SYMMETRIC AND ASYMMETRIC PEAKS IN COMPOSITIONS

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Abstract. Integer compositions and related counting problems are a rich and ubiquitous topic in enumerative combinatorics. In this paper we explore the definition of symmetric and asymmetric peaks and valleys over compositions. In particular, we compute an explicit formula for the generating function for the number of integer compositions according to the number of parts, symmetric, and asymmetric peaks and valleys.

1. Introduction

A *composition* $\sigma = \sigma_1\sigma_2\cdots\sigma_\ell$ of a positive integer n is a sequence of positive integers whose sum is n, that is, $\sigma_1 + \sigma_2 + \cdots + \sigma_\ell = n$. The summands σ_i are called *parts* of the composition and n is referred to as the *weight* of σ. For example, the compositions of 4 are $4, 31, 13, 22, 211, 121, 112, 1111$. Let $\mathcal{C}(n)$ denote the set of compositions of n. It is well-known that $|\mathcal{C}(n)| = 2^{n-1}$ for all $n \geq 1$ (cf. [8]). Moreover, the number of compositions of n with k parts is given by the binomial coefficient $\binom{n-1}{k-1}$.

A *peak* p of a composition $\sigma = \sigma_1\sigma_2\cdots\sigma_\ell$ is a subword $p = \sigma_{i-1}\sigma_i\sigma_{i+1}$, where $2 \leq i \leq \ell - 1$, such that $\sigma_i > \max\{\sigma_{i-1}, \sigma_{i+1}\}$. Similarly, a *valley* is a subword $p = \sigma_{i-1}\sigma_i\sigma_{i+1}$ such that $\sigma_i < \min\{\sigma_{i-1}, \sigma_{i+1}\}$. For example, the composition 3 1 5 2 2 3 1 4 2 has three peaks and two valleys. Blecher et al. [2] determined an explicit formula for the generating function which counts the compositions of n having k parts according to the occurrences of any peak and valley (including weak peaks and weak valleys). These authors also give the following combinatorial formula for the total number of peaks and valleys within all of the members in $\mathcal{C}(n)$ with k parts $2(k-2)\sum_{j=1}^{\lfloor \frac{n}{3} \rfloor}\binom{n-3j}{k-2}$. In [12], Shattuck gives a combinatorial proof of this result. From the main result given by Kitaev et al. in [10], it is possible to obtain a rational generating function for the total number of peaks over all compositions of n.

A peak $p = \sigma_{i-1}\sigma_i\sigma_{i+1}$ is *symmetric* if $\sigma_{i-1} = \sigma_{i+1}$. The peak $p = \sigma_{i-1}\sigma_i\sigma_{i+1}$ is *asymmetric* if $\sigma_{i-1} \neq \sigma_{i+1}$. Analogously, we have the definition of symmetric and asymmetric valleys. For example, the composition 3 5 3 2 4 4 3 4 3 1 2 has two symmetric peaks, one symmetric valley, two asymmetric valleys, and no asymmetric peaks.

Date: December 30, 2023.

2020 *Mathematics Subject Classification.* 05A15, 05A16.

Key words and phrases. Compositions, peaks, symmetric peaks, asymmetric peaks, generating functions.

The study of symmetric peaks was introduced by Asakly [1] in the context of k-ary words. After that Flórez and Ramírez [6] introduced the concept of symmetric and asymmetric peaks in Dyck paths, see also the recent works [3, 4, 5, 7, 13].

In this paper we give a multivariate generating functions, as well as combinatorial formulas to count the distribution of symmetric and asymmetric peaks (resp. valleys) over the set of integer compositions. Finally, we give some considerations for the number of symmetric peaks and valleys in restricted compositions, that is, compositions with parts in a given set.

2. The Symmetric Peak and Valley Statistic

The goal of this section is to find an explicit formula for the generating function $G(x,y,q)$ for the number of compositions of n according to the number of parts and number of symmetric peaks, that is, the coefficient of $x^n y^k q^\ell$ is the number of compositions of n with k parts and ℓ symmetric peaks. In order to do that, we define $G(x,y,q|a_1a_2\cdots a_s)$ to be the generating function for the number of compositions $\pi_1\pi_2\cdots\pi_m$ of n ($m \geq s$) according to the number of parts and number of symmetric peaks such that $\pi_i = a_i$ for all $i = 1,2,\ldots,s$. Hence,

$$G(x,y,q) = 1 + \sum_{a\geq 1} G(x,y,q|a) \tag{1}$$

with

$$\begin{aligned} G(x,y,q|a) &= x^a y + \sum_{j=1}^{a} G(x,y,q|aj) + \sum_{j\geq a+1} G(x,y,q|aj) \\ &= x^a y + x^a y \sum_{j=1}^{a} G(x,y,q|j) + \sum_{j\geq a+1} G(x,y,q|aj), \end{aligned} \tag{2}$$

which implies

$$\sum_{j\geq a+1} G(x,y,q|aj) = G(x,y,q|a) - x^a y - x^a y \sum_{j=1}^{a} G(x,y,q|j). \tag{3}$$

On other hand, for $b > a$ we have

$$\begin{aligned} G(x,y,q|ab) &= x^{a+b}y^2 + \sum_{j=1, j\neq a}^{b} G(x,y,q|abj) + G(x,y,q|aba) + \sum_{j\geq b+1} G(x,y,q|abj) \\ &= x^{a+b}y^2 + x^{a+b}y^2 \sum_{j=1}^{b} G(x,y,q|j) + (q-1)x^{a+b}y^2 G(x,y,q|a) \\ &\quad + x^a y \sum_{j\geq b+1} G(x,y,q|bj). \end{aligned}$$

Thus, by (3), we have

$$G(x,y,q|ab) = x^{a+b}y^2 + x^{a+b}y^2\sum_{j=1}^{b} G(x,y,q|j) + (q-1)x^{a+b}y^2G(x,y,q|a)$$
$$+ x^a y(G(x,y,q|b) - x^b y - x^b y\sum_{j=1}^{b} G(x,y,q|j))$$
$$= (q-1)x^{a+b}y^2G(x,y,q|a) + x^a yG(x,y,q|b).$$

Therefore, by (2), we have

$$G(x,y,q|a) = x^a y + x^a y\sum_{j=1}^{a} G(x,y,q|j) + \sum_{j\geq a+1}((q-1)x^{a+j}y^2G(x,y,q|a) + x^a yG(x,y,q|j))$$
$$= x^a y + x^a y\sum_{j\geq 1} G(x,y,q|j) + \frac{(q-1)x^{2a+1}y^2}{1-x}G(x,y,q|a)$$
$$= x^a yG(x,y,q) + \frac{(q-1)x^{2a+1}y^2}{1-x}G(x,y,q|a).$$

Then

$$G(x,y,q|a) = \frac{x^a y}{1-\frac{(q-1)x^{2a+1}y^2}{1-x}}G(x,y,q).$$

Hence, by (1), we obtain

$$G(x,y,q) - 1 = \sum_{a\geq 1}\frac{x^a y}{1-\frac{(q-1)x^{2a+1}y^2}{1-x}}G(x,y,q),$$

which leads to the following result

Theorem 2.1. *The generating function for the number of compositions of n according to the number of parts and number of symmetric peaks is given by*

$$G(x,y,q) = \frac{1}{1-\sum_{a\geq 1}\frac{x^a y}{1-\frac{(q-1)x^{2a+1}y^2}{1-x}}}.$$

As a series expansion, the generating function $G(x,y,q)$ begins with

$$G(x,y,q) = 1 + yx + (y+y^2)x^2 + (y+2y^2+y^3)x^3 + (y+3y^2+(2+q)y^3+y^4)x^4$$
$$+ (y+4y^2+5y^3+qy^3+\mathbf{(2+2q)y^4}+y^5)x^5 + \cdots$$

The compositions corresponding to the bold coefficients in the above series are

$$\overbrace{2\ 1\ 1\ 1}^{x^5y^4},\quad \overbrace{1\ 2\ 1\ 1}^{x^5y^4q},\quad \overbrace{1\ 1\ 2\ 1}^{x^5y^4q},\quad \overbrace{1\ 1\ 1\ 2}^{x^5y^4}.$$

By using similar techniques as in the proof of Theorem 2.1, we obtain the following theorem.

Theorem 2.2. *The generating function for the number of compositions of n according to the number of parts and number of symmetric valleys is given by*

$$V(x,y,q) = \frac{1}{1-\sum_{a\geq 1}\frac{x^a y}{1-\frac{(q-1)x^{a+1}(1-x^{a-1})y^2}{1-x}}}.$$

Clearly, Theorems 2.1 and 2.2 give $G(x,y,1) = V(x,y,1) = \frac{1-x}{1-x-xy}$, as expected. Let $s(n)$ denote the total number of symmetric peaks in $\mathcal{C}(n)$, and let $s(n,k)$ denote the total number of symmetric peaks in $\mathcal{C}(n)$ with k parts. It is clear that $s(n) = \sum_{k=0}^{n} s(n,k)$.

The generating functions for the sequences $s(n,k)$ and $s(n)$ are given by

$$\frac{\partial}{\partial q}G(x,y,q)\mid_{q=1} = \sum_{n,k\geq 0} s(n,k)x^n y^k = \frac{(1-x)x^4y^3}{(1-x-xy)^2(1-x^3)},$$

$$\frac{\partial}{\partial q}G(x,1,q)\mid_{q=1} = \sum_{n\geq 0} s(n)x^n = \frac{x^4}{(1-2x)^2(1+x+x^2)}.$$

In Theorem 2.3 we give a combinatorial expression for the sequence $s(n,k)$.

Theorem 2.3. *For $n\geq 0$ and $k\geq 3$ we have*

$$s(n,k) = (k-2)\sum_{m=2}^{n-k+2}\left(\sum_{b=1}^{\min\{m-1,t\}}\binom{n-m-2b-1}{k-4}\right), \tag{4}$$

where $t = \lfloor\frac{n-m-k+3}{2}\rfloor$.

Proof. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ be a composition of n with at least three parts, that is, $k\geq 3$. If $p = \sigma_{j-1}\sigma_j\sigma_{j+1} = bmb$ is a symmetric peak in σ, then $2\leq j\leq k-1$, $2\leq m\leq n-(k-2)$, and $1\leq b\leq \min\{m-1,t\}$, where $t = \left\lfloor\frac{n-m-(k-3)}{2}\right\rfloor$. Notice that the value t is reached when $\sigma_i = 1$ for all $i\in[n]-\{j-1,j,j+1\}$. The number of compositions of the form $\sigma' = \sigma_1\cdots\sigma_{i-2}\sigma_{i+2}\cdots\sigma_k$ is given by $\binom{n-m-2b-1}{k-4}$. Therefore, the total number of composition of n with a symmetric peak in the j-th position is

$$\sum_{m=2}^{n-k+2}\left(\sum_{b=1}^{\min\{m-1,t\}}\binom{n-m-2b-1}{k-4}\right).$$

Since j takes values between 2 to $k-1$, then there are $k-2$ options. Multiplying by $k-2$ the last combinatorial expression we obtain the desired result. □

From the above theorem we obtain the first few values of the sequence $s(n,k)$ for $4 \leq n \leq 10$ and $3 \leq k \leq 9$:

$$[s(n,k)]_{n\geq 4, k\geq 3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 & 0 & 0 & 0 \\ 2 & \mathbf{6} & 9 & 4 & 0 & 0 & 0 \\ 2 & 10 & 18 & 16 & 5 & 0 & 0 \\ 2 & 14 & 33 & 40 & 25 & 6 & 0 \\ 3 & 18 & 54 & 84 & 75 & 36 & 7 \end{pmatrix}$$

For example, the symmetric peaks (denoted by red) in the compositions of $n = 7$ with 4 parts are

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1 1 1 4, 1 1 2 3, 1 1 3 2, 1 1 4 1, 1 2 1 3, 1 2 2 2, 1 2 3 1,
1 3 1 2, 1 3 2 1, 1 4 1 1, 2 1 1 3, 2 1 2 2, 2 1 3 1, 2 2 1 2,
2 2 2 1, 2 3 1 1, 3 1 1 2, 3 1 2 1, 3 2 1 1, 4 1 1 1.
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The first few values of the sequence $s(n)$ for $n \geq 4$ are

$$1, 3, 8, 21, 51, 120, 277, 627, 1400, 3093, 6771, \ldots$$

From the rational generating function for the sequence $s(n)$ we can state the following result.

Corollary 2.4. *The total number of symmetric peaks over all compositions of n is given by*

$$s(n) = \frac{7n-17}{49} 2^{n-2} - \frac{12+i\sqrt{3}}{147} \left(\frac{-1+i\sqrt{3}}{2} \right)^n - \frac{12-i\sqrt{3}}{147} \left(\frac{-1-i\sqrt{3}}{2} \right)^n$$

where $i^2 = -1$.

Let $p(n)$ denote the total number of peaks within all of the members in $\mathcal{C}(n)$. From the main result given by Kitaev et al. [10] it is possible to give the following generating function for the sequence $p(n)$

$$\sum_{n\geq 0} p(n)x^n = \frac{x^4(1+x^2)}{(1-x)(1+x)(1-2x)^2(1+x+x^2)}.$$

Theorem 2.5. *Among all peaks of all compositions, the proportion of those that are symmetric is asymptotically*

$$\lim_{n\to\infty} \frac{s(n)}{p(n)} = \frac{3}{5}.$$

Proof. The generating functions of the sequences $s(n)$ and $p(n)$ are rational, therefore we can use the asymptotic analysis for linear recurrences (cf. [11]). First, note that the unique pole $1/\beta$ of the rational generating function

$$\sum_{n\geq 0} s(n)x^n = \frac{x^4}{(1-2x)^2(1+x+x^2)}$$

is $1/2$, with multiplicity 2. Therefore $s(n) \sim \frac{1}{28}2^n n$. Similarly, we have $p(n) \sim \frac{5}{84}2^n n$. From these expressions we obtain the desired result. □

From Theorem 2.2 we can obtain similar results for the symmetric valleys. For example,

$$\frac{\partial}{\partial q} V(x,1,q) \mid_{q=1} = \frac{(1-x)x^5}{(1-2x)^2(1+x)(1-x^3)}$$

is the generating function for the total number of symmetric valleys in $\mathcal{C}(n)$. Moreover, the total number of symmetric valleys over all compositions of n is given by

$$\frac{21n-65}{441}2^{n-2} + \frac{1}{9}(-1)^{n+1} - \frac{15-11i\sqrt{3}}{294}\left(\frac{-1+i\sqrt{3}}{2}\right)^n - \frac{15+11i\sqrt{3}}{294}\left(\frac{-1-i\sqrt{3}}{2}\right)^n,$$

where $i^2 = -1$.

3. The symmetric and asymmetric distributions

By refining the arguments given in the above section we can obtain an explicit formula for the generating function $G(x,y,p,q)$ for the number of compositions of n according to the number of parts, number of symmetric peaks, and number of asymmetric peaks, that is, the coefficient of $x^n y^k p^\ell q^s$ is the number of compositions of n with k parts, ℓ symmetric peaks, and s asymmetric peaks. Again, we define $G(x,y,p,q|a_1a_2\cdots a_s)$ for the number of compositions $\pi_1\pi_2\cdots\pi_m$ of n according to the number of parts, number of symmetric peaks (counted by q), and number of asymmetric peaks (counted by p) such that $\pi_i = a_i$ for all $i = 1,2,\dots,s$. By the definitions, we have

$$G(x,y,q,p) = 1 + \sum_{a\geq 1} G(x,y,q,p|a). \tag{5}$$

Moreover,

$$\begin{aligned} G(x,y,q,p|a) &= x^a y + \sum_{j=1}^{a} G(x,y,q,p|aj) + \sum_{j\geq a+1} G(x,y,q,p|aj) \\ &= x^a y + x^a y \sum_{j=1}^{a} G(x,y,q,p|j) + \sum_{j\geq a+1} G(x,y,q,p|aj). \end{aligned} \tag{6}$$

For all $b \geq a+1$,

$$G(x,y,q,p|ab) = x^{a+b}y^2 + \sum_{j=1,j\neq a}^{b-1} G(x,y,q,p|abj) + G(x,y,q,p|aba)$$
$$+ G(x,y,q,p|abb) + \sum_{j\geq b+1} G(x,y,q,p|abj)$$
$$= x^{a+b}y^2 + px^{a+b}y^2 \sum_{j=1,j\neq a}^{b-1} G(x,y,q,p|j) + qx^{a+b}y^2 G(x,y,q,p|a)$$
$$+ x^{a+b}y^2 G(x,y,q,p|b) + x^a y \sum_{j\geq b+1} G(x,y,q,p|bj)$$
$$= x^{a+b}y^2 + px^{a+b}y^2 \sum_{j=1}^{b-1} G(x,y,q,p|j) + (q-p)x^{a+b}y^2 G(x,y,q,p|a)$$
$$+ x^{a+b}y^2 G(x,y,q,p|b) + x^a y \sum_{j\geq b+1} G(x,y,q,p|bj).$$

Thus, by (6), we have

$$G(x,y,q,p|ab) = x^{a+b}y^2 + px^{a+b}y^2 \sum_{j=1}^{b-1} G(x,y,q,p|j) + (q-p)x^{a+b}y^2 G(x,y,q,p|a)$$
$$+ x^{a+b}y^2 G(x,y,q,p|b) + x^a y\Big(G(x,y,q,p|b) - x^b y - x^b y \sum_{j=1}^{b} G(x,y,q,p|j)\Big)$$
$$= (p-1)x^{a+b}y^2 \sum_{j=1}^{b-1} G(x,y,q,p|j) + (q-p)x^{a+b}y^2 G(x,y,q,p|a)$$
$$+ x^a y G(x,y,q,p|b).$$

Summing over all $b \geq a+1$ we have

$$\sum_{j\geq a+1} G(x,y,q,p|aj) = \sum_{j\geq a+1} (p-1)x^{a+j}y^2 \sum_{k=1}^{j-1} G(x,y,q,p|k) + \frac{(q-p)x^{2a+1}y^2}{1-x} G(x,y,q,p|a)$$
$$+ x^a y \sum_{j\geq a+1} G(x,y,q,p|j).$$

Hence, (6) gives

$$G(x,y,q,p|a) = x^a y + x^a y \sum_{j\geq 1} G(x,y,q,p|j)$$
$$+ \sum_{j\geq a+1} (p-1)x^{a+j}y^2 \sum_{k=1}^{j-1} G(x,y,q,p|k) + \frac{(q-p)x^{2a+1}y^2}{1-x} G(x,y,q,p|a),$$

which, by (5), implies

$$G(x,y,q,p|a) = x^a y G(x,y,q,p) + \frac{(q-p)x^{2a+1}y^2}{1-x} G(x,y,q,p|a) \\ + (p-1)y^2 \sum_{j\geq a+1} x^{a+j} \sum_{k=1}^{j-1} G(x,y,q,p|k).$$

Define $G(x,y,q,p;t) = 1 + \sum_{a\geq 1} G(x,y,q,p|a)t^a$. Then by multiplying by t^a and summing over $a \geq 1$, we obtain

$$\begin{aligned} G(x,y,q,p;t) &= 1 + \frac{xyt}{1-xt} G(x,y,q,p) + \frac{(q-p)xy^2}{1-x}(G(x,y,q,p;x^2t)-1) \\ &+ (p-1)y^2 \sum_{a\geq 1}\sum_{j\geq a+1}\sum_{k=1}^{j-1} G(x,y,q,p|k)x^{a+j}t^a \\ &= 1 + \frac{xyt}{1-xt} G(x,y,q,p) + \frac{(q-p)xy^2}{1-x}(G(x,y,q,p;x^2t)-1) \\ &+ (p-1)y^2 \sum_{k\geq 1}\sum_{j\geq k+1} G(x,y,q,p|k) \frac{x^{j+1}t - x^{2j}t^j}{1-xt} \\ &= 1 + \frac{xyt}{1-xt} G(x,y,q,p) + \frac{(q-p)xy^2}{1-x}(G(x,y,q,p;x^2t)-1) \\ &+ \frac{(p-1)x^2y^2t}{(1-x)(1-xt)}(G(x,y,q,p,x)-1) \\ &- \frac{(p-1)x^2y^2t}{(1-xt)(1-x^2t)}(G(x,y,q,p,x^2t)-1), \end{aligned}$$

which implies

$$G(x,y,q,p;t) - 1 = \alpha(t) + \beta(t)(G(x,y,q,p,x^2t)-1),$$

where

$$\begin{aligned} \alpha(t) &= \frac{xyt}{1-xt} G(x,y,q,p) + \frac{(p-1)x^2y^2t}{(1-x)(1-xt)}(G(x,y,q,p,x)-1), \\ \beta(t) &= \frac{(q-p)xy^2}{1-x} - \frac{(p-1)x^2y^2t}{(1-xt)(1-x^2t)}. \end{aligned}$$

By iterating this equation infinitely many times (here we assume that $|x| < 1$), we obtain

$$\begin{aligned} G(x,y,q,p;t) &= 1 + \sum_{j\geq 0} \alpha(x^{2j}t) \prod_{i=0}^{j-1} \beta(x^{2i}t) \\ &= 1 + G(x,y,q,p) \sum_{j\geq 0} \frac{x^{2j+1}yt}{1-x^{2j+1}t} \prod_{i=0}^{j-1} \beta(x^{2i}t) \\ &+ (G(x,y,q,p,x)-1) \sum_{j\geq 0} \frac{(p-1)x^{2j+2}y^2t}{(1-x)(1-x^{2j+1}t)} \prod_{i=0}^{j-1} \beta(x^{2i}t). \end{aligned}$$

By taking either $t = 1$ or $t = x$, and the solving for $G(x,y,p,q)$, we obtain the following result.

Theorem 3.1. *We have*

$$G(x,y,q,p) = \frac{1-B(x)}{1-A(1)-B(x)-A(x)B(1)+A(1)B(x)},$$

where

$$\begin{aligned} A(t) &= xyt \sum_{j\geq 0} \frac{x^{3j}y^{2j}}{1-x^{2j+1}t} \prod_{i=0}^{j-1} \left(\frac{q-p}{1-x} - \frac{(p-1)x^{2i+1}t}{(1-x^{2i+1}t)(1-x^{2i+2}t)} \right), \\ B(t) &= \frac{(p-1)x^2y^2t}{1-x} \sum_{j\geq 0} \frac{x^{3j}y^{2j}}{1-x^{2j+1}t} \prod_{i=0}^{j-1} \left(\frac{q-p}{1-x} - \frac{(p-1)x^{2i+1}t}{(1-x^{2i+1}t)(1-x^{2i+2}t)} \right). \end{aligned}$$

As a series expansion, the generating function $G(x,y,q,p)$ begins with

$$\begin{aligned} G(x,y,q,p) = 1 &+ xy + (y+y^2)x^2 + y(1+y)^2x^3 + (y^4+(2+q)y^3+3y^2+y)x^4 \\ &+ (y^5 + 2(q+1)y^4 + (5+q)y^3 + 4y^2 + y)x^5 \\ &+ (y^6 + (2+3q)y^5 + (6+4q)y^4 + \boldsymbol{(q+2p+7)y^3} + 5y^2 + y)x^6 + \cdots. \end{aligned}$$

The compositions corresponding to the bold coefficients in the above series are

$$\overbrace{4\ 1\ 1}^{x^6y^3},\quad \overbrace{1\ 4\ 1}^{x^6y^3q},\quad \overbrace{1\ 1\ 4}^{x^6y^3},\quad \overbrace{3\ 2\ 1}^{x^6y^3},\quad \overbrace{3\ 1\ 2}^{x^6y^3},$$
$$\overbrace{2\ 3\ 1}^{x^6y^3p},\quad \overbrace{2\ 1\ 3}^{x^6y^3},\quad \overbrace{1\ 3\ 2}^{x^6y^3p},\quad \overbrace{1\ 2\ 3}^{x^6y^3},\quad \overbrace{2\ 2\ 2}^{x^6y^3}.$$

We define similarly the generating functions $V(x,y,q,p)$, $V(x,y,q,p;t)$, and $V(x,y,q,p|a_1a_2\cdots a_s)$ for symmetric valleys (counted by q) and asymmetric valleys (counted by p). Then, by similar arguments as in the proof of Theorem 3.1, we obtain

$$V(x,y,q,p|a) = x^a y V(x,y,q,p) + \frac{(q-p)x^{a+1}(1-x^{a-1})y^2}{1-x} V(x,y,q,p|a) + (p-1)y^2 \sum_{b=1}^{a-1} x^{a+b} \sum_{j\geq b+1} V(x,y,q,p|j) \tag{7}$$

and then

$$\tilde{V}(x,y,q,p;t) = \frac{xyt}{1-xt} V(x,y,q,p) + \frac{(q-p)y^2}{1-x}(x\tilde{V}(x,y,q,p;xt) - \tilde{V}(x,y,q,p,x^2t)) + \frac{(p-1)xy^2t}{(1-xt)(1-x^2t)}(x^2t\tilde{V}(x,y,q,p) - \tilde{V}(x,y,q,p,x^2t)),$$

where $\tilde{V}(x,y,q,p;t) = V(x,y,q,p;t) - 1$.

3.1. **Total number of asymmetric peaks.** Let $a(n)$ denote the total number of asymmetric peaks in $\mathcal{C}(n)$, and let $a(n,k)$ denote the total number of asymmetric peaks in $\mathcal{C}(n)$ with k parts. The generating functions for the sequences $a(n,k)$ and $a(n)$ are given by

$$\frac{\partial}{\partial p} G(x,y,1,p) \mid_{p=1} = \sum_{n,k\geq 0} a(n,k) x^n y^k = \frac{2x^6y^3}{(1-x-xy)^2(1+x-x^3-x^4)},$$

$$\frac{\partial}{\partial p} G(x,1,1,p) \mid_{p=1} = \sum_{n\geq 0} a(n) x^n = \frac{2x^6}{(1-2x)^2(1+x-x^3-x^4)}. \tag{8}$$

The first few values of the sequence $a(n)$ for $n \geq 6$ are

2, 6, 18, 48, 120, 288, 674, 1542, 3474, 7728, 17016, . . .

In Theorem 3.2 we give a combinatorial expression for the sequence $a(n,k)$.

Theorem 3.2. *For $n \geq 0$ and $k \geq 3$ the sequence $a(n,k)$ is given by*

$$2(k-2) \sum_{m=2}^{n-k+2} \left(\sum_{b=1}^{\min\{m-1,n-m-k+2\}} \left(\sum_{c=1}^{\min\{b-1,n-m-b-k+3\}} \binom{n-m-b-c-1}{k-4} \right) \right).$$

Proof. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ be a composition of n with at least three parts, that is, $k \geq 3$. If $p = \sigma_{j-1}\sigma_j\sigma_{j+1} = bmc$ is an asymmetric peak in σ, then $2 \leq j \leq k-1$, $2 \leq m \leq n-(k-2)$, $c \neq b$, and $\max\{b,c\} < m$. Notice that $1 \leq b \leq \min\{m-1,t\}$, where $t = n-m-(k-2)$. The value t is reached when $\sigma_i = 1$ for all $i \in [n] - \{j-1,j\}$. If $c < b$, then $c < \min\{b-1, n-m-b-(k+3)\}$. The number of compositions of the form $\sigma' = \sigma_1\cdots\sigma_{i-2}\sigma_{i+2}\cdots\sigma_k$ is given by $\binom{n-m-b-c-1}{k-4}$. Therefore, the total number of composition of n with an asymmetric peak in the j-th position is given by

$$\sum_{m=2}^{n-k+2} \left(\sum_{b=1}^{\min\{m-1,n-m-k+2\}} \left(\sum_{c=1}^{\min\{b-1,n-m-b-k+3\}} \binom{n-m-b-c-1}{k-4} \right) \right).$$

Since j takes values between 2 to $k-1$, then there are $k-2$ options. The option $b < c$ is analogous. Therefore, multiplying by $2(k-2)$ the last combinatorial expression we obtain the desired result. □

From Theorem 3.2 we obtain the first few values of the sequence $a(n,k)$ for $6 \leq n \leq 12$ and $3 \leq k \leq 9$:

$$[a(n,k)]_{n\geq 6, k\geq 3} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 8 & 6 & 0 & 0 & 0 & 0 \\ 6 & 16 & 18 & 8 & 0 & 0 & 0 \\ 8 & 28 & 42 & 32 & 10 & 0 & 0 \\ 10 & 44 & 84 & 88 & 50 & 12 & 0 \\ 14 & 64 & 150 & 200 & 160 & 72 & 14 \end{pmatrix}$$

For example, the asymmetric peaks (denoted by red) in the compositions of $n = 7$ with 4 parts are

1 1 1 4, 1 1 2 3, 1 1 3 2, 1 1 4 1, 1 2 1 3, 1 2 2 2, 1 2 3 1,
1 3 1 2, 1 3 2 1, 1 4 1 1, 2 1 1 3, 2 1 2 2, 2 1 3 1, 2 2 1 2,
2 2 2 1, 2 3 1 1, 3 1 1 2, 3 1 2 1, 3 2 1 1, 4 1 1 1.

From the rational generating function, see (8), for the sequence $a(n)$ we can state the following result.

Corollary 3.3. *The total number of asymmetric peaks over all compositions of n is given by*

$$a(n) = \frac{3+(-1)^n}{9} + \frac{21n-107}{441}2^{n-1} + \frac{13-3\sqrt{3}i}{147}\left(\frac{1-\sqrt{3}i}{-2}\right)^n + \frac{13+3\sqrt{3}i}{147}\left(\frac{1+\sqrt{3}i}{-2}\right)^n,$$

where $i^2 = -1$.

Corollary 3.4. *Among all peaks of all compositions, the proportion of those that are asymmetric is asymptotically*

$$\lim_{n\to\infty} \frac{a(n)}{p(n)} = \frac{2}{5}.$$

4. Symmetric Peaks over Restricted Compositions.

Let $\mathcal{C}_A(n)$ denote the set of compositions of n with parts in the set $A \subseteq \mathbb{N}$. For example, it is well-known that $|\mathcal{C}_{\{1,2\}}(n)| = F_{n+1}$, where F_n is the n-th Fibonacci number. In [9], Janjić gave several combinatorial expressions for the sequence $|\mathcal{C}_A(n)|$. From the same argument as in the previous section, it is possible to obtain expressions for the generating function $G_A(x, y, q)$ for the number of compositions with parts in $A \subseteq \mathbb{N}$ according to the number of parts and symmetric peaks.

For example, for $\ell \geq 2$ we have

$$G_{\{1,2,\dots,\ell\}}(x,y,q) := G_\ell(x,y,q) = \frac{1}{1-\sum_{a=1}^{\ell} \frac{x^a y}{1-\frac{(q-1)y^2(x^{2a+1}-x^{a+\ell+1})}{1-x}}}.$$

In particular, for $\ell = 2$ we have

$$\begin{aligned} G_2(x,y,q) &= \frac{1+x^3y^2-qx^3y^2}{1-xy-x^2y+x^3y^2-qx^3y^2-x^5y^3+qx^5y^3} \\ &= 1+yx+(y+y^2)x^2+(2y^2+y^3)x^3+(y^2+(2+q)y^3+y^4)x^4 \\ &+ \mathbf{(3y^3+(2+2q)y^4+y^5)}x^5+\cdots \end{aligned}$$

The compositions corresponding to the bold coefficients in the above series are

$$\overbrace{2\ 1\ 1\ 1}^{x^5y^4},\quad \overbrace{1\ 2\ 1\ 1}^{x^5y^4q},\quad \overbrace{1\ 1\ 2\ 1}^{x^5y^4q},\quad \overbrace{1\ 1\ 1\ 2}^{x^5y^4}.$$

Let $s_A(n)$ denote the total number of symmetric peaks in $\mathcal{C}_A(n)$, and let $s_A(n,k)$ denote the total number of symmetric peaks in $\mathcal{C}_A(n)$ with k parts. For example, for $\ell = 2$

$$\frac{\partial}{\partial q} G_2(x,y,q)\mid_{q=1} = \sum_{n,k\geq 0} s_{\{1,2\}}(n,k)x^n y^k = \frac{x^4y^3}{(1-(x+x^2)y)^2},$$

$$\frac{\partial}{\partial q} G_2(x,1,q)\mid_{q=1} = \sum_{n\geq 0} s_{\{1,2\}}(n)x^n = \frac{x^4}{(1-x-x^2)^2}.$$

In Theorem 4.1 we generalize the combinatorial expression given in Theorem 2.3.

Theorem 4.1. *We have*

$$s_A(n,k) = (k-2)\sum_{m\in M\cap A}\left(\sum_{b\in B_s\cap A} c_A(n-m-2b,k-3)\right),$$

where $M = \{2,3,\dots,n-k+1\}$, $B_s = \{1,2,\dots,\min\{m-1,\lfloor\frac{n-m-k+3}{2}\rfloor\}\}$, *and* $c_A(n,k)$ *denote the number of compositions of* n *with* k *parts in* A.

For example, for $A = \{1,2,3\}$ we obtain the array

$$[s_{\{1,2,3\}}(n,k)]_{n\geq 4,k\geq 3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 3 & 0 & 0 & 0 & 0 \\ 1 & \mathbf{4} & 9 & 4 & 0 & 0 & 0 \\ 0 & 4 & 15 & 16 & 5 & 0 & 0 \\ 0 & 2 & 18 & 36 & 25 & 6 & 0 \\ 0 & 2 & 15 & 56 & 70 & 36 & 7 \end{pmatrix}$$

In particular, the symmetric peaks (denoted by red) in the compositions of $n = 7$ with 4 parts in $A = \{1,2,3\}$ are 3121, 2131, 1312, 1213.

The first few values of the sequence $s_{\{1,2,3\}}(n)$ for $n \geq 4$ are

$$1, 3, 7, 18, 40, 87, 186, 388, 799, 1627, 3281, \ldots$$

Moreover,

$$\frac{\partial}{\partial q} G_3(x, y, q) \mid_{q=1} = \sum_{n,k \geq 0} s_{\{1,2,3\}}(n,k) x^n y^k = \frac{x^4(1 + x + x^3)y^3}{(1 - (x + x^2 + x^3)y)^2},$$

$$\frac{\partial}{\partial q} G_3(x, 1, q) \mid_{q=1} = \sum_{n \geq 0} s_{\{1,2,3\}}(n) x^n = \frac{x^4(1 + x + x^3)}{(1 - x - x^2 - x^3)^2}.$$

Similarly, we obtain the following result.

Theorem 4.2. *We have*

$$a_A(n,k) = 2(k-2) \sum_{m \in M \cap A} \left(\sum_{b \in B_a \cap A} \left(\sum_{c \in C_a \cap A} c_A(n - m - b - c, k - 3) \right) \right),$$

where

$$B_a = \{1, 2, \ldots, \min\{m - 1, n - m - k + 2\}\} \quad \textit{and}$$
$$C_a = \{1, 2, \ldots, \min\{b - 1, n - m - b - k + 3\}\}.$$

References

[1] W. Asakly, *Enumerating symmetric and non-symmetric peaks in words*, Online J. Anal. Comb. 13 (2018) 7 pp.

[2] A. Blecher, C. Brennan, and T. Mansour, *Compositions of n as alternating sequences of weakly increasing and strictly decreasing partitions*, Cent. Eur. J. Math. 10(2) (2012), 788–796.

[3] S. Elizalde, *Symmetric peaks and symmetric valleys in Dyck paths*, Discrete Math. 344 (2021), 112364.

[4] S. Elizalde, R. Flórez, and J. L. Ramírez, *Enumerating symmetric peaks in non-decreasing Dyck paths*, Ars Math. Contemp. 21 (2021), Article #P2.04.

[5] R. Flórez, L. Junes, and J. L. Ramírez, *Counting asymmetric weighted pyramids in non-decreasing Dyck paths*, Australas. J. Combin. 79.1 (2021), 123–140.

[6] R. Flórez and J. L. Ramírez, *Enumerating symmetric and asymmetric peaks in Dyck paths*, Discrete Math. 343 (2020) 112118.

[7] R. Flórez and J. L. Ramírez, *Enumerating symmetric pyramids in Motzkin paths*, Ars Math. Contemp. 23 (2023), Article #P4.06.

[8] S. Heubach and T. Mansour, *Combinatorics of Compositions and Words*, CRC Press, 2010.

[9] M. Janjić, *Binomial coefficients and enumeration of restricted words*, J. Integer Seq. (2016), Article 16.7.3.

[10] S. Kitaev, T. B. McAllister, and T. K. Petersen, *Enumerating segmented patterns in compositions and encoding by restricted permutations*, Integers 6 (2006), Article A34.

[11] R. Sedgewick and P. Flajolet, *An Introduction to the Analysis of Algorithms* 2nd ed., Addison-Wesley, 2013.

[12] M. Shattuck, *Recounting the number of peaks and valleys in compositions of integers*, Ars Combin. 132 (2017), 323–329.

[13] Y. Sun, W. Shi, and D. Zhao, *Symmetric and asymmetric peaks or valleys in (partial) Dyck paths*, Enumer. Combin. Appl. 2 (2022), article #S2R24.

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