A COMBINATORIAL CHARACTERIZATION OF THE GENERALIZED EXPONENTIAL AND FUBINI POLYNOMIALS

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Abstract. In this paper, we show that the generalized exponential polynomials and the generalized Fubini polynomials satisfy certain binomial identities and that these identities characterize the mentioned polynomials (up to an affine transformation of the variable) among the class of the normalized Sheffer sequences.

1. Introduction

The *generalized exponential polynomials* $S_n^{(\nu)}(x)$ are defined by the exponential generating series

$$S^{(\nu)}(x;t) = \sum_{n\geq 0} S_n^{(\nu)}(x)\,\frac{t^n}{n!} = \mathrm{e}^{\nu t}\mathrm{e}^{x(\mathrm{e}^t-1)} \tag{1}$$

and can be written explicitly as

$$S_n^{(\nu)}(x) = \sum_{k=0}^{n} S_{n,k}^{(\nu)} x^k$$

where the coefficients $S_{n,k}^{(\nu)}$ are the *generalized Stirling numbers of the second kind*. For $\nu = r \in \mathbb{N}$, we have the *r-exponential polynomials* [7], whose coefficients are the *r-Stirling numbers of the second kind* [2]. In particular, for $\nu = 0$, we have the ordinary *exponential polynomials* $S_n(x)$ [15, p. 63]. From series (1), we have that the generalized exponential polynomials can be expressed in terms of the ordinary exponential polynomials, namely

$$S_n^{(\nu)}(x) = \sum_{k=0}^{n} \binom{n}{k} \nu^{n-k}\, S_k(x)\,.$$

The *generalized Bell numbers* $b_n^{(\nu)}$ are defined by setting $x = 1$ in the generalized exponential polynomials, i.e. $b_n^{(\nu)} = S_n^{(\nu)}(1) = \sum_{k=0}^{n} S_{n,k}^{(\nu)}$, and consequently have exponential generating series

$$\sum_{n\geq 0} b_n^{(\nu)}\,\frac{t^n}{n!} = \mathrm{e}^{\nu t}\mathrm{e}^{\mathrm{e}^t-1}\,. \tag{2}$$

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For $\nu = r \in \mathbb{N}$, we have the *r-Bell numbers* [7], and, in particular, for $\nu = 0$, we have the ordinary *Bell numbers* b_n [4, p. 210] [18, A000110]. Notice that the generalized Bell numbers can be expressed in terms of the ordinary Bell numbers:

$$b_n^{(\nu)} = \sum_{k=0}^{n} \binom{n}{k} \nu^{n-k} b_k \,.$$

The *generalized Fubini polynomials* $F_n^{(\nu)}(x)$ are defined by the exponential generating series

$$F^{(\nu)}(x;t) = \sum_{n\geq 0} F_n^{(\nu)}(x)\, \frac{t^n}{n!} = \frac{e^{\nu t}}{1 - x(e^t - 1)} \tag{3}$$

and can be written explicitly as

$$F_n^{(\nu)}(x) = \sum_{k=0}^{n} S_{n,k}^{(\nu)}\, k!\, x^k \,.$$

For $\nu = r \in \mathbb{N}$, we have the *r-Fubini polynomials* [8, 9] and, in particular, for $\nu = 0$, we have the ordinary *Fubini polynomials* $F_n(x)$ [19] (or *ordered Bell polynomials*, or *De Morgan polynomials* [5, 11]). From series (3), we have that the generalized Fubini polynomials can be expressed in terms of the ordinary Fubini polynomials, namely

$$F_n^{(\nu)}(x) = \sum_{k=0}^{n} \binom{n}{k} \nu^{n-k}\, F_k(x) \,.$$

The *generalized Fubini numbers* $F_n^{(\nu)}$ are defined by setting $x = 1$ in the generalized Fubini polynomials, i.e. $F_n^{(\nu)} = F_n^{(\nu)}(1) = \sum_{k=0}^{n} S_{n,k}^{(\nu)}\, k!$, and consequently have exponential generating series

$$\sum_{n\geq 0} F_n^{(\nu)}\, \frac{t^n}{n!} = \frac{e^{\nu t}}{2 - e^t} \,. \tag{4}$$

For $\nu = r \in \mathbb{N}$, we have the *r-Fubini numbers* [9], and, in particular, for $\nu = 0$, we have the ordinary *Fubini numbers* F_n [3] [4, p. 228] [6] [18, A000670], or *ordered Bell numbers* [10]. Notice that the generalized Fubini numbers can be expressed in terms of the ordinary Fubini numbers:

$$F_n^{(\nu)} = \sum_{k=0}^{n} \binom{n}{k} F_{n-k} \nu^k \,.$$

A *Sheffer sequence* [1, 15, 16, 17] is a polynomial sequence $\{s_n(x)\}_{n\geq 0}$ having exponential generating series

$$\sum_{n\geq 0} s_n(x)\, \frac{t^n}{n!} = g(t)\, e^{x f(t)} \tag{5}$$

where $g(t) = \sum_{n\geq 0} g_n \frac{t^n}{n!}$ and $f(t) = \sum_{n\geq 0} f_n \frac{t^n}{n!}$ are two exponential series with $g_0 \neq 0$, $f_0 = 0$ and $f_1 \neq 0$. A *normalized Sheffer sequence* is a Sheffer sequence with $g_0 = 1$.

A *Sheffer matrix* is an infinite lower triangular matrix $S = [s_{n,k}]_{n,k\geq 0}$ whose k-th column has exponential generating series

$$\sum_{n\geq k} s_{n,k} \frac{t^n}{n!} = g(t)\frac{f(t)^k}{k!}$$

where $g(t) = \sum_{n\geq 0} g_n \frac{t^n}{n!}$ and $f(t) = \sum_{n\geq 0} f_n \frac{t^n}{n!}$ are two exponential series with $g_0 \neq 0$, $f_0 = 0$ and $f_1 \neq 0$. In this case, we say that S has *spectrum* $(g(t), f(t))$ and we write $S = (g(t), f(t))$. The row polynomials of S are defined by

$$s_n(x) = \sum_{k=0}^{n} s_{n,k}\, x^k$$

and form a Sheffer sequence $\{s_n(x)\}_{n\geq 0}$ with exponential generating series (5). On the other hand, given a Sheffer sequence $\{s_n(x)\}_{n\geq 0}$, the matrix $S = [s_{n,k}]_{n,k\geq 0}$ of the coefficients is the Sheffer matrix with spectrum $(g(t), f(t))$.

Given a Sheffer sequence $\{s_n(x)\}_{n\geq 0}$, with Sheffer matrix $S = [s_{n,k}]_{n,k\geq 0} = (g(t), f(t))$, the *associated sequence* $\{s_n^*(x)\}_{n\geq 0}$ is defined by the polynomials

$$s_n^*(x) = \sum_{k=0}^{n} s_{n,k} k!\, x^k$$

with exponential generating series

$$\sum_{n\geq 0} s_n^*(x) \frac{t^n}{n!} = \frac{g(t)}{1 - xf(t)}\,. \tag{6}$$

The associated polynomials $s_n^*(x)$ are essentially the Laplace transform of the polynomials $s_n(x)$, since we have the following integral relation

$$s_n^*(x) = \int_0^{+\infty} s_n(xz)\, \mathrm{e}^{-z}\, \mathrm{d}z\,.$$

Notice that the space $\mathcal{S}$ of the Sheffer sequences and the space $\mathcal{S}^*$ of the associated Sheffer sequences are both closed under affine transformations of the variable. Indeed, if $\{s_n(x)\}_{n\geq 0}$ is a (normalized) Sheffer sequence with spectrum $(g(t), f(t))$, then, for every constant $\alpha \neq 0$ and β, the sequence $\{s_n(\alpha x + \beta)\}_{n\geq 0}$ is a (normalized) Sheffer sequence with spectrum $(g(t)\mathrm{e}^{\beta f(t)}, \alpha f(t))$, since

$$\sum_{n\geq 0} s_n(\alpha x + \beta)\frac{t^n}{n!} = g(t)\, \mathrm{e}^{(\alpha x+\beta)f(t)} = g(t)\, \mathrm{e}^{\beta f(t)} \cdot \mathrm{e}^{\alpha x f(t)}\,.$$

Similarly, if $\{s_n^*(x)\}_{n\geq 0}$ is an associated Sheffer sequence with spectrum $(g(t), f(t))$, then, for every constant $\alpha \neq 0$ and β, the sequence $\{s_n^*(\alpha x + \beta)\}_{n\geq 0}$ is an associated Sheffer sequence with spectrum $\left(\frac{g(t)}{1-\beta f(t)}, \frac{\alpha f(t)}{1-\beta f(t)}\right)$, since

$$\sum_{n\geq 0} s_n^*(\alpha x + \beta)\frac{t^n}{n!} = \frac{g(t)}{1 - (\alpha x + \beta)f(t)} = \frac{g(t)}{1 - \beta f(t)}\, \frac{1}{1 - x\frac{\alpha f(t)}{1-\beta f(t)}}\,.$$

Finally, notice that the generalized exponential polynomials form a Sheffer sequence with spectrum $(\mathrm{e}^{\nu t}, \mathrm{e}^t - 1)$, while the generalized Fubini polynomials are the associated sequence. In particular, as just remarked, these polynomials are related by the identity

$$F_n^{(\nu)}(x) = \int_0^{+\infty} S_n^{(\nu)}(xz)\, \mathrm{e}^{-z}\, \mathrm{d}z\,.$$

In this paper, we show that the generalized exponential polynomials and the generalized Fubini polynomials satisfy certain binomial identities and that these identities characterize the mentioned polynomials (up to an affine transformation of the variable) in the class of the normalized Sheffer sequences.

2. First characterization

We start by proving (in Theorem 1) that the generalized exponential polynomials satisfy a binomial identity. Since this identity is of the form $F(s_0(x), s_1(x), \ldots, s_n(x))$, with $F(x_0, x_1, \ldots, x_n) \in \mathbb{R}[x_0, x_1, \ldots, x_n]$, it is satisfied by all sequences of the form $\{s_n(\alpha x + \beta)\}_{n\geq 0}$. However, we will prove (in Theorem 3) that such an identity characterizes the generalized exponential polynomials in the class of normalized Sheffer sequences, up to an affine transformation of the variable.

Theorem 1. *The generalized exponential polynomials satisfy the binomial identity*

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}S_{k+1}^{(\nu)}(x)S_{n-k}^{(\nu)}(x) + \sum_{k=0}^{n}\binom{n}{k}S_{k+1}^{(\nu)}(x)S_{n-k+1}^{(\nu)}(x) &= \\ = \sum_{k=0}^{n}\binom{n}{k}S_{k+2}^{(\nu)}(x)S_{n-k}^{(\nu)}(x) + \nu\sum_{k=0}^{n}\binom{n}{k}S_{k}^{(\nu)}(x)S_{n-k}^{(\nu)}(x)\,.\end{aligned} \tag{7}$$

In particular, for $\nu = 0$, we have that the exponential polynomials satisfy the identity

$$\sum_{k=0}^{n}\binom{n}{k}S_{k+1}(x)S_{n-k}(x) + \sum_{k=0}^{n}\binom{n}{k}S_{k+1}(x)S_{n-k+1}(x) = \sum_{k=0}^{n}\binom{n}{k}S_{k+2}(x)S_{n-k}(x)\,. \tag{8}$$

Proof. Consider series (1) and, for simplicity, set $S = S^{(\nu)}(x;t)$. Differentiating two times with respect to t, we have

$$\frac{\partial S}{\partial t} = \nu\, \mathrm{e}^{\nu t}\mathrm{e}^{x(\mathrm{e}^t-1)} + x\mathrm{e}^t\mathrm{e}^{\nu t}\mathrm{e}^{x(\mathrm{e}^t-1)} = \nu\, S + x\mathrm{e}^t S$$

and

$$\frac{\partial^2 S}{\partial t^2} = \nu\,\frac{\partial S}{\partial t} + x\mathrm{e}^t S + x\mathrm{e}^t\frac{\partial S}{\partial t}\,.$$

From the first equation, we have

$$x\mathrm{e}^t S = \frac{\partial S}{\partial t} - \nu\, S\,. \tag{$*$}$$

So, by replacing this expression in the second equation, we have

$$\frac{\partial^2 S}{\partial t^2} = (\nu+1)\,\frac{\partial S}{\partial t} - \nu\, S + x\mathrm{e}^t \frac{\partial S}{\partial t}\,.$$

Then, by multiplying both sides by S, we have

$$\frac{\partial^2 S}{\partial t^2}\, S = (\nu+1)\,\frac{\partial S}{\partial t}\, S - \nu\, S^2 + x\mathrm{e}^t S\,\frac{\partial S}{\partial t}\,.$$

Using again identity $(*)$, we have

$$\frac{\partial^2 S}{\partial t^2}\, S = \frac{\partial S}{\partial t}\, S - \nu\, S^2 + \Big(\frac{\partial S}{\partial t}\Big)^2$$

or

$$\frac{\partial S}{\partial t}\, S + \Big(\frac{\partial S}{\partial t}\Big)^2 = \frac{\partial^2 S}{\partial t^2}\, S + \nu\, S^2\,.$$

This equation is equivalent to identity (7). □

As an immediate consequence of Theorem 1, for $x = 1$, we have the following result.

Theorem 2. *The generalized Bell numbers satisfy the identity*

$$\sum_{k=0}^{n}\binom{n}{k} b_{k+1}^{(\nu)} b_{n-k}^{(\nu)} + \sum_{k=0}^{n}\binom{n}{k} b_{k+1}^{(\nu)} b_{n-k+1}^{(\nu)} = \sum_{k=0}^{n}\binom{n}{k} b_{k+2}^{(\nu)} b_{n-k}^{(\nu)} + \nu\sum_{k=0}^{n}\binom{n}{k} b_{k}^{(\nu)} b_{n-k}^{(\nu)}\,. \tag{9}$$

In particular, for $\nu = 0$, we have that the Bell numbers satisfy the identity

$$\sum_{k=0}^{n}\binom{n}{k} b_{k+1} b_{n-k} + \sum_{k=0}^{n}\binom{n}{k} b_{k+1} b_{n-k+1} = \sum_{k=0}^{n}\binom{n}{k} b_{k+2} b_{n-k}\,. \tag{10}$$

Now, we prove the converse of Theorem 1, i.e. that identity (7) essentially characterizes the generalized exponential polynomials.

Theorem 3. *If $\{s_n(x)\}_{n\in\mathbb{N}}$ is a normalized Sheffer sequence satisfying the binomial identity*

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k} s_{k+1}(x) s_{n-k}(x) + \sum_{k=0}^{n}\binom{n}{k} s_{k+1}(x) s_{n-k+1}(x) &= \\ = \sum_{k=0}^{n}\binom{n}{k} s_{k+2}(x) s_{n-k}(x) + \nu\sum_{k=0}^{n}\binom{n}{k} s_{k}(x) s_{n-k}(x)\,, \end{aligned} \tag{11}$$

then

$$s_n(x) = S_n^{(\nu)}(\alpha x + \beta) \tag{12}$$

where α and β are arbitrary constants (with $\alpha \neq 0$).

Proof. Since $\{s_n(x)\}_{n\in\mathbb{N}}$ is a normalized Sheffer sequence, its exponential generating series is

$$S(x;t) = \sum_{n\geq 0} s_n(x)\,\frac{t^n}{n!} = g(t)\,\mathrm{e}^{xf(t)}$$

where $g_0 = 1$. Identity (11) is equivalent to the equation

$$\frac{\partial S}{\partial t}\, S + \left(\frac{\partial S}{\partial t}\right)^2 = \frac{\partial^2 S}{\partial t^2}\, S + \nu\, S^2$$

where $S = s(x;t)$. Since

$$\begin{aligned}\frac{\partial S}{\partial t} &= g' e^{xf} + xgf' e^{xf}\\ \frac{\partial^2 S}{\partial t^2} &= g'' e^{xf} + 2xg'f' e^{xf} + xgf'' e^{xf} + x^2 g f'^2 e^{xf}\,,\end{aligned}$$

the above equation becomes

$$(g' + xgf')\, g e^{2xf} + (g' + xgf')^2 e^{2xf} = (g'' + 2xg'f' + xgf'' + x^2 g f'^2)\, g e^{2xf} + \nu\, g^2 e^{2xf}\,.$$

Simplifying by the exponential e^{2xf}, we obtain the equation

$$gg' + xg^2 f' + g'^2 + 2xgg'f' + x^2 g^2 f'^2 = gg'' + 2xgg'f' + xg^2 f'' + x^2 g^2 f'^2 + \nu\, g^2\,,$$

that is the equation

$$gg' + g'^2 - gg'' - \nu\, g^2 + x(g^2 f' + 2gg'f' - 2gg'f' - g^2 f'') = 0$$

which is equivalent to the differential system

$$(**) \qquad \begin{cases} gg'' - g'^2 - gg' + \nu\, g^2 = 0\\ g^2 f'' - g^2 f' = 0\,. \end{cases}$$

Let us consider the first equation, depending only on g. Setting $z = g'/g$, we have $z' = (gg'' - g'^2)/g^2$, and consequently

$$g^2 z' - g^2 z + \nu\, g^2 = 0\,.$$

Simplifying by g^2, we obtain the differential equation

$$z' = z - \nu\,.$$

Hence, we have

$$\int \frac{dz}{z-\nu} = \int dt$$

that is

$$\ln(z - \nu) = t + c \qquad \text{or} \qquad z = \nu + \beta e^t$$

where β is an arbitrary constant with respect to t. Since $z = g'/g$, we have

$$\frac{g'}{g} = \nu + \beta e^t$$

that is

$$\ln g = \nu t + \beta e^t + c$$

that is

$$g(t) = C\, e^{\nu t}\, e^{\beta(e^t - 1)}$$

where C is an arbitrary constant with respect to t. Since $g_0 = 1$, we have $C = 1$ and

$$g(t) = \mathrm{e}^{\nu t}\, \mathrm{e}^{\beta(\mathrm{e}^t - 1)}\,.$$

Let us now consider the second equation of the differential system $(**)$. Simplifying by g^2, we have $f'' = f'$, that is $f' = f + \alpha$. Hence, we have

$$\int \frac{f'(t)}{f(t) + \alpha}\,\mathrm{d}t = \int \mathrm{d}t$$

that is

$$\ln(f(t) + \alpha) = t + K = t + \ln\alpha = \ln \alpha\mathrm{e}^t$$

being $f_0 = 0$. So

$$f(t) = \alpha(\mathrm{e}^t - 1)\,.$$

In conclusion, we have the generating series

$$S(x;t) = \mathrm{e}^{\nu t}\mathrm{e}^{\beta(\mathrm{e}^t-1)}\,\mathrm{e}^{x\alpha(\mathrm{e}^t-1)} = \mathrm{e}^{\nu t}\,\mathrm{e}^{(\alpha x+\beta)(\mathrm{e}^t-1)} = S^{(\nu)}(\alpha x + \beta; t)$$

and consequently identity (12). □

3. Second characterization

Also in this case, we start by proving (in Theorem 4) that the generalized Fubini polynomials satisfy a binomial identity, and then we prove (in Theorem 6) that such an identity characterizes these polynomials among the normalized Sheffer sequences, up to an affine transformation of the variable.

Theorem 4. *The generalized Fubini polynomials satisfy the binomial identity*

$$\begin{aligned}(1-2\nu)\sum_{k=0}^{n}\binom{n}{k}F_{k+1}^{(\nu)}(x)F_{n-k}^{(\nu)}(x) + 2\sum_{k=0}^{n}\binom{n}{k}F_{k+1}^{(\nu)}(x)F_{n-k+1}^{(\nu)}(x) = \\ = \sum_{k=0}^{n}\binom{n}{k}F_{k+2}^{(\nu)}(x)F_{n-k}^{(\nu)}(x) + \nu(1-\nu)\sum_{k=0}^{n}\binom{n}{k}F_{k}^{(\nu)}(x)F_{n-k}^{(\nu)}(x)\,.\end{aligned} \tag{13}$$

In particular, for $\nu = 0$, we have that the Fubini polynomials satisfy the identity

$$\sum_{k=0}^{n}\binom{n}{k}F_{k+1}(x)F_{n-k}(x) + 2\sum_{k=0}^{n}\binom{n}{k}F_{k+1}(x)F_{n-k+1}(x) = \sum_{k=0}^{n}\binom{n}{k}F_{k+2}(x)F_{n-k}(x)\,. \tag{14}$$

Proof. Consider series (3) and, for simplicity, set $F = F^{(\nu)}(x;t)$. Differentiating two times with respect to t, we have

$$\frac{\partial F}{\partial t} = \frac{\nu\,\mathrm{e}^{\nu t}}{1 - x(\mathrm{e}^t - 1)} + \frac{x\mathrm{e}^t\mathrm{e}^{\nu t}}{(1 - x(\mathrm{e}^t - 1))^2} = \nu\, F + \frac{x\mathrm{e}^t}{1 - x(\mathrm{e}^t - 1)}\, F$$

and

$$\frac{\partial^2 F}{\partial t^2} = \nu\,\frac{\partial F}{\partial t} + \frac{x\mathrm{e}^t}{1 - x(\mathrm{e}^t - 1)}\,F + \frac{x^2\mathrm{e}^{2t}}{(1 - x(\mathrm{e}^t - 1))^2}\,F + \frac{x\mathrm{e}^t}{1 - x(\mathrm{e}^t - 1)}\,\frac{\partial F}{\partial t}\,.$$

From the first equation, we have

$$(\star)\qquad \frac{x\mathrm{e}^t}{1-x(\mathrm{e}^t-1)}\,F = \frac{\partial F}{\partial t} - \nu\,F\,.$$

So, by replacing this expression in the second equation, we have

$$\frac{\partial^2 F}{\partial t^2} = (\nu+1)\,\frac{\partial F}{\partial t} - \nu\,F + \frac{x^2\mathrm{e}^{2t}}{(1-x(\mathrm{e}^t-1))^2}\,F + \frac{x\mathrm{e}^t}{1-x(\mathrm{e}^t-1)}\,\frac{\partial F}{\partial t}\,.$$

Then, by multiplying both sides by F, we have

$$\frac{\partial^2 F}{\partial t^2}\,F = (\nu+1)\,\frac{\partial F}{\partial t}\,F - \nu\,F^2 + \left(\frac{x\mathrm{e}^t}{1-x(\mathrm{e}^t-1)}\,F\right)^2 + \frac{x\mathrm{e}^t}{1-x(\mathrm{e}^t-1)}\,F\,\frac{\partial F}{\partial t}\,.$$

Using again identity $(\star)$, we have

$$\begin{aligned}\frac{\partial^2 F}{\partial t^2}\,F &= (\nu+1)\,\frac{\partial F}{\partial t}\,F - \nu\,F^2 + \left(\frac{\partial F}{\partial t} - \nu\,F\right)^2 + \left(\frac{\partial F}{\partial t} - \nu\,F\right)\frac{\partial F}{\partial t}\\ &= (1-2\nu)\,\frac{\partial F}{\partial t}\,F - \nu(1-\nu)\,F^2 + 2\left(\frac{\partial F}{\partial t}\right)^2\end{aligned}$$

or

$$(1-2\nu)\,\frac{\partial F}{\partial t}\,F + 2\left(\frac{\partial F}{\partial t}\right)^2 = \frac{\partial^2 F}{\partial t^2}\,F + \nu(1-\nu)\,F^2\,.$$

This equation is equivalent to identity (13). □

From Theorem 4, with $x = 1$, we immediately have the following specializations.

Theorem 5. *The generalized Fubini numbers satisfy the binomial identity*

$$(15)\qquad \begin{aligned}&(1-2\nu)\sum_{k=0}^{n}\binom{n}{k}F_{k+1}^{(\nu)}F_{n-k}^{(\nu)} + 2\sum_{k=0}^{n}\binom{n}{k}F_{k+1}^{(\nu)}F_{n-k+1}^{(\nu)} =\\ &\qquad = \sum_{k=0}^{n}\binom{n}{k}F_{k+2}^{(\nu)}F_{n-k}^{(\nu)} + \nu(1-\nu)\sum_{k=0}^{n}\binom{n}{k}F_{k}^{(\nu)}F_{n-k}^{(\nu)}\,.\end{aligned}$$

In particular, for $\nu = 0$, we have that the Fubini numbers satisfy the identity

$$(16)\qquad \sum_{k=0}^{n}\binom{n}{k}F_{k+1}F_{n-k} + 2\sum_{k=0}^{n}\binom{n}{k}F_{k+1}F_{n-k+1} = \sum_{k=0}^{n}\binom{n}{k}F_{k+2}F_{n-k}\,.$$

Now, we can prove the converse of Theorem 4.

Theorem 6. *If $\{f_n(x)\}_{n\in\mathbb{N}}$ is a sequence associated with a normalized Sheffer matrix and satisfy the identity*

$$(17)\qquad \begin{aligned}&(1-2\nu)\sum_{k=0}^{n}\binom{n}{k}f_{k+1}(x)f_{n-k}(x) + 2\sum_{k=0}^{n}\binom{n}{k}f_{k+1}(x)f_{n-k+1}(x) =\\ &\qquad = \sum_{k=0}^{n}\binom{n}{k}f_{k+2}(x)f_{n-k}(x) + \nu(1-\nu)\sum_{k=0}^{n}\binom{n}{k}f_{k}(x)f_{n-k}(x)\,,\end{aligned}$$

then

$$f_n(x) = F_k^{(\nu)}(\alpha x + \beta) \tag{18}$$

where α and β are arbitrary constants (with $\alpha \neq 0$).

Proof. By hypothesis, the exponential generating series of the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is of the form

$$F(x;t) = \sum_{n\geq 0} f_n(x)\,\frac{t^n}{n!} = \frac{g(t)}{1 - xf(t)}$$

where $g(t)$ and $f(t)$ are two exponential series with $g_0 = 1$, $f_0 = 0$ and $f_1 \neq 0$. Then, setting $F = F(x;t)$ for simplicity, identity (17) is equivalent to the equation

$$(1-2\nu)\,\frac{\partial F}{\partial t}\,F + 2\left(\frac{\partial F}{\partial t}\right)^2 = \frac{\partial^2 F}{\partial t^2}\,F + \nu(1-\nu)\,F^2\,. \tag{$\star$}$$

Differentiating two times the generating series F with respect to t, we have

$$\frac{\partial F}{\partial t} = \frac{g'(t)}{1-xf(t)} + x\frac{g(t)f'(t)}{(1-xf(t))^2}$$

and

$$\frac{\partial^2 F}{\partial t^2} = \frac{g''(t)}{1-xf(t)} + x\,\frac{2g'(t)f'(t)+g(t)f''(t)}{(1-xf(t))^2} + x^2\frac{2g(t)f'(t)^2}{(1-xf(t))^3}\,.$$

Then, by replacing these expressions in equation ($\star$), we have

$$\begin{aligned}(1-2\nu)\left(\frac{g'}{1-xf} + x\,\frac{gf'}{(1-xf)^2}\right)\frac{g}{1-xf} + 2\left(\frac{g'}{1-xf} + x\,\frac{gf'}{(1-xf)^2}\right)^2 = \\ = \left(\frac{g''}{1-xf} + x\,\frac{2g'f'+gf''}{(1-xf)^2} + x^2\frac{2gf'^2}{(1-xf)^3}\right)\frac{g}{1-xf} + \nu(1-\nu)\left(\frac{g}{1-xf}\right)^2\end{aligned}$$

that is

$$\begin{aligned}(1-2\nu)\Big(gg'(1-xf) + xg^2f'\Big)(1-xf) + 2\Big(g'(1-xf) + xgf'\Big)^2 = \\ = \Big(g''(1-xf)^2 + x(2g'f'+gf'')(1-xf) + 2x^2gf'^2\Big)g + \nu(1-\nu)g^2(1-xf)^2\end{aligned}$$

that is

$$\begin{aligned}\Big((1-2\nu)gg' + 2g'^2 - gg'' - \nu(1-\nu)g^2\Big)(1-xf)^2 + \\ + x\Big(g^2f' + 4gg'f' - (2gg'f' + g^2f'')\Big)(1-xf) = 0\,.\end{aligned}$$

By simplifying by $1-xf$, we obtain the equation

$$\big((1-2\nu)gg' + 2g'^2 - gg'' - \nu(1-\nu)g^2\big)(1-xf) + x\big((1-2\nu)g^2f' + 2gg'f' - g^2f''\big) = 0$$

which is equivalent to the differential system

$$\begin{cases} gg'' - 2g'^2 - (1-2\nu)gg' + \nu(1-\nu)g^2 = 0 \\ g^2f'' - 2gg'f' - (1-2\nu)g^2f' = 0\,. \end{cases} \tag{$\star\star$}$$

Let us consider the first equation, which depends only on g:

$$gg'' - 2g'^2 - (1-2\nu)gg' + \nu(1-\nu)g^2 = 0\,.$$

Setting $z = g'/g$, we have $z' = (gg'' - g'^2)/g^2$, and the above equation becomes

$$g^2z' - g^2z^2 - (1-2\nu)g^2z + \nu(1-\nu)g^2 = 0\,.$$

Simplifying by g^2, we obtain the differential equation

$$z' = z^2 + (1-2\nu)z - \nu(1-\nu) = (z-\nu)(z-\nu+1)$$

from which we have

$$\int \frac{\mathrm{d}z}{(z-\nu)(z-\nu+1)} = \int \mathrm{d}t$$

that is

$$\ln\frac{z-\nu}{z-\nu+1} = t + c \qquad \text{or} \qquad \frac{z-\nu}{z-\nu+1} = \lambda \mathrm{e}^t \qquad \text{or} \qquad z = \frac{\nu + (1-\nu)\lambda\mathrm{e}^t}{1-\lambda\mathrm{e}^t}$$

where λ is an arbitrary constant with respect to t, $\lambda \neq 1$. Hence, being $z = g'/g$, we have

$$\frac{g'}{g} = \frac{\nu + (1-\nu)\lambda\mathrm{e}^t}{1-\lambda\mathrm{e}^t} = \nu - \frac{\lambda\mathrm{e}^t}{1-\lambda\mathrm{e}^t}\,.$$

So, integrating, we have

$$\ln g = \nu t - \ln(1-\lambda\mathrm{e}^t) + C = \ln\frac{h\mathrm{e}^{\nu t}}{1-\lambda\mathrm{e}^t}$$

that is

$$g(t) = \frac{h\mathrm{e}^{\nu t}}{1-\lambda\mathrm{e}^t}$$

where h is an arbitrary constant with respect to t. Since $g_0 = 1$, we have $h = 1-\lambda$, and consequently

$$g(t) = \frac{(1-\lambda)\,\mathrm{e}^{\nu t}}{1-\lambda\mathrm{e}^t} = \frac{(1-\lambda)\,\mathrm{e}^{\nu t}}{1-\lambda-\lambda(\mathrm{e}^t-1)} = \frac{\mathrm{e}^{\nu t}}{1-\frac{\lambda}{1-\lambda}(\mathrm{e}^t-1)}\,.$$

Setting $\beta = \frac{\lambda}{1-\lambda}$, we have

$$g(t) = \frac{\mathrm{e}^{\nu t}}{1-\beta(\mathrm{e}^t-1)}$$

where β is an arbitrary constant with respect to t.

Let us now consider the second equation of the differential system ($\star\star$). Simplifying by g, we have

$$gf'' - 2g'f' - (1-2\nu)gf' = 0\,.$$

Then, dividing by gf' both the sides, we have

$$\frac{f''}{f'} = 2\,\frac{g'}{g} + 1 - 2\nu\,.$$

Integrating, we have

$$\ln f' = 2\,\ln g + (1-2\nu)t + a = \ln \alpha g^2 \mathrm{e}^{(1-2\nu)t}$$

where $a = \ln\alpha$. So

$$f'(t) = \alpha\, g(t)^2\, \mathrm{e}^{(1-2\nu)t} = \frac{\alpha\, \mathrm{e}^t}{(1-\beta(\mathrm{e}^t-1))^2}\,.$$

For $\beta \neq 0$, integrating we have

$$f(t) = \frac{\alpha}{\beta}\,\frac{1}{1-\beta(\mathrm{e}^t-1)} + K\,.$$

Since $f_0 = 0$, we have $K = \frac{\alpha}{\beta}$, and consequently

$$f(t) = \frac{\alpha}{\beta}\,\frac{1}{1-\beta(\mathrm{e}^t-1)} - \frac{\alpha}{\beta} = \frac{\alpha(\mathrm{e}^t-1)}{1-\beta(\mathrm{e}^t-1)}\,.$$

In conclusion, we have the generating series

$$F(x;t) = \frac{\mathrm{e}^{\nu t}}{1-\beta(\mathrm{e}^t-1)}\,\frac{1}{1-x\frac{\alpha(\mathrm{e}^t-1)}{1-\beta(\mathrm{e}^t-1)}} = \frac{\mathrm{e}^{\nu t}}{1-(\alpha x+\beta)(\mathrm{e}^t-1)} = F^{(\nu)}(\alpha x+\beta;t)$$

and consequently we have identity (18). Similarly, for $\beta = 0$. □

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