GENERALIZED FIBONACCI-PELL HYBRINOMIALS

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Abstract. Hybrid numbers are generalization of complex, hyperbolic and dual numbers. In this paper we introduce and study Fibonacci-Pell hybrinomials, i.e. polynomials, which are a generalization of hybrid numbers of the Fibonacci type.

1. Introduction and preliminary results

Let F_n be the nth Fibonacci number defined recursively by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with initial terms $F_0 = 0$, $F_1 = 1$, see for details [9]. There are many generalizations of Fibonacci numbers related to them, the survey can be found in [1].

Among the number of distinct generalizations and variants of Fibonacci numbers an important role play numbers defined by the second order linear recurrence relations which are named as numbers of the Fibonacci type, see [15]. In this paper apart classical Fibonacci numbers we investigate Pell numbers and their special second-order generalizations.

Let P_n be the nth Pell number defined by $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$ with $P_0 = 0$, $P_1 = 1$.

For numbers F_n and P_n direct formulas named as Binet formula and Binet formula for Pell numbers, respectively, have the form

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}.$$

In [7] Falcon and Plaza gave the following generalization of Fibonacci numbers and Pell numbers. Let $k \geq 1$ be an integer. Then

$$F_n^k = k \cdot F_{n-1}^k + F_{n-2}^k \quad \text{for } n \geq 2 \tag{1}$$

with $F_0^k = 0$ and $F_1^k = 1$.

Clearly $F_n^1 = F_n$ and $F_n^2 = P_n$. Consequently numbers F_n^k are named as Fibonacci-Pell numbers. For their combinatorial properties see [6, 7].

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Interesting results for numbers of the Fibonacci type were obtain quite recently in [2, 3]. Polynomials defined by a recurrence relations $f_n(x) = h(x)f_{n-1}(x) + g(x)f_{n-2}$, for $n \geq 2$ with fixed f_0, f_1 are Fibonacci type polynomials, see for details [15]. They can be considered as a generalization of numbers of the Fibonacci type. A special case of $h(x)$-Fibonacci polynomials were introduced and studied in [11]. For any variable quantity x and an integer $k \geq 1$ generalized Fibonacci polynomials $F_n^k(x)$ are defined as

$$F_n^k(x) = kx \cdot F_{n-1}^k(x) + F_{n-2}^k(x) \text{ for } n \geq 2 \tag{2}$$

with $F_0^k(x) = 0$ and $F_1^k(x) = 1$. Then $F_n^1(x) = F_n(x)$ is the nth Fibonacci polynomial and $F_n^2(x) = P_n(x)$ is the nth Pell polynomial.

In this paper we define a family of polynomials which includes the classical Fibonacci polynomials, Pell polynomials and their generalizations given by (2). Based on this family we introduce and study a generalization of Fibonacci and Pell hybrinomials.

Let $k \geq 1$ be an integer and $\alpha, q \in \mathbb{R}$. For any variable quantity x, the (k, α, q)-Fibonacci-Pell polynomial $F_n^k(\alpha, q; x)$ is defined as

$$F_n^k(\alpha, q; x) = kx \cdot F_{n-1}^k(\alpha, q; x) + F_{n-2}^k(\alpha, q; x) \text{ for } n \geq 2 \tag{3}$$

with $F_0^k(\alpha, q; x) = 1$ and $F_1^k(\alpha, q; x) = \alpha(1 + q)x$.

Using the above definition, we can write initial (k, α, q)-Fibonacci-Pell polynomials

$$\begin{aligned}
F_2^k(\alpha, q; x) &= \alpha(1 + q)kx^2 + 1, \\
F_3^k(\alpha, q; x) &= \alpha(1 + q)k^2x^3 + (\alpha(1 + q) + k)x, \\
F_4^k(\alpha, q; x) &= \alpha(1 + q)k^3x^4 + (2\alpha(1 + q) + k)kx^2 + 1, \\
F_5^k(\alpha, q; x) &= \alpha(1 + q)k^4x^5 + (3\alpha(1 + q) + k)k^2x^3 + (\alpha(1 + q) + 2k)x.
\end{aligned}$$

The formula (3) generalizes Fibonacci polynomials and Pell polynomials, simultaneously. It is interesting that there are infinitely many pairs (α, q) such that $F_n^k(\alpha, q; x)$ give Fibonacci polynomials and Pell polynomials. If $\alpha(1 + q) = k$ then $F_n^k(\alpha, q; x) = F_{n+1}^k(x)$ and consequently for $k = 1, 2$ we have $F_n^1(\alpha, q; x) = F_{n+1}(x)$ and $F_n^2(\alpha, q; x) = P_{n+1}(x)$. Moreover we can observe that $F_n^k(0, q; x) = F_{n+2}^k(x)$ for an arbitrary $q \in \mathbb{R}$.

In particular if α is a positive root of the characteristic equation of the recurrence relation of the Fibonacci sequence, the Pell sequence and k-Fibonacci-Pell sequence, respectively, then

$$F_n^1\left(\frac{1+\sqrt{5}}{2}, \frac{-2}{3+\sqrt{5}}; x\right) = F_{n+1}(x),$$

$$F_n^2\left(1+\sqrt{2}, -\frac{-1}{3+2\sqrt{2}}; x\right) = P_{n+1}(x),$$

and

$$F_n^k\left(\frac{k+\sqrt{k^2+4}}{2}, \frac{-2}{k^2+2+k\sqrt{k^2+4}}; x\right) = F_{n+1}^k(x).$$

Roots of the characteristic equation of the Horadam recurrence relation were used in the concept of generalized hybrid numbers, see for details [8].

The characteristic equation of the relation (3) is $t^2(x) - kx \cdot t(x) - 1 = 0$ so roots of it are $t_1(x) = \dfrac{kx+\sqrt{k^2x^2+4}}{2}$ and $t_2(x) = \dfrac{kx-\sqrt{k^2x^2+4}}{2}$.

Consequently if $n \geq 0$ then the Binet formula for $F_n^k(\alpha, q; x)$ has the form

$$F_n^k(\alpha, q; x) = \frac{\alpha(1+q)x - t_2(x)}{t_1(x)-t_2(x)} t_1^n(x) - \frac{\alpha(1+q)x - t_1(x)}{t_1(x)-t_2(x)} t_2^n(x). \tag{4}$$

In particular, for special values of k, α and q we obtain Binet formulas for Fibonacci polynomials, Pell polynomials and k-Fibonacci polynomials, respectively. Let $\alpha(1+q) = k$. Then

$$F_n^1(\alpha, q; x) = \frac{t_1^{n+1}(x) - t_2^{n+1}(x)}{t_1(x)-t_2(x)},$$

where $t_1(x) = \frac{1}{2}\left(x+\sqrt{x^2+4}\right)$ and $t_2(x) = \frac{1}{2}\left(x-\sqrt{x^2+4}\right)$;

$$F_n^2(\alpha, q; x) = \frac{t_1^{n+1}(x) - t_2^{n+1}(x)}{t_1(x)-t_2(x)},$$

where $t_1(x) = x+\sqrt{x^2+1}$ and $t_2(x) = x-\sqrt{x^2+1}$;

$$F_n^k(\alpha, q; x) = \frac{t_1^{n+1}(x) - t_2^{n+1}(x)}{t_1(x)-t_2(x)},$$

where $t_1(x) = \dfrac{kx+\sqrt{k^2x^2+4}}{2}$ and $t_2(x) = \dfrac{kx-\sqrt{k^2x^2+4}}{2}$.

Now we will give some identities such as Catalan identity, Cassini identity and d'Ocagne identity for (k, α, q)-Fibonacci-Pell polynomials. These identities can be proved using Binet formula (4). For simplicity of notation let $A = \dfrac{\alpha(1+q)x - t_2(x)}{t_1(x)-t_2(x)}$ and $B = \dfrac{\alpha(1+q)x - t_1(x)}{t_1(x)-t_2(x)}$. Then we can write (4) as $F_n^k(\alpha, q; x) = At_1^n(x) - Bt_2^n(x)$.

Theorem 1.1. *(Catalan identity for (k, α, q)-Fibonacci-Pell polynomials)*
Let $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then for an integer $k \geq 1$ and $\alpha, q \in \mathbb{R}$ holds

$$F_{n-r}^k(\alpha, q; x) \cdot F_{n+r}^k(\alpha, q; x) - \left(F_n^k(\alpha, q; x)\right)^2 =$$
$$= ABt_1^n(x)t_2^n(x)\left(2 - \left(\frac{t_1(x)}{t_2(x)}\right)^r - \left(\frac{t_2(x)}{t_1(x)}\right)^r\right).$$

Proof. For integers $n \geq 0$, $r \geq 0$ and $n \geq r$ we have

$$\begin{aligned}
&F_{n-r}^{k}(\alpha,q;x) \cdot F_{n+r}^{k}(\alpha,q;x) - \left(F_{n}^{k}(\alpha,q;x)\right)^{2} \\
&= \left(At_1^{n-r}(x) - Bt_2^{n-r}(x)\right) \cdot \left(At_1^{n+r}(x) - Bt_2^{n+r}(x)\right) \\
&\ - \left(At_1^{n}(x) - Bt_2^{n}(x)\right)^2 \\
&= 2ABt_1^n(x)t_2^n(x) - ABt_1^{n+r}(x)t_2^{n-r}(x) - ABt_1^{n-r}(x)t_2^{n+r}(x) \\
&= ABt_1^n(x)t_2^n(x)\left(2 - \left(\frac{t_1(x)}{t_2(x)}\right)^r - \left(\frac{t_2(x)}{t_1(x)}\right)^r\right),
\end{aligned}$$

which ends the proof. □

Note that if $r = 1$ then we get Cassini identity for (k, α, q)-Fibonacci-Pell polynomials.

Corollary 1.2. *(Cassini identity for (k, α, q)-Fibonacci-Pell polynomials)*
Let $n \geq 0$, $k \geq 1$ be integers. Then for $\alpha, q \in \mathbb{R}$ holds

$$\begin{aligned}
&F_{n-1}^{k}(\alpha,q;x) \cdot F_{n+1}^{k}(\alpha,q;x) - \left(F_{n}^{k}(\alpha,q;x)\right)^{2} = \\
&= ABt_1^n(x)t_2^n(x)\left(2 - \frac{t_1(x)}{t_2(x)} - \frac{t_2(x)}{t_1(x)}\right).
\end{aligned}$$

Analogously as in Theorem 1.1 we can prove

Theorem 1.3. *(d'Ocagne identity for (k, α, q)-Fibonacci-Pell polynomials)*
Let $m \geq 0$, $n \geq 0$ be integers such that $m \geq n$. Then for an integer $k \geq 1$ and $\alpha, q \in \mathbb{R}$ holds

$$\begin{aligned}
&F_{m}^{k}(\alpha,q;x) \cdot F_{n+1}^{k}(\alpha,q;x) - F_{m+1}^{k}(\alpha,q;x) \cdot F_{n}^{k}(\alpha,q;x) = \\
&= AB(t_1(x) - t_2(x))\left(t_1^m(x)t_2^n(x) - t_1^n(x)t_2^m(x)\right).
\end{aligned}$$

2. Generalizations of Fibonacci hybrinomials

In [12] Özdemir introduced the set of hybrid numbers denoted by $\mathbb{K}$ as a new generalization of complex, hyperbolic and dual numbers. The set $\mathbb{K}$ of hybrid numbers $\mathbf{Z}$ has the form

$$\begin{aligned}
\mathbb{K} = \{\mathbf{Z} :\ &\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h},\ a,b,c,d \in \mathbb{R}, \\
&\mathbf{i}^2 = -1,\ \varepsilon^2 = 0,\ \mathbf{h}^2 = 1,\ \mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}\}.
\end{aligned}$$

Let $\mathbf{Z}_1 = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}$ and $\mathbf{Z}_2 = a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h}$ be two hybrid numbers. Then

$\mathbf{Z}_1 = \mathbf{Z}_2$ if and only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$ (equality)
$\mathbf{Z}_1 + \mathbf{Z}_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\varepsilon + (d_1 + d_2)\mathbf{h}$ (addition)
$\mathbf{Z}_1 - \mathbf{Z}_2 = (a_1 - a_2) + (b_1 - b_2)\mathbf{i} + (c_1 - c_2)\varepsilon + (d_1 - d_2)\mathbf{h}$ (subtraction)
$s\mathbf{Z}_1 = sa_1 + sb_1\mathbf{i} + sc_1\varepsilon + sd_1\mathbf{h}$ (multiplication by scalar $s \in \mathbb{R}$).

The hybrid numbers multiplication is made analogously as the multiplication of algebraic expressions using rules for the multiplications of operators $\mathbf{i}$, ε and $\mathbf{h}$ given in Table 1.

$\cdot$	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	-1	$1-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	$\mathbf{h}+1$	0	$-\varepsilon$
$\mathbf{h}$	$-\varepsilon-\mathbf{i}$	ε	1

Table 1. The hybrid number multiplication.

The multiplication of hybrid numbers is not commutative but it is associative. The addition of hybrid numbers is commutative and associative. Zero $\mathbf{0} = 0 + 0\mathbf{i} + 0\varepsilon + 0\mathbf{h}$ is the null element. With respect to the addition operation, the inverse element of $\mathbf{Z}$ is $-\mathbf{Z} = -a - b\mathbf{i} - c\varepsilon - d\mathbf{h}$. Consequently $(\mathbb{K}, +)$ is an Abelian group. Moreover, $(\mathbb{K}, +, \cdot)$ is non-commutative ring, with identity element $\mathbf{1} = 1 + 0\mathbf{i} + 0\varepsilon + 0\mathbf{h}$.

In recent works special families of hybrid numbers related to the Fibonacci type sequences were studied. In [14] Horadam hybrid numbers were introduced and consequently their special cases related to the Fibonacci type numbers were investigated in [5, 18, 19, 20].

For future investigations we recall that $FH_n = F_n + \mathbf{i}F_{n+1} + \varepsilon F_{n+2} + \mathbf{h}F_{n+3}$ is the nth Fibonacci hybrid number and $PH_n = P_n + \mathbf{i}P_{n+1} + \varepsilon P_{n+2} + \mathbf{h}P_{n+3}$ is the nth Pell hybrid number. Moreover based on (1) we define generalized Fibonacci hybrid numbers as $FH_n^k = F_n^k + \mathbf{i}F_{n+1}^k + \varepsilon F_{n+2}^k + \mathbf{h}F_{n+3}^k$ and we called FH_n^k as the nth k-Fibonacci hybrid number.

The Fibonacci type hybrinomials being a generalization of Fibonacci type hybrid numbers were introduced recently in [16]. Results concerning Pell hybrinomials are included in [10] and they are a sequel of the Fibonacci hybrinomials concept. The last survey of these results is included in [15]. We recall that for $n \geq 0$ the nth Fibonacci hybrinomials $FH_n(x)$ are defined by

$$FH_n(x) = F_n(x) + \mathbf{i}F_{n+1}(x) + \varepsilon F_{n+2}(x) + \mathbf{h}F_{n+3}(x)$$

where $F_n(x)$ is the nth Fibonacci polynomial, and $\mathbf{i}$, ε, $\mathbf{h}$ are hybrid operators. Analogously nth Pell hybrinomials $PH_n(x)$ are defined by

$$PH_n(x) = P_n(x) + \mathbf{i}P_{n+1}(x) + \varepsilon P_{n+2}(x) + \mathbf{h}P_{n+3}(x)$$

where $P_n(x)$ is the nth Pell polynomial. In the same way we define nth k-Fibonacci hybrinomial $FH_n^k(x)$ as

$$FH_n^k(x) = F_n^k(x) + \mathbf{i}F_{n+1}^k(x) + \varepsilon F_{n+2}^k(x) + \mathbf{h}F_{n+3}^k(x).$$

Clearly $FH_n(1) = FH_n$, $PH_n(1) = PH_n$ and $FH_n^k(1) = FH_n^k$, for $k \geq 1$. For other Fibonacci type hybrinomials and their properties see [15].

In this paper we define a wide generalization of Fibonacci hybrinomials, which includes also Fibonacci type hybrinomials not defined yet.

Let $k \geq 1$, $n \geq 0$ be integers, $\alpha, q \in \mathbb{R}$. Then (k, α, q)-Fibonacci-Pell hybrinomials $FH_n^k(\alpha, q; x)$ are defined by

$$FH_n^k(\alpha, q; x) = F_n^k(\alpha, q; x) + \mathbf{i}F_{n+1}^k(\alpha, q; x) + \varepsilon F_{n+2}^k(\alpha, q; x) + \mathbf{h}F_{n+3}^k(\alpha, q; x) \tag{5}$$

where $F_n^k(\alpha, q; x)$ is the nth (k, α, q)-Fibonacci-Pell polynomial, and $\mathbf{i}$, ε, $\mathbf{h}$ are hybrid operators.

For special values of k, α and q we obtain the special Fibonacci type hybrinomials. Let $\alpha(1+q) = k$. Then $FH_n^1(\alpha, q; x) = FH_{n+1}(x)$, $FH_n^2(\alpha, q; x) = PH_{n+1}(x)$ and $FH_n^k(\alpha, q; x) = FH_{n+1}^k(x)$. Moreover $FH_n^1(\alpha, q; 1) = FH_{n+1}$, $FH_n^2(\alpha, q; 1) = PH_{n+1}$ and $FH_n^k(\alpha, q; 1) = FH_{n+1}^k$.

Theorem 2.1. *Let $k \geq 1$ be an integer and $\alpha, q \in \mathbb{R}$. Then for any variable quantity x we have*

$$FH_n^k(\alpha, q; x) = kx \cdot FH_{n-1}^k(\alpha, q; x) + FH_{n-2}^k(\alpha, q; x) \text{ for } n \geq 2$$

with

$$\begin{aligned} FH_0^k(\alpha, q; x) &= 1 + \mathbf{i}\alpha(1+q)x + \varepsilon(\alpha(1+q)kx^2 + 1) \\ &\quad + \mathbf{h}(\alpha(1+q)k^2x^3 + (\alpha(1+q) + k)x) \end{aligned} \tag{6}$$

and

$$\begin{aligned} FH_1^k(\alpha, q; x) &= \alpha(1+q)x + \mathbf{i}(\alpha(1+q)kx^2 + 1) \\ &\quad + \varepsilon(\alpha(1+q)k^2x^3 + (\alpha(1+q) + k)x) \\ &\quad + \mathbf{h}(\alpha(1+q)k^3x^4 + (2\alpha(1+q) + k)kx^2 + 1) \end{aligned} \tag{7}$$

Proof. If $n = 2$ we have

$$\begin{aligned} FH_2^k(\alpha, q; x) &= kx \cdot FH_1^k(\alpha, q; x) + FH_0^k(\alpha, q; x) \\ &= kx \cdot \Big[\alpha(1+q)x + \mathbf{i}(\alpha(1+q)kx^2 + 1) \\ &\quad + \varepsilon(\alpha(1+q)k^2x^3 + (\alpha(1+q) + k)x) \\ &\quad + \mathbf{h}(\alpha(1+q)k^3x^4 + (2\alpha(1+q) + k)kx^2 + 1)\Big] \\ &\quad + 1 + \mathbf{i}\alpha(1+q)x + \varepsilon(\alpha(1+q)kx^2 + 1) \\ &\quad + \mathbf{h}(\alpha(1+q)k^2x^3 + (\alpha(1+q) + k)x) \\ &= \alpha(1+q)kx^2 + 1 \\ &\quad + \mathbf{i}(\alpha(1+q)k^2x^3 + (\alpha(1+q) + k)x) \\ &\quad + \varepsilon(\alpha(1+q)k^3x^4 + (2\alpha(1+q) + k)kx^2 + 1) \\ &\quad + \mathbf{h}(\alpha(1+q)k^4x^5 + (3\alpha(1+q) + k)k^2x^3 + (\alpha(1+q) + 2k)x). \end{aligned}$$

If $n \geq 3$ then using the definition of (k, α, q)-Fibonacci-Pell polynomials we have

$$\begin{aligned} FH_n^k(\alpha,q;x) &= kx \cdot F_{n-1}^k(\alpha,q;x) + F_{n-2}^k(\alpha,q;x) \\ &\quad + \mathbf{i}(kx \cdot F_n^k(\alpha,q;x) + F_{n-1}^k(\alpha,q;x)) \\ &\quad + \varepsilon(kx \cdot F_{n+1}^k(\alpha,q;x) + F_n^k(\alpha,q;x)) \\ &\quad + \mathbf{h}(kx \cdot F_{n+2}^k(\alpha,q;x) + F_{n+1}^k(\alpha,q;x)) \\ &= kx \cdot FH_{n-1}^k(\alpha,q;x) + FH_{n-2}^k(\alpha,q;x) \end{aligned}$$

which ends the proof. □

Theorem 2.2. *(Binet formula for the (k, α, q)-Fibonacci-Pell hybrinomials)*
Let $n \geq 0$, $k \geq 1$ be integers and $\alpha, q \in \mathbb{R}$. Then

$$\begin{aligned} &FH_n^k(\alpha,q;x) = \\ &= \frac{\alpha(1+q)x - t_2(x)}{t_1(x)-t_2(x)} t_1^n(x)\left(1 + \mathbf{i}t_1(x) + \varepsilon t_1^2(x) + \mathbf{h}t_1^3(x)\right) \\ &\quad - \frac{\alpha(1+q)x - t_1(x)}{t_1(x)-t_2(x)} t_2^n(x)\left(1 + \mathbf{i}t_2(x) + \varepsilon t_2^2(x) + \mathbf{h}t_2^3(x)\right), \end{aligned} \tag{8}$$

where $t_1(x) = \dfrac{kx + \sqrt{k^2x^2+4}}{2}$ *and* $t_2(x) = \dfrac{kx - \sqrt{k^2x^2+4}}{2}$.

Proof. Using (4) and (5) we have

$$\begin{aligned} &FH_n^k(\alpha,q;x) \\ &= F_n^k(\alpha,q;x) + \mathbf{i}F_{n+1}^k(\alpha,q;x) + \varepsilon F_{n+2}^k(\alpha,q;x) + \mathbf{h}F_{n+3}^k(\alpha,q;x) \\ &= \frac{\alpha(1+q)x - t_2(x)}{t_1(x)-t_2(x)} t_1^n(x) - \frac{\alpha(1+q)x - t_1(x)}{t_1(x)-t_2(x)} t_2^n(x) \\ &+ \mathbf{i}\left(\frac{\alpha(1+q)x - t_2(x)}{t_1(x)-t_2(x)} t_1^{n+1}(x) - \frac{\alpha(1+q)x - t_1(x)}{t_1(x)-t_2(x)} t_2^{n+1}(x)\right) \\ &+ \varepsilon\left(\frac{\alpha(1+q)x - t_2(x)}{t_1(x)-t_2(x)} t_1^{n+2}(x) - \frac{\alpha(1+q)x - t_1(x)}{t_1(x)-t_2(x)} t_2^{n+2}(x)\right) \\ &+ \mathbf{h}\left(\frac{\alpha(1+q)x - t_2(x)}{t_1(x)-t_2(x)} t_1^{n+3}(x) - \frac{\alpha(1+q)x - t_1(x)}{t_1(x)-t_2(x)} t_2^{n+3}(x)\right) \end{aligned}$$

and after calculations the result (8) follows. □

For simplicity of notation let
$\hat{t_1}(x) = 1 + \mathbf{i}t_1(x) + \varepsilon t_1^2(x) + \mathbf{h}t_1^3(x)$,
$\hat{t_2}(x) = 1 + \mathbf{i}t_2(x) + \varepsilon t_2^2(x) + \mathbf{h}t_2^3(x)$.

Then we can write (8) as $FH_n^k(\alpha,q;x) = At_1^n(x)\hat{t_1}(x) - Bt_2^n(x)\hat{t_2}(x)$, where $A = \dfrac{\alpha(1+q)x - t_2(x)}{t_1(x)-t_2(x)}$ and $B = \dfrac{\alpha(1+q)x - t_1(x)}{t_1(x)-t_2(x)}$. Using Binet formula (8) one can obtain

Catalan identity, Cassini identity and d'Ocagne identity for (k,α,q)-Fibonacci-Pell hybrinomials.

Theorem 2.3. *(Catalan identity for (k,α,q)-Fibonacci-Pell hybrinomials)*
Let $n\geq 0$, $r\geq 0$ be integers such that $n\geq r$. Then for integer $k\geq 1$ and $\alpha,q\in\mathbb{R}$ holds

$$FH^k_{n-r}(\alpha,q;x)\cdot FH^k_{n+r}(\alpha,q;x)-\left(FH^k_n(\alpha,q;x)\right)^2=$$
$$=ABt_1^n(x)t_2^n(x)\left[\left(1-\frac{t_2^r(x)}{t_1^r(x)}\right)\hat{t_1}(x)\hat{t_2}(x)+\left(1-\frac{t_1^r(x)}{t_2^r(x)}\right)\hat{t_2}(x)\hat{t_1}(x)\right].$$

Proof. For integers $n\geq 0$, $r\geq 0$ and $n\geq r$ we have

$$\begin{aligned}
&FH^k_{n-r}(\alpha,q;x)\cdot FH^k_{n+r}(\alpha,q;x)-\left(FH^k_n(\alpha,q;x)\right)^2\\
&=\left(At_1^{n-r}(x)\hat{t_1}(x)-Bt_2^{n-r}(x)\hat{t_2}(x)\right)\cdot\left(At_1^{n+r}(x)\hat{t_1}(x)-Bt_2^{n+r}(x)\hat{t_2}(x)\right)\\
&-\left(At_1^{n}(x)\hat{t_1}(x)-Bt_2^{n}(x)\hat{t_2}(x)\right)\cdot\left(At_1^{n}(x)\hat{t_1}(x)-Bt_2^{n}(x)\hat{t_2}(x)\right)\\
&=-At_1^{n-r}(x)Bt_2^{n+r}(x)\hat{t_1}(x)\hat{t_2}(x)+At_1^n(x)Bt_2^n(x)\hat{t_1}(x)\hat{t_2}(x)\\
&-Bt_2^{n-r}(x)At_1^{n+r}(x)\hat{t_2}(x)\hat{t_1}(x)+Bt_2^n(x)At_1^n(x)\hat{t_2}(x)\hat{t_1}(x)\\
&=-ABt_1^n(x)t_2^n(x)t_1^{-r}(x)t_2^r(x)\hat{t_1}(x)\hat{t_2}(x)+ABt_1^n(x)t_2^n(x)\hat{t_1}(x)\hat{t_2}(x)\\
&-ABt_1^n(x)t_2^n(x)t_1^r(x)t_2^{-r}(x)\hat{t_2}(x)\hat{t_1}(x)+ABt_1^n(x)t_2^n(x)\hat{t_2}(x)\hat{t_1}(x)\\
&=ABt_1^n(x)t_2^n(x)\hat{t_1}(x)\hat{t_2}(x)\left(1-\frac{t_2^r(x)}{t_1^r(x)}\right)\\
&+ABt_1^n(x)t_2^n(x)\hat{t_2}(x)\hat{t_1}(x)\left(1-\frac{t_1^r(x)}{t_2^r(x)}\right)\\
&=ABt_1^n(x)t_2^n(x)\left[\left(1-\frac{t_2^r(x)}{t_1^r(x)}\right)\hat{t_1}(x)\hat{t_2}(x)+\left(1-\frac{t_1^r(x)}{t_2^r(x)}\right)\hat{t_2}(x)\hat{t_1}(x)\right],
\end{aligned}$$

which ends the proof. □

For $r=1$ we obtain Cassini type identity for (k,α,q)-Fibonacci-Pell hybrinomials.

Corollary 2.4. *(Cassini identity for (k,α,q)-Fibonacci-Pell hybrinomials)*
Let $n\geq 0$, $k\geq 1$ be integers. Then for $\alpha,q\in\mathbb{R}$ holds

$$FH^k_{n-1}(\alpha,q;x)\cdot FH^k_{n+1}(\alpha,q;x)-\left(FH^k_n(\alpha,q;x)\right)^2=$$
$$=ABt_1^n(x)t_2^n(x)\left[\left(1-\frac{t_2(x)}{t_1(x)}\right)\hat{t_1}(x)\hat{t_2}(x)+\left(1-\frac{t_1(x)}{t_2(x)}\right)\hat{t_2}(x)\hat{t_1}(x)\right].$$

Theorem 2.5. *(d'Ocagne identity for (k, α, q)-Fibonacci-Pell hybrinomials)*
Let $m \geq 0$, $n \geq 0$ be integers such that $m \geq n$. Then for integer $k \geq 1$ and $\alpha, q \in \mathbb{R}$ holds

$$\begin{aligned}
&FH_m^k(\alpha,q;x) \cdot FH_{n+1}^k(\alpha,q;x) - FH_{m+1}^k(\alpha,q;x) \cdot FH_n^k(\alpha,q;x) = \\
&= ABt_1^m(x)t_2^n(x)\hat{t_1}(x)\hat{t_2}(x)\left(t_1(x) - t_2(x)\right) \\
&\ - ABt_1^n(x)t_2^m(x)\hat{t_2}(x)\hat{t_1}(x)\left(t_1(x) - t_2(x)\right).
\end{aligned}$$

Proof. For integers $m \geq 0$, $m \geq 0$ and $m \geq n$ we have

$$\begin{aligned}
&FH_m^k(\alpha,q;x) \cdot FH_{n+1}^k(\alpha,q;x) - FH_{m+1}^k(\alpha,q;x) \cdot FH_n^k(\alpha,q;x) \\
&= \left(At_1^m(x)\hat{t_1}(x) - Bt_2^m(x)\hat{t_2}(x)\right) \cdot \left(At_1^{n+1}(x)\hat{t_1}(x) - Bt_2^{n+1}(x)\hat{t_2}(x)\right) \\
&- \left(At_1^{m+1}(x)\hat{t_1}(x) - Bt_2^{m+1}(x)\hat{t_2}(x)\right) \cdot \left(At_1^n(x)\hat{t_1}(x) - Bt_2^n(x)\hat{t_2}(x)\right) \\
&= -At_1^m(x)Bt_2^{n+1}(x)\hat{t_1}(x)\hat{t_2}(x) + At_1^{m+1}(x)Bt_2^n(x)\hat{t_1}(x)\hat{t_2}(x) \\
&- Bt_2^m(x)At_1^{n+1}(x)\hat{t_2}(x)\hat{t_1}(x) + Bt_2^{m+1}(x)At_1^n(x)\hat{t_2}(x)\hat{t_1}(x) \\
&= -ABt_1^m(x)t_2^n(x)t_2(x)\hat{t_1}(x)\hat{t_2}(x) + ABt_1^m(x)t_2^n(x)t_1(x)\hat{t_1}(x)\hat{t_2}(x) \\
&- ABt_1^n(x)t_2^m(x)t_1(x)\hat{t_2}(x)\hat{t_1}(x) + ABt_1^n(x)t_2^m(x)t_2(x)\hat{t_2}(x)\hat{t_1}(x) \\
&= ABt_1^m(x)t_2^n(x)\hat{t_1}(x)\hat{t_2}(x)\left(t_1(x) - t_2(x)\right) \\
&- ABt_1^n(x)t_2^m(x)\hat{t_2}(x)\hat{t_1}(x)\left(t_1(x) - t_2(x)\right),
\end{aligned}$$

which ends the proof. □

Theorem 2.6. *The generating function for the (k, α, q)-Fibonacci-Pell hybrinomial sequence $\{FH_n^k(\alpha, q; x)\}$ is*

$$G(t) = \frac{FH_0^k(\alpha,q;x) + \left(FH_1^k(\alpha,q;x) - FH_0^k(\alpha,q;x)kx\right)t}{1 - kxt - t^2},$$

where $FH_0^k(\alpha, q; x)$ and $FH_1^k(\alpha, q; x)$ are given by (6) *and* (7)*.*

Proof. Assume that the generating function of the (k, α, q)-Fibonacci-Pell hybrinomial sequence $\{FH_n^k(\alpha, q; x)\}$ has the form $G(t) = \sum_{n=0}^{\infty} FH_n^k(\alpha, q; x)t^n$. Then

$$G(t) = FH_0^k(\alpha,q;x) + FH_1^k(\alpha,q;x)t + FH_2^k(\alpha,q;x)t^2 + \dots$$

Multiply the above equality on both sides by $-kxt$ and then by $-t^2$ we obtain

$$-G(t)kxt = -FH_0^k(\alpha,q;x)kxt - FH_1^k(\alpha,q;x)kxt^2 - FH_2^k(\alpha,q;x)kxt^3 - \dots$$

$$-G(t)t^2 = -FH_0^k(\alpha,q;x)t^2 - FH_1^k(\alpha,q;x)t^3 - FH_2^k(\alpha,q;x)t^4 - \dots$$

By adding the three equalities above, we will get

$$G(t)(1 - kxt - t^2) = FH_0^k(\alpha,q;x) + \left(FH_1^k(\alpha,q;x) - FH_0^k(\alpha,q;x)kx\right)t$$

since $FH_n^k(\alpha,q;x) = kx \cdot FH_{n-1}^k(\alpha,q;x) + FH_{n-2}^k(\alpha,q;x)$ and the coefficients of t^n for $n \geq 2$ are equal to zero. This ends the proof. $\square$

Matrix generators are a complement to the theory of (k,α,q)-Fibonacci-Pell hybrinomials.

Theorem 2.7. *Let $n \geq 0$, $k \geq 1$ be integers. Then for $\alpha, q \in \mathbb{R}$ holds*

$$\begin{bmatrix} FH_{n+2}^k(\alpha,q;x) & FH_{n+1}^k(\alpha,q;x) \\ FH_{n+1}^k(\alpha,q;x) & FH_{n}^k(\alpha,q;x) \end{bmatrix} = \\ = \begin{bmatrix} FH_{2}^k(\alpha,q;x) & FH_{1}^k(\alpha,q;x) \\ FH_{1}^k(\alpha,q;x) & FH_{0}^k(\alpha,q;x) \end{bmatrix} \cdot \begin{bmatrix} kx & 1 \\ 1 & 0 \end{bmatrix}^n .$$

Proof. (by induction on n)
If $n = 0$ then assuming that the matrix to the power 0 is the identity matrix the result is obvious. Now assume that for any $n \geq 0$ holds

$$\begin{bmatrix} FH_{n+2}^k(\alpha,q;x) & FH_{n+1}^k(\alpha,q;x) \\ FH_{n+1}^k(\alpha,q;x) & FH_{n}^k(\alpha,q;x) \end{bmatrix} = \\ = \begin{bmatrix} FH_{2}^k(\alpha,q;x) & FH_{1}^k(\alpha,q;x) \\ FH_{1}^k(\alpha,q;x) & FH_{0}^k(\alpha,q;x) \end{bmatrix} \cdot \begin{bmatrix} kx & 1 \\ 1 & 0 \end{bmatrix}^n .$$

We shall show that

$$\begin{bmatrix} FH_{n+3}^k(\alpha,q;x) & FH_{n+2}^k(\alpha,q;x) \\ FH_{n+2}^k(\alpha,q;x) & FH_{n+1}^k(\alpha,q;x) \end{bmatrix} = \\ = \begin{bmatrix} FH_{2}^k(\alpha,q;x) & FH_{1}^k(\alpha,q;x) \\ FH_{1}^k(\alpha,q;x) & FH_{0}^k(\alpha,q;x) \end{bmatrix} \cdot \begin{bmatrix} kx & 1 \\ 1 & 0 \end{bmatrix}^{n+1} .$$

By simple calculation using induction's hypothesis we have

$$\begin{bmatrix} FH_{2}^k(\alpha,q;x) & FH_{1}^k(\alpha,q;x) \\ FH_{1}^k(\alpha,q;x) & FH_{0}^k(\alpha,q;x) \end{bmatrix} \cdot \begin{bmatrix} kx & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} kx & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} FH_{n+2}^k(\alpha,q;x) & FH_{n+1}^k(\alpha,q;x) \\ FH_{n+1}^k(\alpha,q;x) & FH_{n}^k(\alpha,q;x) \end{bmatrix} \cdot \begin{bmatrix} kx & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} kx \cdot FH_{n+2}^k(\alpha,q;x) + FH_{n+1}^k(\alpha,q;x) & FH_{n+2}^k(\alpha,q;x) \\ kx \cdot FH_{n+1}^k(\alpha,q;x) + FH_{n}^k(\alpha,q;x) & FH_{n+1}^k(\alpha,q;x) \end{bmatrix}$$

$$= \begin{bmatrix} FH_{n+3}^k(\alpha,q;x) & FH_{n+2}^k(\alpha,q;x) \\ FH_{n+2}^k(\alpha,q;x) & FH_{n+1}^k(\alpha,q;x) \end{bmatrix},$$

which ends the proof. $\square$

Conclusion

Numbers of the Fibonacci type have many interesting applications in different branches of mathematics not only in combinatorics and number theory and what is interesting also in the theory of complex variables, see for example [13], where using a special q-derivative operator a new class of q-starlike functions associated with generalized-Fibonacci numbers was defined and studied. In spite of hybrid numbers and hybrinomials were introduced quite recently their connections with numbers of the Fibonacci type also are intensively studied. Related to results obtained in [4] and [17] it seem to be interesting to consider also Mersenne hybrinomials and their properties.

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Compliance with Ethical Standards

Conflict of Interest: The authors declare that they have no conflict of interest.

References

[1] U. Bednarz, I. Włoch, M. Wołowiec-Musiał, *Total graph interpretation of the numbers of the Fibonacci type*, Journal of Applied Mathematics, Vol. 2015, Article ID 837917, 7 pages.

[2] A. Boussayoud, *On some identities and generating functions for Pell-Lucas numbers*, Online Analytic Combinatorics Journal, 12 (2017), pp. 1–10.

[3] A. Boussayoud, S. Boughaba, *On some identities and symmetric functions for k-Pell sequences and Chebychev polynomials*, Online Analytic Combinatorics Journal 14, (2019), pp. 1–13.

[4] A. Boussayoud, M. Chelgham, S. Boughaba, *On some identities and generating functions for Mersenne numbers and polynomials*, Turkish Journal of Analysis and Number Theory, Vol. 6, No. 3 (2018), pp. 93–97.

[5] P. Catarino, *On k-Pell hybrid numbers*, Journal of Discrete Mathematical Sciences and Cryptography, 22(1) (2019), pp. 83–89.

[6] S. Falcon, A. Plaza, *On the Fibonacci k-numbers*, Chaos, Solitons & Fractals, 32(5) (2007), pp. 1615–1624.

[7] S. Falcon, A. Plaza, *The k-Fibonacci sequence and the Pascal* 2*-triangle*, Chaos, Solitons & Fractals, 33(1) (2007), pp. 38–49.

[8] C. Kızılateş, *A new generalization of Fibonacci hybrid and Lucas hybrid numbers*, Chaos, Solitons & Fractals, 130 (2020) 109449, https://doi.org/10.1016/j.chaos.2019.109449.

[9] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, New York-Toronto 2001.

[10] M. Liana, A. Szynal-Liana, I. Włoch, *On Pell hybrinomials*, Miskolc Mathematical Notes, 20.2 (2019), pp. 1051–1062.

[11] A. Nalli, P. Haukkanen, *On generalized Fibonacci and Lucas polynomials*, Chaos, Solitons & Fractals, 42(5) (2009), pp. 3179–3186.

[12] M. Özdemir, *Introduction to Hybrid Numbers*, Advances in Applied Clifford Algebras, 28 (2018), https://doi.org/10.1007/s00006-018-0833-3.

[13] M. Shafiq, H. M. Srivastava, N. Khan, Q. Z. Ahmad, M. Darus, S. Kiran, *An upper bound of the third Hankel determinant for a subclass of q-starlike functions associated with k-Fibonacci numbers*, Symmetry, 12 (2020), Article ID 1043, pp. 1–17.

[14] A. Szynal-Liana, *The Horadam hybrid numbers*, Discussiones Mathematicae. General Algebra and Applications, 38, No. 1 (2018), pp. 91–98.

[15] A. Szynal-Liana, I. Włoch, *Hypercomplex numbers of the Fibonacci type*, Oficyna Wydawnicza Politechniki Rzeszowskiej, Rzeszów 2019.

[16] A. Szynal-Liana, I. Włoch, *Introduction to Fibonacci and Lucas hybrinomials*, Complex Variables and Elliptic Equations, https://doi.org/10.1080/17476933.2019.1681416.

[17] A. Szynal-Liana, I. Włoch, *On generalized Mersenne hybrid numbers*, Annales UMCS, sectio A, in press.

[18] A. Szynal-Liana, I. Włoch, *On Jacobsthal and Jacobsthal-Lucas Hybrid Numbers*, Annales Mathematicae Silesianae, 33 (2019), pp. 276–283.

[19] A. Szynal-Liana, I. Włoch, *On Pell and Pell-Lucas Hybrid Numbers*, Commentationes Mathematicae, Vol. 58, No 1-2 (2018), pp. 11–17.

[20] A. Szynal-Liana, I. Włoch, *The Fibonacci hybrid numbers*, Utilitas Mathematica, 110 (2019), pp. 3–10.

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