leq n-1,\ 1 \leq j \leq m-1\}$ [6]. A broom graph $B_{p,d}$ consists of a path P_d, together with $(p-d)$ pendant vertices all adjacent to the same pendant vertex of P_d [17]. A spider graph G_s is a tree which is constructed by subdividing each edge once in $K_{1,p-1}, p \geq 3$ [5]. In the present work, the basic properties of hub edge-integrity and of $HEI-$sets, are explored, and bounds as well as relationships between hub edge-integrity and other graph parameters are considered. Finally, the hub edge-integrity of families of some trees are determined.

The following results are needed to prove the main results.

Theorem 1.1. [7] *For any graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

Lemma 1.1. *For any graph G, $\beta_1(G) \leq \alpha(G)$.*

Theorem 1.2. [12] *For any graph G, $HI(G) \geq \chi(G)$.*

2. Main results

Let $e = (u,v)$ and $f = (u',v')$. A path between the two edges e and f is a path between one end vertex of e and another end vertex of f such that $d(e,f) = min\{d(u,u'), d(u,v'), d(v,u'), d(v,v')\}$. The internal edges of a path between two edges e and f are all the edges of the path except e and f. Suppose that $S \subseteq E(G)$. An S-path between edges e and f is a path for which all its edges except e and f are in S. (This definition allows for the cases when the path contains only two adjacent edges or single edge. In such cases, the S-path is trivial.) Here a new concept is introduced, namely, an edge hub set. A subset $S \subseteq E(G)$ is called an edge hub set of G if every pair of edges $e, f \in E-S$ is connected by a path where all internal edges are from

S. The minimum cardinality of an edge hub set is called the edge hub number of G, and is denoted by $h_e(G)$. Some studies on edge hub number were found in the papers [9, 10] If G is a disconnected graph then any edge hub set must contain all of the edges in all but one of the components, as well as an edge hub set in the remaining component. We have integrated the concepts of edge hub set and edge-integrity to get a new concept. Motivated by this, we introduce hub edge-integrity as a new measure of the stability of a graph G called the hub edge-integrity of a graph G, denoted by $HEI(G)$, defined as $HEI(G) = \min\{|S| + m(G-S), S \textit{ is an edge hub set of } G\}$, where $m(G-S)$ is the order of a maximum component of $G-S$. Any set $S \subseteq E(G)$ with the property that $|S| + m(G-S) = HEI(G)$ is called an $HEI-$ set of G. For any disconnected graph G having k components $G_1, G_2, ..., G_k$ of sizes $q_1, q_2, ..., q_{k-1}, q_k$, respectively, then any an $HEI(G)$-set must be union of the set of all edges belonging to all components except one component and the HEI-set of the remaining component. For more details we refer the reader to [15]. The definition shows that $HEI(G) \geq I'(G)$. This bound is sharp for $G \cong K_{1,p-1}$.
The following result is the straight forward consequence of the definition of hub edge-integrity.

Proposition 2.1.

(a): For any complete graph K_p, $p \geq 3$, $HEI(K_p) = 2p-3$.
(b): For any path P_p with $p \geq 4$, $HEI(P_p) = p-1$.
(c): For any cycle C_p, $p \geq 3$,

$$HEI(C_p) = \begin{cases} p, \text{ if } p = 3,4,5\ ; \\ p-1, \text{ if } p \geq 6. \end{cases}$$

(d): For the star $K_{1,p-1}$, $HEI(K_{1,p-1}) = p$.
(e): For the double star $S_{n,m}$, $HEI(S_{n,m}) = 1 + max\{n, m\}$.
(f): For the complete bipartite graph $K_{n,m}$, $n, m > 2$,

$$HEI(K_{n,m}) = \begin{cases} 3n-1, \text{ if } n = m\ ; \\ \lfloor \frac{3(n+m)}{2} \rfloor - 1, \text{ if } n \neq m. \end{cases}$$

(g): For the wheel graph $W_{1,p-1}$, $p \geq 5$,

$$HEI(W_{1,p-1}) = \begin{cases} \frac{3p-3}{2} + 1, \text{ if } p-1 \textit{ is even}\ ; \\ \lceil \frac{3p-3}{2} \rceil + 1, \text{ if } p-1 \textit{ is odd}. \end{cases}$$

Observation 2.1. *If G is a nontrivial connected graph of order p, $G \neq K_2$, then*

$$2 \leq HI(G) \leq HEI(G) \leq 2p-3.$$

But if G is disconnected, this relation need not be true. For example, consider the graph $K_2 \cup K_2 \cup K_2 \cup K_2 = 4K_2$, $HI(4K_2) = 8, HEI(4K_2) = 5$, so $HEI(G) < HI(G)$ in this case. The star graph $K_{1,p-1}$ shows a beautiful contrast between the two parameters. Removing edges from $K_{1,p-1}$ decreases the order of the largest component by only one for each edge, thus $HEI(K_{1,p-1}) = p$. In comparison, removing the central vertex from the star $K_{1,p-1}$ leaves only isolated vertices, from which we get $HI(K_{1,p-1}) = 2$. On the other hand, the two parameters are equal for paths: for the path P_p with $p \geq 4$, $HI(P_p) = HEI(P_p) = p-1$. We remove $p-3$ edges from P_p, then $p-2$ components remain, with two components of order 2 and the other components having only one vertex. Therefore, $HEI(P_p) = p-1$.

Let G_1 and G_2 be graphs. Then the question arises: is the hub edge-integrity a suitable measure of stability? In other words, does the hub edge-integrity discriminate between G_1 and G_2? There are many examples of graphs which suggest that $HEI(G)$ is a suitable measure of stability which is able to discriminate between graphs. For example, consider the graphs G_1, G_2 and G_3 in Figure 1.

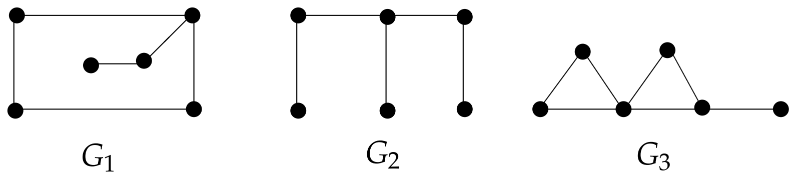

Figure 1: G_1, G_2, and G_3.

We have $HI(G_1) = HI(G_2) = HI(G_3) = 4$, the hub-integrity does not discriminate between graphs G_1, G_2 and G_3. But $HEI(G_1) = 5, HEI(G_2) = 4$ and $HEI(G_3) = 6$, so that $HEI(G_1) \neq HEI(G_2) \neq HEI(G_3)$, and the hub edge-integrity discriminates between graphs G_1, G_2 and G_3.

Remark 2.1. *We have*

- $HEI(G) \neq 0$, *by the definition of hub edge-integrity,* $m(G-S) \geq 1$ *for any* $S \subseteq E(G)$.
- $HEI(G) = 2$ *if and only if* $G \cong K_2$.
- $HEI(G) = 3$ *if and only if* $G \cong 2K_2$, *or* $G \cong P_3$, $G \cong P_4$ *or* $G \cong K_3$.

Proposition 2.2. *Let G be a graph with p vertices. Then* $1 \leq HEI(\overline{G}) \leq p+2$, *if* $4 \leq p \leq 7$ *and* $HEI(\overline{G}) \leq \frac{p^2}{8} + \frac{3}{4}p - 3$, *if* $p \geq 8$, *and the upper bound is sharp for* $G \cong K_{\frac{p}{2},\frac{p}{2}}$, *and p is even.*

Proof. By Observation 2.1, $HEI(G) \leq 2p-3$, if $G = K_p, p \neq 2$. So to find the upper bound of $HEI(\overline{G})$, we should choose any graph G such that $\overline{G}$ is union of complete graphs. Hence we discuss the following cases:
Case 1: $4 \leq p \leq 7$. If $p = 4$, then we have only one graph $G = 4K_1$ such that $\overline{G} = K_4$, and $HEI(K_4) = 5$. Now, if $p = 5, 6$ or 7, we have $G = 5K_1, K_{1,4}, 6K_1, K_{1,5}, K_{3,4}$ or $K_{1,6}$ such that $\overline{G} = K_5, K_1 \cup K_4, K_6, K_1 \cup K_5, K_3 \cup K_4$ or $K_1 \cup K_6$, respectively. Then $HEI(K_5) = HEI(K_1 \cup K_4) = 5, HEI(K_6) = 6, HEI(K_1 \cup K_5) = 7$ and $HEI(K_3 \cup K_4) = 8$, $HEI(K_1 \cup K_6) = 9$. Hence, $HEI(\overline{G}) \leq p+2$.
Case 2: $p \geq 8$, Since the greatest value of HEI is achieved for any graph G if $G = K_p$, and to get the most value of the complement graph $\overline{G}$ of G, we should select the graph G such that $\overline{G} = K_p$. Since $\overline{K_{n,m}} = K_n \cup K_m$, the greatest value of $HEI(\overline{G})$ is achieved if $G = K_{n,m}$, $n+m = p$ and p is even. Then, $G = K_{\frac{p}{2},\frac{p}{2}}$ and $\overline{G} = K_{\frac{p}{2}} \cup K_{\frac{p}{2}}$. By definition

of edge hub-integrity of disconnected graph, we have

$$\begin{aligned} HEI(\overline{G}) &= |E(K_{\frac{p}{2}})| + HEI(K_{\frac{p}{2}}) \\ &= \frac{\frac{p}{2}(\frac{p}{2}-1)}{2} + 2\frac{p}{2} - 3 \\ &= \frac{p}{4}(\frac{p}{2}-1) + p - 3 \\ &= \frac{p^2}{8} + \frac{3p}{4} - 3. \end{aligned} \tag{1}$$

□

Remark 2.2. *In general, the inequality $HEI(G') \leq HEI(G)$ is not true for a subgraph G' of G. For example, for the graph G and a subgraph G' of G shown in Figure 2, $HEI(G) = 7$, while $HEI(G') = 8$.*

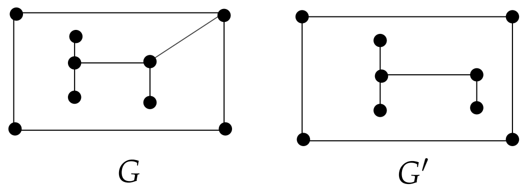

Figure 2: G and G'

So, it follows that the number of edges need not necessarily grow with the stability of a graph.

Theorem 2.1. *Let G be a graph, and $D \subseteq E$. Then $HEI(G - D) \geq HEI(G) - |D|$.*

Proof. Let S be an HEI-set of $G - D$, then S is an edge hub set of $G - D$ and $HEI(G - D) = |S| + m((G - D) - S)$. Let $S' = S \cup D$, then S' is an edge hub set of G and $m(G - S') = m((G - D) - S)$. Therefore,

$$\begin{aligned} HEI(G) &\leq |S'| + m(G - S') \\ &= |S| + |D| + m[(G - D) - S] \\ &= HEI(G - D) + |D|. \end{aligned}$$

Then $HEI(G - D) \geq HEI(G) - |D|$. □

Corollary 2.1. *For any graph G, $e \in E(G)$, $HEI(G - e) \geq HEI(G) - 1$, the bound is sharp for $G = K_{1,p-1}$.*

Theorem 2.2. *Let S be an HEI-set of G. Then $m(G - S) \leq HEI(G - S)$.*

Proof. Let S' be an HEI-set of $G - S$,

$$\begin{aligned} |S| + m(G - S) &= HEI(G) \\ &\leq m(G - (S \cup S')) + |S \cup S'| \\ &= |S| + m((G - S) - S') + |S'| \\ &= |S| + HEI(G - S). \end{aligned}$$

Then $m(G-S) \leq HEI(G-S)$. □

Observation 2.2. *(i)* $HEI(G) = HI(G) = |E(G)|$ *if* $G \cong P_p, p \geq 4$.
(ii) $HEI(G) = HI(G) = |E(G)| = |V(G)|$ *if and only if* $G \cong rC_3, r \geq 1$.

Lemma 2.1. *For any connected graph* G *of order* p *and size* q, $G \neq K_2$,

$$HEI(G) \leq 2q - 1.$$

This bound is sharp for $G = P_3$.

Proof. By Observation 2.1, and since $p-1 \leq q$ for any connected graph, we have, $HEI(G) \leq 2p-3 = 2(p-1)-1 \leq 2q-1$. □

Theorem 2.3. *Let* G *be a connected graph. Then every edge of* G *is an* HEI*-set of* G *if and only if* $G \cong K_{1,p-1}$.

Proof. Let every edge of G constitute an HEI-set of G. Let $e \in E(G)$, then $\{e\}$ is an HEI-set of G. Therefore,

$$1 + m(G-e) = HEI(G). \tag{2}$$

Hence, $m(G-e) = HEI(G) - 1$, for every $e \in E(G)$. Since G is connected, every edge of G must be incident to a pendant vertex. Then

$$m(G-e) = m(G) - 1. \tag{3}$$

From (2) and (3), $HEI(G) = m(G) = p$. So $HEI(G) = p$, which implies that G is a star.
Conversely, if G is a star $K_{1,p-1}$, then consider $S = \{e\}$ for any $e \in E(G)$. We have $m(G-S) = p-1$ so that $HEI(G) = p$, hence every edge is an $HEI-$set of G. □

Proposition 2.3. *If a connected graph* G *is isomorphic to its line graph, then* $HEI(G) = HEI(L(G))$. *The converse is not true, for example, see the graph* G *in Figure 3.*

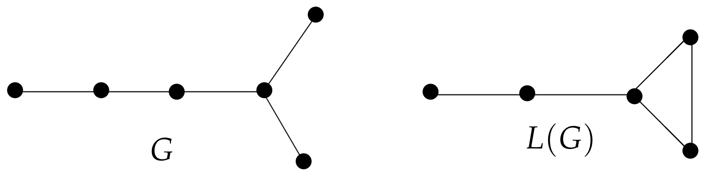

Figure 3: $G, L(G)$

$HEI(G) = 5 = HEI(L(G))$, *but* G *and* $L(G)$ *are not isomorphic.*

Proposition 2.4. *Let* G *be a connected graph with* $\Delta(G) \leq 2$. *Then* $HEI(G) = |E(G)|$ *if and only if* $G = P_p, p \geq 4$ *or* $G = C_p, 3 \leq p \leq 5$.

Proof. G is a path or cycle, since G is a connected graph with $\Delta(G) \leq 2$, and from Proposition 2.1, $HEI(G) = |E(G)|$. Conversely, suppose that $HEI(G) = |E(G)|$. Since $\Delta \leq 2$, $G = P_p, p \geq 2$ or $C_p, p \geq 3$.
Case 1: $G = P_p, p \geq 2$. If $G = P_2$, then $HEI(G) = 2 \neq |E(G)| = 1$ and if $G = P_3$, then $HEI(G) = 3 \neq |E(G)| = 2$.
Case 2: $G = C_p, p \geq 3$. If $p \geq 6$, then $HEI(G) = p-1$ whereas $|E(G)| = p$, then $3 \leq p \leq 5$. □

Lemma 2.2. *For any graph G, $HEI(G) \geq \delta(G) + 1$.*

Proof. Consider S an $HEI-$ set of G, i.e. $HEI(G) = |S| + m(G - S)$. Since $m(G - S) \geq \delta(G - S) + 1 \geq \delta(G) - |S| + 1$, we have, $HEI(G) = |S| + m(G - S) \geq |S| + \delta(G) - |S| + 1$. □

Lemma 2.3. *For any graph G, $HEI(G) \geq \Delta(G) + 1$.*

Proof. The proof is similar to the proof of Lemma 2.2. □

Proposition 2.5. *For any graph G,*

(1) $HEI(G) \geq \lambda(G) + 1$.
(2) $HEI(G) \geq \kappa(G) + 1$.

Proof. The proofs follow from Theorem 1.1 and Lemma 2.2. □

Lemma 2.4. *For any graph G, $HEI(G) \geq \alpha_1(G)$.*

Proof. Since an edge hub set is an edge covering set, this completes the proof and the bound is sharp for $G = S_{2,2}$. □

Lemma 2.5. *For any graph G, $HEI(G) \geq \beta(G)$. The bound is sharp for $G = S_{2,3}$.*

Observation 2.3. *We have*

(1) $I(G) = I'(G) = HI(G) = HEI(G)$ *if and only if* $G \cong P_4, G \cong K_2$ *or* $G \cong K_3$.
(2) $I(G) = I'(G) = HI(G) = HEI(G) = |V(G)|$ *if and only if* $G \cong K_2, G \cong K_3$.
(3) $I(G) = I'(G) = HI(G) = HEI(G) = |E(G)|$ *if and only if* $G \cong P_4$, *or* $G \cong K_3$.
(4) $I(G) = I'(G) = HI(G) = HEI(G) = |V(G)| = |E(G)|$ *if and only if* $G \cong K_3$.

Lemma 2.6. *For any connected graph G, $HEI(G) \geq \chi(G)$.*

Proof. Since $HEI(G) \geq HI(G)$ for any connected graph G, and by Theorem 1.2, we get the result. □

Proposition 2.6. *For every integer $r \geq 2$, there exists graph G such that $HEI(G) = r$.*

Proof. Suppose $r = 2$, let $G = K_{1,1}$. Then $HEI(K_{1,1}) = 2$.
For $r = 3$, let $G = K_{1,2}$. Then $HEI(K_{1,2}) = 3$.
For $r = 4$, let $G = K_{1,3}$. Then $HEI(K_{1,1}) = 4$.
And so on, for $r = n - 1$, let $G = K_{1,n-1}$. Then $HEI(K_{1,n-1}) = n - 1$. □

Since $1 \leq HI(G) \leq p$ and $1 \leq HEI(G) \leq 2p - 3, G \neq K_2$, the proof of the following result is straight forward.

Lemma 2.7. *For any graph $G \neq K_2$,*

(1) $p + 1 \leq HI(G) + HEI(G) \leq 3p - 3$.
(2) $p \leq HI(G)HEI(G) \leq 2p^2 - 3p$.

The upper bound is sharp for $G = K_p$, and the lower bound is sharp for $G = K_1$.

Proposition 2.7. *For any connected graph G, $HEI(G) = HI(G)$ if G is one of the following graphs: $G \cong P_p, p \geq 4, G \cong K_2, G \cong K_3$, G_1, G_2 or G_3 shown in Figure 4.*

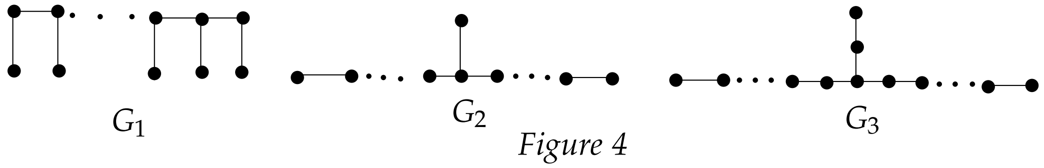

Figure 4

Interestingly, for a given positive integer n, we can get a graph G whose order, maximum degree and HEI are related to n. So, we have the following theorem.

Theorem 2.4. *For any positive integer n, there exists a graph G such that $HEI(G) - \lceil \frac{p}{\Delta+1} \rceil = n$.*

Proof. For $n = 1$, let $G = K_2$. Then, $HEI(G) - \lceil \frac{p}{\Delta+1} \rceil = 2 - 1 = 1$.
For $n = 2$, let $G = P_3$. Then, $HEI(G) - \lceil \frac{p}{\Delta+1} \rceil = 3 - 1 = 2$.
For $n \geq 3$, let $G = K_{1,n}$ such that $p = n + 1$, then $HEI(G) = n + 1$ and $\lceil \frac{p}{\Delta+1} \rceil = \lceil \frac{n+1}{n+1} \rceil = 1$. Thus, $HEI(G) - \lceil \frac{p}{\Delta+1} \rceil = 1 + n - 1 = n$. □

Remark 2.3. *If G is disconnected and $G \cong mP_p, p \geq 4, m \geq 2$, and $G \cong mC_p, m \geq 2, 3 \leq p \leq 5$. Then $HEI(G) = |E(G)|$.*

Remark 2.4. *If G is star graph or $L(B_{p,d})$ with $p - d = 2$ or $G \cong C_p, 3 \leq p \leq 5$, or G is one of the following graphs in Figure 5, then $HEI(G) = p$.*

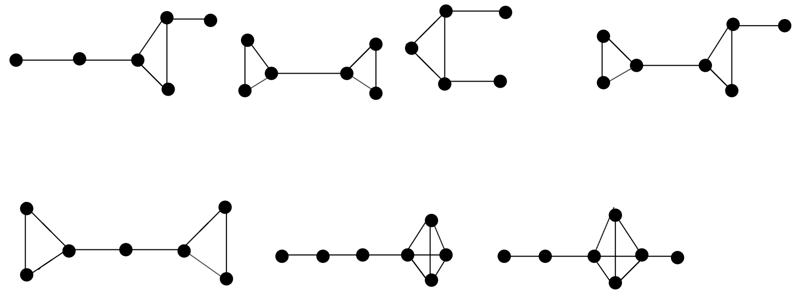

Figure 5: Graphs for Remark 2.4

3. Hub edge-integrity of a tree

Firstly, the behaviour of parameters hub-integrity and hub edge-integrity for star and path graphs are compared. A path has the greatest hub-integrity among trees of a given order and a star the least, while a star has the greatest hub edge-integrity, but a path does not have the least. The hub edge-integrity of path is $p-1, p \geq 4$. There are other graphs having this value, for example, a broom graph $B_{p,d}$ with $p-d=2$ and $d \geq 4$ and $S_{2,m}$, $m \geq 3$, $HEI(S_{2,m}) = HEI(B_{p,d}) = p-1$.

Theorem 3.1. *For any tree T, with p vertices, $h_e(T) = q - q_1$, where q_1 is the number of pendant edges.*

Proof. Suppose that set S consists of all internal edges in T. Clearly, S is an edge hub set, since for any $x, y \in E - S$, there is an S-path in T between x and y. Every edge belonging to S is a bridge, hence any proper subset of S cannot be an edge hub set. S is a minimum edge hub set considered. In case any edge e from S is removed, then there does not exist path between any two pendant edges $x, y \in E - S$, so $S - \{e\}$ is not an edge hub set for any $e \in S$, therefore S must be a minimum, and $h_e(T) = q - q_1$. □

Theorem 3.2. $HEI(T) = q - q_1 + l + 1$, *for any tree T of order p, where l is the number of trivial components and q_1 is the number of pendant edges.*

Proof. Let $S \subseteq E(T)$ such that $HEI(T) = |S| + m(T-S)$. Theorem 3.1 demonstrates that $|S| = q - q_1$. Consider $X = \{x_1, x_2, ..., x_t\}$ be the strong steam of T. Suppose that u is a vertex with a maximum degree of X and $T - u$ consists of l trivial components and z nontrivial components. Since every internal edge is a bridge, if all $q - q_1$ edges are deleted, we get $m(T-S) \geq 2$, so the largest component of $T - S$ has order $l+1$. Then $HEI(T) = |S| + m(T-S) = q - q_1 + l + 1$. □

Definition 3.1. [18] *The binomial tree B_p is an ordered tree defined recursively. The binomial tree B_0 consists of a single vertex. The binomial tree B_p consists of two binomial trees B_{p-1} that are linked together: the root of one is the leftmost child of the root of the other.*

Theorem 3.3. *Let $n \geq 2$, be a positive integer. Then $HEI(B_n) = |E(B_{n-1})| + 2$.*

Proof. The number vertices of B_n is 2^n and the number of edges is $2^n - 1$. Let S be an HEI-set of B_n. The internal edges in B_n form a minimum edge hub set of B_n, so it leads to $|S| = |E(B_{n-1})|$, since for any binomial tree B_n, the number of internal edges is equal to the number of edges in B_{n-1}, and removing it from B_n, results in 2^{n-1} components of order 2, thus $m(B_n - E(B_{n-1})) = 2$. Then, $HEI(B_n) = |E(B_{n-1})| + 2$. □

Definition 3.2. [19] *A galaxy graph G is a forest in which each component is a star.*

Theorem 3.4. *For a (p,q) galaxy graph, $G = \cup_{i=1}^{k} K_{1,q_i}$,*

$$HEI(G) = \begin{cases} q+1, \text{ if } q_1 = q_2 = \cdots = q_k \,; \\ \sum_{i=1}^{k} q_k + 1, \text{ otherwise.} \end{cases}$$

Proof. Suppose that G consists of k components $G_1, G_2, G_3, \cdots, G_k$ of sizes $q_1, q_2, q_3, \cdots, q_k$, respectively. We consider the following two cases:

Case 1: $q_1 = q_2 = q_3 = \cdots = q_k = \frac{q}{k}$, then $HEI(G) = \sum_{i=1}^{k-1}(q_i) + HEI(G_k) = (k-1)\frac{q}{k} + \frac{q}{k} + 1 = q + 1$.
Case 2: $q_1 \leq q_2 \leq q_3 \leq \cdots \leq q_k$, then $HEI(G) = q_1 + q_2 + q_3 + \cdots + q_{k-1} + HEI(G_k) = q_1 + q_2 + q_3 + \cdots + q_{k-1} + q_k + 1 = \sum_{i=1}^{k} q_i + 1$. □

Definition 3.3. [11] *A tree is called a binary tree if it has one vertex of degree 2 and each of the remaining vertices of degree 1 or 3. Clearly, P_3 is the smallest binary tree.*

Theorem 3.5. *If a tree T is a binary tree of order p, then $HEI(T) = \lceil \frac{p}{2} \rceil + 1$.*

Proof. Let $S \subseteq E(T)$ such that $HEI(T) = |S| + m(T - S)$. Since the edge hub set of any binary tree consists of all internal edges, $|S| = q_1 - 2$, where q_1 is the number of pendant edges of T. Removing $q_1 - 2$ internals edges from binary tree T, results in components of order 2 or 3. Therefore, $HEI(T) = q_1 - 2 + 3 = q_1 + 1$. Since the number of pendant edges in any binary tree equal $\lceil \frac{p}{2} \rceil$, we have $q_1 = \lceil \frac{p}{2} \rceil$. Therefore $HEI(T) = \lceil \frac{p}{2} \rceil + 1$. □

Theorem 3.6. *Let G_s be a spider graph with $2p - 1$ vertices. Then*

$$HEI(G_s) = HI(G_s) = p + 1.$$

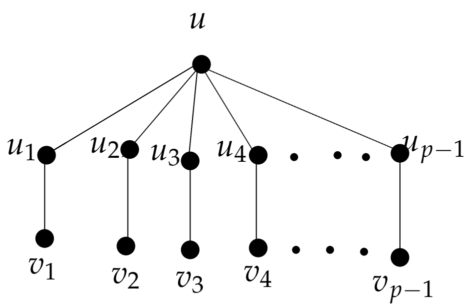

Figure 6: G_s

Proof. Let G_s be a spider graph shown in Figure 6, with $|V(G_s)| = 2p - 1$ and $|E(G_s)| = 2p - 2$. Consider $S = \{u, u_1, u_2, \cdots, u_{p-1}\}$, a hub set of G_s. Then $m(G_s - S) = 1$, therefore,

$$HI(G_s) \leq |S| + m((G_s) - S) = p + 1. \tag{4}$$

Consider S_1 any hub set other than S such that $m(G_s - S_1) = 0$, then $|S_1| \geq 2p - 1$. This implies that

$$|S_1| + m(G_s - S_1) > |S| + m(G_s - S). \tag{5}$$

Assume that S_2 is any hub set other than S such that $m(G_s - S_2) \geq 1$, then

$$|S_2| + m(G_s - S_2) \geq p + 1. \tag{6}$$

Therefore, 4, 5 and 6, lead to $HI(G_s) = p + 1$.
Now consider $S' = \{(u, u_1), (u, u_2), \cdots, (u, u_{p-1})\}$, an edge hub set of G_s. Then $m(G_s - S') = 2$. Thus,

$$HEI(G_s) \leq |S'| + m(G_s - S') = p + 1. \tag{7}$$

If S'_1 is any edge hub set other than S' such that $m(G_s - S'_1) = 1$, then $|S'_1| \geq 2p - 2$. This implies that

$$|S'_1| + m(G_s - S'_1) > 2p - 2. \tag{8}$$

Suppose that S'_2 is any edge hub set other than S' and $m(G_s - S'_2) \geq 2$, then

$$|S'_2| + m(G_s - S'_2) \geq p + 1. \tag{9}$$

Therefore, 7, 8 and 9, lead to $HEI(G_s) = p + 1$. □

Corollary 3.1. *For every integer $n \geq 2$, there exists a graph G of order $p \geq n$ with $HEI(G) = HI(G) = n$.*

Proof. For $n = 2, 3$, the graphs K_2, C_3 have the required property. For $n \geq 4$, the spider graph in Theorem 3.6 has the same property. This completes the proof. □

4. Conclusion

In this paper, we introduced the concept of the hub edge-integrity of graphs. We obtained the bounds and some properties for hub edge-integrity of graphs. Relationships between hub edge-integrity and some other parameters were established.

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