SHADE IN PARTITIONS

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Abstract. Integer partitions of n are viewed as bargraphs (i.e., Ferrers diagrams rotated anticlockwise by 90 degrees) in which the ith part of the partition x_i is given by the ith column of the bargraph with x_i cells. The sun is at infinity in the north west of our two dimensional model and each partition casts a shadow in accordance with the rules of physics. The number of unit squares in this shadow but not being part of the partition is found through a bivariate generating function in q tracking partition size and u tracking shadow. To do this we define triangular q-binomial coefficients which are analogous to standard q-binomial coefficients and we obtain a formula for these. This is used to obtain a generating function for the total number of shaded cells in (weakly decreasing) partitions of n.

1. Introduction

Integer compositions of n with k parts have recently been modelled by a bargraph with k columns in which the ith part, say x_i, is represented by column i of the bargraph of height x_i. We adopt this convention here also for integer partitions, which implies that such weakly decreasing partitions are represented as Ferrers diagrams with k parts rotated anticlockwise by 90 degrees. See Figure 1 for the geometry of our representations. [1] and [2] are good sources for general information about partitions.

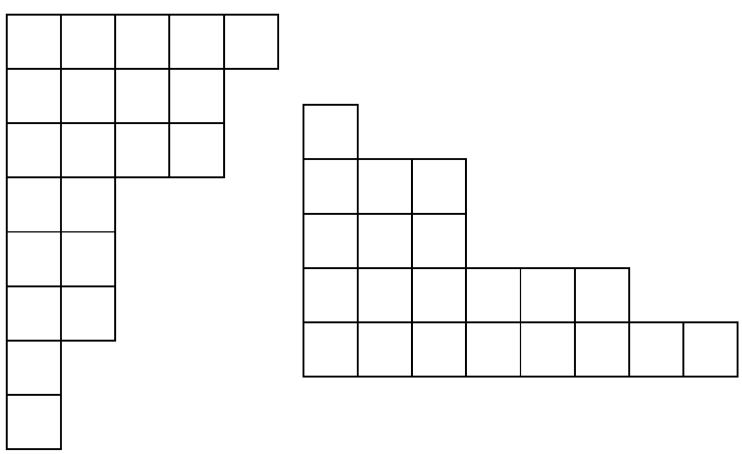

Figure 1. 5+4+4+2+2+2+1+1 as a Ferrers diagram and as a decreasing bargraph

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In this model, each part of the (weakly decreasing) partition of size r is represented by r vertically stacked square cells. We position the sun at infinity in the north west and consider the problem of how many of the cells above or to the right of the bargraph representation of a weakly decreasing partition are in shade. Any shade in a partition can only occur when there is a descent of two or more (and possibly to the right of this part also). For example, figure 2 shows that the partition 7+7+5+5+5+2+2+2+2+2 has 5 cells in the shade, as shown below. The partition in figure 1 would only have one shaded cell, at position (4,3).

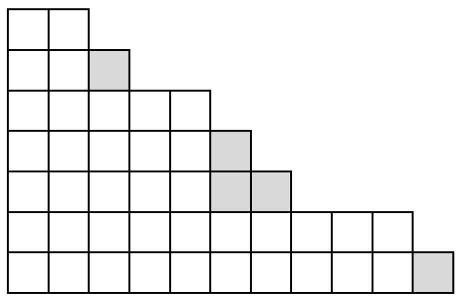

FIGURE 2. 7+7+5+5+5+2+2+2+2+2 has 5 cells in the shade from a NW sun

In Sections 3 and 4, we develop a bivariate generating function tracking shade with parameter u and partition size with parameter q. Note that u tracks only empty cells which are in the shade and are therefore not part of the underlying partition which causes the shade. In [4], the water capacity of integer compositions also counts cells which are not in the underlying combinatorial object and it employed a global as opposed to a local method for its solution. A global approach is essayed again in Section 4. The idea of lit cells has been studied before (see [3]) in the paradigm of words over a fixed alphabet k and also in [6] where the paradigm was compositions of n. However, in this paper we are concerned with the complementary concept of shade and not light.

The general problem covering all partitions is dealt with in Section 4. The strategy we use is (as we already said) a global approach in which maximal sub-partitions of a given partition are characterised as cases 1-4. To implement this characterisation, we make use of the bijection specified below by Prodinger in [14]. This bijection sends each weakly decreasing partition of n with j parts to the super-diagonal (or skew, in Prodinger's terms) composition of $n + \binom{j}{2}$. It is defined by mapping each original ith part x_i to the new ith part $x_i + i - 1$ (and its reverse for the inverse). The image objects for this bijection are super-diagonal compositions (i.e., compositions in which the ith part $\geq i$) which are so called 1-compositions (i.e., compositions in which successive parts are not allowed to increase by more than 1). We call this bijection the *skew bijection* which we will refer to by name, later in the paper.

See an example in Figure 3.

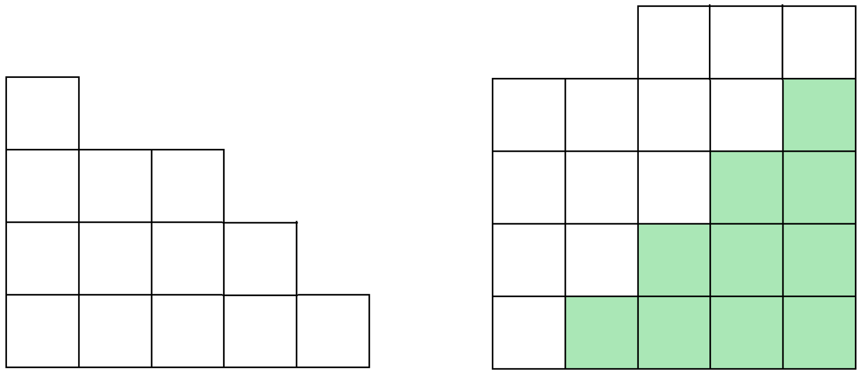

FIGURE 3. The skew bijection maps 4+3+3+2+1 to 4+4+5+5+5 with added staircase shown in green

Other papers dealing with skew or super-diagonal bargraphs are found in [8] and [17]. A more general question linked to the geometry of graphical representations is considered in [5], in which the number of points of degree 2 is enumerated.

In the next section, by analogy with q-binomial coefficients, we introduce triangular q-binomial coefficients. These are necessary for the solution of the problem posed in section 4, namely finding a bivariate generating function tracking the number of shade cells (with parameter u) in weakly decreasing partitions of n tracked by q.

Our results are in the tradition of analysis of partition statistics exemplified by [4], [10], [11], [13], [18], and [19], and the more basic topic of classifying partitions by the nature of their parts as in [9], [12], [13], and [15]. It shares an approach with [4] of counting cells exterior to those in the graphical representation, rather than counting a distinguished subset of cells within the graphical representation.

2. Triangular Binomial Coefficients

Consider weakly decreasing partitions with r parts.

In general, see [2], a standard q-binomial coefficient $\binom{N+m}{m}_q$ is the partition generating function

$$\binom{N+m}{m}_q = \sum_{n\geq 0} p(n : \leq m \text{ parts, each} \leq N)q^n$$

where $p(n : \leq m \text{ parts, each} \leq N)$ counts partitions of n into at most m parts, each $\leq N$. For example,

$$\binom{7}{3}_q = \binom{4+3}{3}_q = 1+q+2q^2+3q^3+4q^4+4q^5+5q^6+\mathbf{4q^7}+4q^8+3q^9+2q^{10}+q^{11}+q^{12}$$

where the term in bold represents the 4 partitions of 7, namely $\{43, 421, 331, 322\}$ with a maximum of 3 parts and a maximum part size of 4.

In [7], the current authors modified this generating function in order to account for partitions contained in a triangle, rather than in a rectangle. For the convenience of the reader we reiterate the process leading to this modification, albeit, in an abbreviated form. Firstly, we define triangular q-binomial coefficients $t[r,i]_q$ to be the generating function for all weakly decreasing partitions with exactly i parts that are contained in a rhombus or modified rectangle in which the upper boundary is a decreasing staircase with column heights $r, r-1, ..., r-i+1$. We use q to track the size of partitions that fit inside the modified rectangle.

We set up a recursion for $t[r,s]_q = t[r,s]$. We obtain the following

Proposition 1. *The generating function $t[r,s]$ satisfies the recursion*

$$t[r,s] = q^s \sum_{i=0}^{s} t[r-1,i]. \tag{1}$$

The initial conditions for the recursion are:

$$t[1,1] = q;\ t[2,1] = q(1+q);$$
$$t[r,0] = 1 \text{ if } r \geq 0;$$
$$t[r,s] = 0 \text{ if } s > r \geq 0. \tag{2}$$

For the proof of this proposition, see [7].

For example,

$$t[4,3] = q^3 + q^4 + 2q^5 + \mathbf{3q^6} + 3q^7 + 3q^8 + q^9$$

where the term in bold represents the 3 partitions $\{411, 321, 222\}$ of 6 with three parts that fit inside the modified triangle with columns of heights $4, 3, 2$, as shown below.

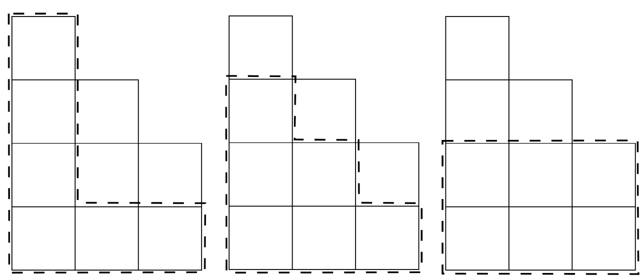

FIGURE 4. The three partitions of 6 with exactly three parts that fit in a modified triangle

In [7], the previous Proposition is presented in an alternative form as

Proposition 2. *The triangular q-binomial coefficients $t[r,s]$ are also given by the following formula:*

$$t[r,s] = q^s \binom{r+s-1}{s}_q - q^{s+1} \sum_{i=2}^{s} t[i-1,i-1] q^{i(r-i)} \binom{r-2i+1+s}{s-i}_q \tag{3}$$

where $t[r,s]$ satisfies the initial conditions given in (2).

The proof of this proposition is also to be found in [7].

And finally, we specify the generating function $T_q(r)$ for all weakly decreasing partitions with a maximum of r parts contained in the triangle $r, r-1, ..., 1$.

This is given by

Definition 3. *$T_q(r) := \sum_{s=0}^{r} t[r,s]$ which is the generating function for all decreasing partitions with a maximum of r parts contained in the triangle $r, r-1, ..., 1$.*

For example,

$$T_q(4) = 1 + q + 2q^2 + 3q^3 + 5q^4 + 5q^5 + \mathbf{7q^6} + 7q^7 + 6q^8 + 4q^9 + q^{10}$$

where the term in bold represents the 7 partitions of 6 which fit in the triangle 4321, namely

$$\{42, 411, 33, 321, 3111, 222, 2211\}.$$

3. Shade in Partitions with Distinct Parts

We first consider the easier problem of decreasing partitions with distinct parts. We note that a distinct part partition of n with largest part m must lie within the triangle with columns $m, m-1 \cdots 2, 1$. This triangle represents the maximal shade area that is produced by the largest part m of the partition. From this area of $m(m+1)/2$ we must subtract the area of the distinct part partition that lies within the triangle. The difference is the area of the shade produced by the specific distinct part partition of n lying within the triangle.

The generating function that counts the number of distinct part partitions of n (given by $q(n,m)$) where the largest part is m is given by

$$\sum_{n=m}^{\frac{1}{2}m(m+1)} q(n,m)q^n = q^m \prod_{i=1}^{m-1} \left(1 + q^i\right) \tag{4}$$

Now, we define a bivariate generating function $F_m(q,u)$ in which q tracks the size of the partition and u tracks the area of the shade.

We obtain

$$F_m(q,u) = \sum_{n=m}^{\frac{1}{2}m(m+1)} q(n,m)u^{\frac{1}{2}m(m+1)} \left(\frac{q}{u}\right)^n = u^{\frac{1}{2}m(m-1)} q^m \prod_{i=1}^{m-1} \left(1 + \left(\frac{q}{u}\right)^i\right) \tag{5}$$

Then

$$F(q,u) = \sum_{m=0}^{\infty} F_m(q,u) = \sum_{m=0}^{\infty} u^{\frac{1}{2}m(m-1)} q^m \prod_{i=1}^{m-1} \left(1 + \left(\frac{q}{u}\right)^i\right) \tag{6}$$

The series expansion of $F(q,u)$ begins

$$1+q+uq^2+\left(1+u^3\right)q^3+\left(u^2+u^6\right)q^4+\left(u+u^5+u^{10}\right)q^5+\left(1+u^4+u^9+u^{15}\right)q^6$$
$$+\left(\mathbf{2u^3}+u^8+u^{14}+u^{21}\right)q^7+\left(u^2+2u^7+u^{13}+u^{20}+u^{28}\right)q^8+$$
$$\left(u+2u^6+2u^{12}+u^{19}+u^{27}+u^{36}\right)q^9+\left(1+2u^5+2u^{11}+2u^{18}+u^{26}+u^{35}+u^{45}\right)q^{10}.$$

For example $2u^3q^7$, partially set in boldface above, corresponds to the distinct part partitions of $n=7$; 4 + 3 and 4 + 2 + 1 each with respective shade areas of 3, as illustrated below.

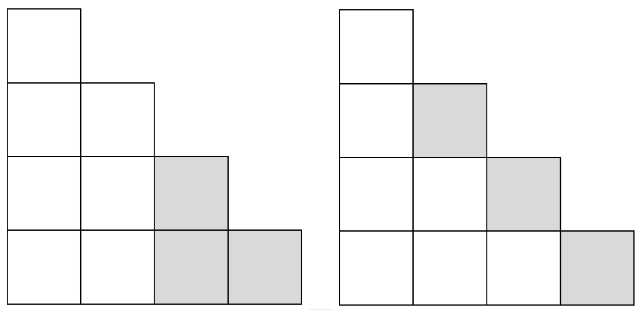

FIGURE 5. Two partitions of 7 in a triangular grid

The total shade generating function is

$$\frac{\partial}{\partial u}F(q,u)\Big|_{u=1}=\sum_{m=0}^{\infty}\left(\frac{1}{2}m(m-1)q^m\prod_{i=1}^{m-1}\left(1+q^i\right)+q^m\frac{\partial}{\partial u}\prod_{i=1}^{m-1}\left(1+\left(\frac{q}{u}\right)^i\right)\right). \tag{7}$$

4. SHADE IN UNRESTRICTED DECREASING PARTITIONS

Our goal here is to extend the result of the previous section to find the bivariate generating function $F(q,u)$, tracking partition size using q and tracking shade using u. We need to consider several cases, with their associated generating functions, and combine the cases in our main theorem. The cases involve the "shape" of the partition, in terms of a bounding region containing its parts.

Any shade in a partition can only occur when there is a descent of two or more (and possibly to the right of this part also). We can see this in Figure 2 and Figure 5, and also later in Figure 8 and Figure 9. This implies that the analysis which follows must carefully track partitions by paying attention to those parts which constitute a descent of two or more. So the bivariate generating function tracking shade with parameter u and partition size with parameter q will be split up into four disjoint cases where the various situations representing a drop of two or more are defined. Note that u tracks only empty cells which are in the shade and are therefore not part of the underlying partition which causes the shade. Some of these cases may be concatenated with each other and this too will be explained.

For an arbitrary partition, we let the largest part be M_1. We consider firstly all parts prior the first descent of 2 or more. We let p_0 be the last part in this initial block of size M_2. The generating function for this initial block is given by

$$\sum_{M_1=3}^{v} \sum_{M_2=3}^{M_1} \prod_{i=M_2}^{M_1} \frac{q^i}{1-q^i} \tag{8}$$

where v marks the largest possible size for the largest part.

4.1. **Cases 1 and 2.** These two cases have in common that after the initial block, all parts as well as any shade must be contained in the triangle of size $p_0 - 2, p_0 - 3, \ldots, 1$, i.e., under the *skew bijection* the image of these parts is less than the image of p_0. Also, any of these cases may follow an occurrence of case 4 (discussed later) and in order to keep the index names disjoint from each other, we replaced the M indices used in Equation (8) with N indices. We distinguish the following three cases.

Case 1a has at least one part after the initial block (it must be a descent of at least two) and using the *skew partition* bijection defined in the introduction, the images of all these parts are less than image p_0. This is why the parts themselves must be contained in the triangle specified in the previous paragraph.

This case has generating function

$$case1a(v) = \sum_{N_1=3}^{v} \sum_{N_2=3}^{N_1} \left(\prod_{i=N_2}^{N_1} \frac{q^i}{1-q^i} \right) \underbrace{u^{\binom{N_2}{2}}}_{A} \underbrace{\left(T_q(N_2-2)-1\right)|_{q \to \frac{q}{u}}}_{B}, \tag{9}$$

where underbrace A is the generating function for the shade that would be cast by the initial block assuming that there were no parts following the initial block. However there is at least one following part and these parts must be subtracted from the total shade given in underbrace A. This is precisely what the generating function in underbrace B achieves. The operation $|_{q \to \frac{q}{u}}$ used above means replace q by $\frac{q}{u}$ in the preceding bracket.

Case 1b is defined by the initial block being the whole partition but the last part is greater than 1.

$$case1b(v) = \sum_{N_1=2}^{v} \sum_{N_2=2}^{N_1} \left(\prod_{i=N_2}^{N_1} \frac{q^i}{1-q^i} \right) u^{\binom{N_2}{2}}. \tag{10}$$

Combining the above two cases, we obtain the case 1 generating function for all such partitions which have at least one drop of 2 or more, anywhere after the initial block or at the end. So we have proved our first lemma.

Lemma 4. *The generating function for all partitions which after the initial block ending with part p_0 are contained in the triangle $p_0 - 2, p_0 - 3, \ldots, 1$ and which have at least one drop of 2* *or more is given by*

$$case1(v) = \sum_{N_1=2}^{v} \sum_{N_2=2}^{N_1} \left(\prod_{i=N_2}^{N_1} \frac{q^i}{1-q^i} \right) u^{\binom{N_2}{2}} \left(T_q(N_2-2) \right) |_{q \to \frac{q}{u}}. \tag{11}$$

Case 2 is defined by the last part of the initial block being 1. The initial block is then the entire partition.

Lemma 5. *The generating function for all partitions in which the initial block ends with 1 is*

$$case2(v) = \sum_{N_1=1}^{v} \prod_{i=1}^{N_1} \frac{q^i}{1-q^i} \tag{12}$$

4.2. **Case 3.** Case 3 has at least one part after the initial block (it must be a descent of at least two) and we call the first such part p_1 and the part immediately to its left of size N_2 we call p_0; here we let the the first part of the initial block have size N_1 and the last part have size N_2; case 3 must also satisfy the following: converting all the parts to their respective images using the *skew bijection* defined in the introduction, there must be at least one part to the right of p_1, with size equal to image p_0, but no parts to the right of p_1 whose image size is larger than image p_0. Let the leftmost such part with image equal to image p_0 be s_3 places to the right of p_0 and let it be named p_2. So, in a similar manner to cases 1 and 2, the parts that follow the initial block must lie in the rhombus $p_0 - 2, p_0 - 2, p_0 - 3, \ldots, 1$.

We illustrate these parameters in figure 6 for the partition 5+5+3+2+2+1. The initial block has $N_1 = 5$ and ends at the second summand, so $N_2 = 5$ as well. The next part has the first descent of 2 or more, so p_0 is 5 and p_1 is 3. The column labelled p_2 is the 5th column here because under the skew bijection image p_2=image p_0=6 and everything between p_0 and p_2 has a lesser image. The last column also has the same image and not a larger one as required. s_3 spans the third to 5th column so $s_3 = 3$.

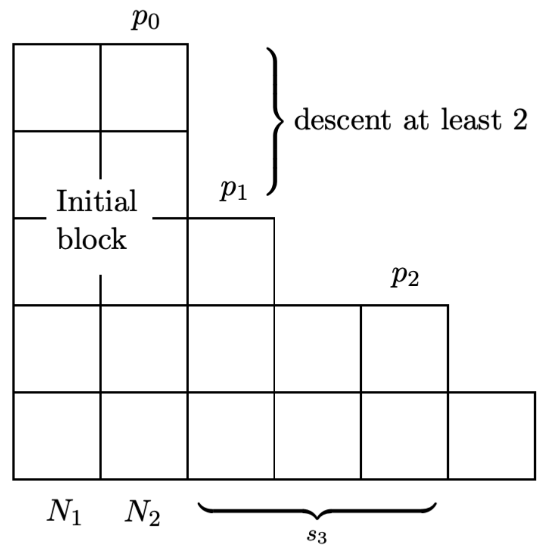

FIGURE 6. Case 3 parameters

The generating function for this case is given by the following lemma:

Lemma 6. *The generating function for case 3 as described above is*

$$case3(v) := \underbrace{\sum_{N_2=3}^{N_1}\sum_{s_3=2}^{N_2-1}\left(\prod_{i=N_2}^{N_1}\frac{q^i}{1-q^i}\right)}_{A}\underbrace{u^{\binom{N_2}{2}-\binom{N_2-s_3}{2}}}_{B}\times$$

$$\times\underbrace{\left(q^{(N_2-s_3)s_3}\left(T_q(s_3-2)\right)\right)|_{q\to\frac{q}{u}}}_{C}\underbrace{u^{\binom{N_2-s_3}{2}}\left(T_q(N_2-s_3-1)\right)|_{q\to\frac{q}{u}}}_{D}. \tag{13}$$

Proof. We explain each underbraced part in this generating function.

A is the generating function for the initial block and includes a sum over the index s_3 whose range is given in the sum limits.

B is the shade that would be cast by the initial block up to the part p_2 assuming that there were no parts following the initial block.

However there are following parts at least for these s_3 positions to the right of p_0 and these s_3 parts must be subtracted from the total shade given in underbrace B.

So for underbrace C, the part p_2 has size $N_2 - s_3$. This size follows from the fact that under the *skew bijection* part p_2 has the same image as part p_0. From this it follows that all parts in the partition lying between p_0 and p_2 are at least of size $N_2 - s_3$. This is captured in the generating function C by $q^{(N_2-s_3)s_3}$ (i.e., the generating function for the bottom rectangle). The parts in the partition between p_0 and p_2 but above the bottom rectangle themselves lie in a triangle of width $s_3 - 2$. The width of this triangle is determined by the fact that under the *skew bijection* all these parts have image less than image of p_0 and so have generating function $T_q(s_3 - 2)$. With these adjustments C is the generating function for all parts from p_1 to p_2. But as already said, these need to be removed from the total shade captured by the generating function B. This removal is achieved by the replacement of q in the generating function for C by $\frac{q}{u}$.

We can modify figure 6 to show the relevant parameters. The dotted box indicates the parts of the partition handled by the generating function of underbrace C, for all parts between p_0 and p_2, the second summand of 5 and the second summand of 2 (the size is $p_2 = 2 = 5 - 3 = N_2 - s_3$).

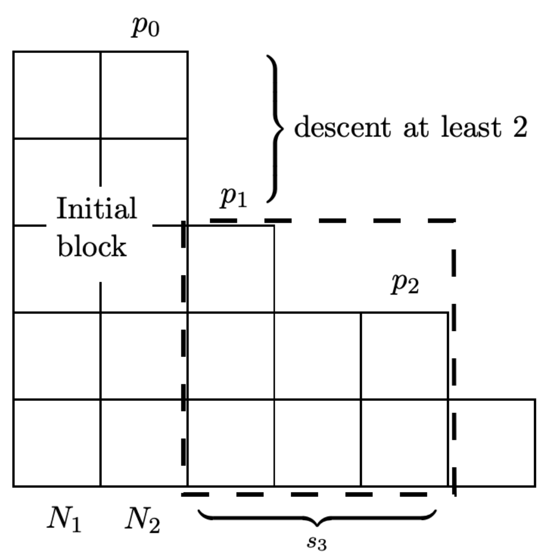

FIGURE 7. Underbrace C details

The generating function for D is obtained in a similar manner to those for B and C. The reader should bear in mind that all parts to the right of p_2 have image under the *skew bijection* of maximum size equal to image of p_0 (i.e., also equal to image of p_2) and therefore as parts in the domain of the *skew bijection* must lie in the triangle with maximum height $N_2 - s_3 - 1$, captured in the generating function by $T_q(N_2 - s_3 - 1)$. As before the replacement of q by $\frac{q}{u}$ is required. □

4.3. **Case 4.** Case 4 begins like Case 3. It has at least one part after the initial block (which must also be a descent of at least two) and once again, we call the first such part p_1 and as stated for an initial block, the part immediately to its left is of size M_2 and we call this part p_0; thus we let the the first part of the initial block have size M_1 and the last part p_0 have size M_2; unlike Case 3, Case 4 must also satisfy the following: converting all the parts to their respective images using the *skew bijection* defined in the introduction, there must be at least one part to the right of p_1, with image size greater than image p_0. Let the part immediately to the left of the first such part be p_2, occurring at say s_2 places to the right of p_0. By definition of the *skew bijection* in the introduction $p_2 = M_2 - s_2$.

In fact Case 4 splits into subcases.

For all these subcases we first need to define the generating function $p1(M_1, M_2, s_2)$ which tracks shade and parts in the partition up to p_2:

$$p1(M_1, M_2, s_2) := \left(\prod_{i=M_2}^{M_1} \frac{q^i}{1-q^i}\right) u^{\binom{M_2}{2}-\binom{M_2-s_2}{2}} \times$$
$$\times \underbrace{\left(q^{M_2-s_2}\left(q^{(M_2-s_2)s_2}\left(T_q(s_2-1) - q^{s_2-1}T_q(s_2-2)\right)\right)|_{q\to\frac{q}{u}}\right)}_{A}. \tag{14}$$

For underbrace A, the part p_2 has size $M_2 - s_2$. The generating function for part p_2 is therefore $q^{M_2-s_2}$. From this it follows that all parts in the partition lying between p_1 and p_2 (inclusive) are at least of size $M_2 - s_2$. This is captured in the generating function A by $q^{(M_2-s_2)s_2}$ (i.e., the generating function for the bottom rectangle).

The parts in the partition between p_1 and p_2 but above the bottom rectangle themselves lie in a triangle of width $s_2 - 1$ with generating function $T_q(s_2 - 1)$. But this allows the first of these parts to be $s_2 - 1$, which is not possible because for the first part p_1, there has to be a drop of at least two. Hence all such partitions must be removed from the generating function resulting in the subtraction of $q^{s_2-1}T_q(s_2 - 2)$. With these adjustments A is the generating function for all parts from p_1 to p_2. These need to be removed from the total shade captured in the preceding u - term. This removal is achieved by the replacement of q in the generating function for A by $\frac{q}{u}$.

From the definition of Cases 1, 2 and 3, we see firstly that the partitions described by any of these are disjoint from any other and the union of these cases is the set of all partitions with the following property: after the initial block described by Equation (8), the rest of the partition is entirely contained in the rhombus $p_0 - 2, p_0 - 2, p_0 - 3, \ldots, 1$. If $p_0 - 2 \leq 0$, we interpret the rhombus as being empty. This means that these cases are terminal because if the partition has any parts outside this rhombus, by definition it is describing a Case 4 and not any of these cases. Also, for the same reason any of these cases following a Case 4 is also terminal in the sense that these cases are then describing the end of the partition. So, to summarise, any partition must be ended by an occurrence of one of Cases 1 to 4. What remains is to describe the multiple block recurrence of Case 4, as well as the generating function for these terminated by any of Cases 1-4. We let $case4_k$ be the generating function for Case 4 repeated k times and then terminated by any of Cases 1-4.

To this end, we define

$$lastblock(v) := case1(v) + case2(v) + case3(v). \tag{15}$$

We will also need

$$\Delta(v) := lastblock(v) - lastblock(v-1). \tag{16}$$

and from this definition we see that the $\Delta(v)$ generating function calls the terminating cases where the largest part size of the partitions tracked is precisely v.

What follows respectively in Equations (17), (18) and (19) below is the generating function where $case4_k$ for $k \in \{2,3,4\}$ is ended by one of the terminal cases.

$$case4_1 := \underbrace{\sum_{M_1=3}^{\infty} \sum_{M_2=3}^{M_1} \sum_{s_2=2}^{M_2-1} \frac{p1(M_1, M_2, s_2)}{q^{M_2-s_2}}}_{A} \underbrace{\Delta(M_2 - s_2)}_{B}. \tag{17}$$

The last part of the partition tracked by underbrace A is p_2 which, as already stated, has size $M_2 - s_2$. The generating function for this part is removed from the generating function in underbrace A, and becomes the starting value for one of the terminating cases tracked in underbrace B.

We repeat this process to obtain the generating function for $case4_2$:

$$\begin{aligned} case4_2 :=& \underbrace{\sum_{M_1=3}^{\infty} \sum_{M_2=3}^{M_1} \sum_{s_2=2}^{M_2-1} \frac{p1(M_1, M_2, s_2)}{q^{M_2-s_2}}}_{A} \times \\ &\times \underbrace{\sum_{M_3=3}^{M_2-s_2} \sum_{s_3=2}^{M_3-1} \frac{p1(M_2-s_2, M_3, s_3)}{q^{M_3-s_3}} \Delta(M_3 - s_3)}_{B}. \end{aligned} \tag{18}$$

The explanation for this case is just an extension of the one for the previous case, except that the indices and parameters are renamed appropriately so as not to confuse the second occurrence of case 4 with the first.

The same argument is applied to three occurrences of case 4 as below.

$$\begin{aligned} case4_3 :=& \sum_{M_1=3}^{\infty} \sum_{M_2=3}^{M_1} \sum_{s_2=2}^{M_2-1} \frac{p1(M_1, M_2, s_2)}{q^{M_2-s_2}} \sum_{M_3=3}^{M_2-s_2} \sum_{s_3=2}^{M_3-1} \frac{p1(M_2-s_2, M_3, s_3)}{q^{M_3-s_3}} \times \\ &\times \sum_{M_4=3}^{M_3-s_3} \sum_{s_4=2}^{M_4-1} \frac{p1(M_3-s_3, M_4, s_4)}{q^{M_4-s_4}} \Delta(M_4 - s_4). \end{aligned} \tag{19}$$

For the equations above, because the indices and limits of the sums can be read off from the variables of the summand we abbreviate $\sum_{M_2=3}^{M_1} \sum_{s_2=2}^{M_2-1} \frac{f(M_1,M_2,s_2)}{q^{M_2-s_2}} = p_1^*(M_1, M_2, s_2)$.

Equation (19) is then abbreviated as

$$case4_3 := \sum_{M_1=3}^{\infty} p1^*(M_1, M_2, s_2)p1^*(M_2 - s_2, M_3, s_3)\times$$

$$\times \sum_{M_4=3}^{M_3-s_3} \sum_{s_4=2}^{M_4-1} p1(M_3 - s_3, M_4, s_4)\Delta(M_4 - s_4), \tag{20}$$

and if we generalise this process, we have proved that the generating function for $case4_k$ is given by by the following lemma:

Lemma 7. *Case 4 is characterised by the property that after the initial block ending with part p_0, there is a first part lying outside the rhombus $p_0 - 2, p_0 - 2, p_0 - 3, \ldots, 1$ which is where case 4 ends. This may be followed recursively, by more occurrences of case 4. The generating function for exactly k such recurrences is given by*

$$case4_k = \sum_{M_1=3}^{\infty} p1^*(M_1, M_2, s_2)p1^*(M_2 - s_2, M_3, s_3) \ldots p1^*(M_{k-1} - s_{k-1}, M_k, s_k))\times$$

$$\times \sum_{M_{k+1}=3}^{M_k-s_k} \sum_{s_{k+1}=2}^{M_{k+1}-1} p1(M_k - s_k, M_{k+1}, s_{k+1})\Delta(M_{k+1} - s_{k+1}). \tag{21}$$

4.4. **Overall Generating Function.** To obtain the bivariate generating function $F(q, u)$, tracking partition size using q and shade using u, we note that any of Cases 1-3 may occur on their own or any of these cases may follow $case4_k$.

On their own the generating function is found using Equation (15), i.e., $lastblock[\infty]$. The alternative is given by Equation (21), i.e., $\sum_{k\geq 1} case4_k$ and so we have proved our main result.

Theorem 8. *The bivariate generating function $F(q, u)$ for all partitions whose size is tracked by q, where the shade created by each partition is tracked by the variable u is*

$$F(q, u) = lastblock[\infty] + \sum_{k\geq 1} case4_k.$$

We illustrate this series up to q^8 below:

$$q + (1 + u)q^2 + \left(2 + u^3\right) q^3 + \left(2 + u + u^2 + u^6\right) q^4 + \left(3 + 2u + u^5 + u^{10}\right) q^5$$
$$+ \left(4 + 2u + u^3 + 2u^4 + u^9 + u^{15}\right) q^6 + \left(5 + 2u + u^2 + 3u^3 + 2u^8 + u^{14} + u^{21}\right) q^7$$
$$+ \left(6 + 4u + 3u^2 + u^3 + u^6 + \mathbf{3u^7} + 2u^{13} + u^{20} + u^{28}\right) q^8. \tag{22}$$

Next, we illustrate the bold term above. This coefficient comes from the following three partitions of 8: 5+3, 5+2+1 and 5+1+1+1, as illustrated below.

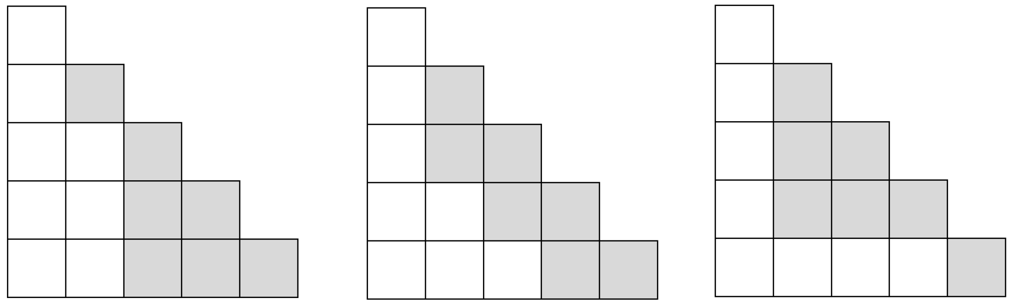

FIGURE 8. 5+3, 5+2+1 and 5+1+1+1 all generate 7 shaded cells

Setting $u = 1$, the series for $F(q,1)$ yields the partition numbers beginning with
$q+2q^2+3q^3+5q^4+7q^5+11q^6+15q^7+22q^8+30q^9+42q^{10}+56q^{11}+77q^{12}+101q^{13}+135q^{14}$.

The total shade according to size of partition is found from $\frac{\partial F(q,u)}{\partial u}|_{u=1}$ and this series begins
$q^2+3q^3+\mathbf{9q^4}+17q^5+37q^6+64q^7+114q^8+186q^9+308q^{10}+470q^{11}+734q^{12}+1087q^{13}+1618q^{14}$.

The bold term accounts for the nine shaded cells that arise among all five partitions of 4, as illustrated in Figure 9.

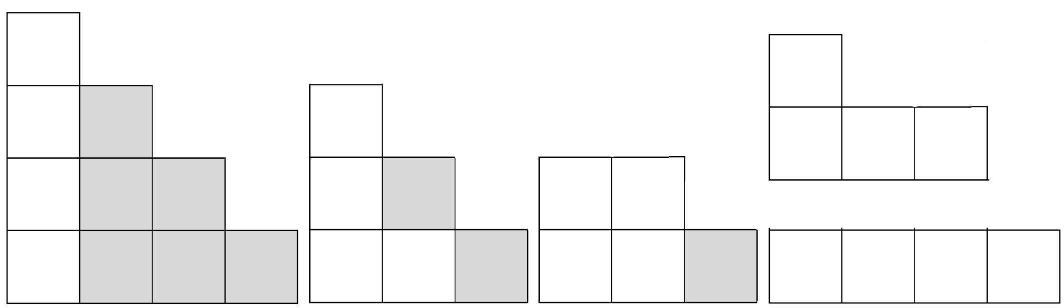

FIGURE 9. Nine shaded cells among all partitions of 4

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