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Integer distances in vector spaces over finite fields

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ABSTRACT

The Erdős-Anning Theorem states that an integer distance set in the Euclidean plane must have all of its points on a single line or is finite. However, this is not true if we consider area sets. That is, if (x_1, y_1) and (x_2, y_2) are any two vectors contained in the integer lattice, then the area of the parallelogram determined by the two vectors is an integer, showing that the points do not have to lie on a line. We prove a finite field version of these results for $d = 2$ and $d = 3$, showing that if $E \subset \mathbb{F}_q^d, q = p^2$, where p is an odd prime and the distance set of E is $\mathbb{F}_p$, then the size of E is at most p^d. Furthermore, we prove that if the area set of E is a subset of $\mathbb{F}_p$, then the size of E is at most p^2 in two dimensions.

Keywords: Erdős-Anning theorem, finite fields

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1. Introduction

An integer distance set is a set in which the distance between every pair of points is an integer. In this paper, we establish a finite field version of a well-known result by Paul Erdős and Norman Anning which states that an infinite integer distance set must be contained in a line [1]. In [3], Erdős found an upper bound of $4(\delta + 1)^2$ points for a non-collinear integer distance set with a diameter of δ, and showed that non-collinear integer distance sets can be constructed of arbitrarily large finite size.

Erdős and Anning's argument relies heavily on algebraic geometry. In particular, the argument uses the fact that three non-collinear points in the integer distance set produce a finite family of hyperbolas due to the fact the the difference of the distances is an integer. Using two of the same points along with a fourth point yields another finite family of

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hyperbolas. By Bezout's theorem, each pair of hyperbolas can intersect in at most four points. Since an arbitrary point satisfying the theorem must lie on an intersection point of the two families, it follows the integer distance set must be finite.

A question often contributed to Erdős concerns the size of a distance set if no three points are collinear and no four points lie on the same circle. In [5], Kreisel and Kurz found seven points in the Euclidean plane satisfying these criteria, the largest set found to date. Recently, Greenfeld, Iliopoulou, and Peluse proved that an integer distance set in the Euclidean plane has all but a small number of points lying on a single line or circle [4]. As a consequence, they proved that if $S \subset [-N, N]^2$ is an integer distance set with no three points on a line and no four points on the same circle, then $|S|= O\left((\log N)^{O(1)}\right)$, thus showing that such a set is indeed scarce.

We may consider integer distance sets using metrics other than the Euclidean distance. There are several obstacles to contend with here. First, as mentioned above, Erdős' and Anning's usual argument relies heavily on the geometry of hyperbolas. Also, the Erdős-Anning Theorem is not true when using some metrics. The integer lattice is an example of an infinite non-collinear set whose L_1 distance forms an integer distance set.

In [2], Eppstein used non-Euclidean distances to prove several versions of the Erdős-Anning Theorem, including one for strictly convex distance functions on the plane, for two-dimensional complete Riemannian manifolds of bounded genus, and for the geodesic distance on the boundary of every three-dimensional Euclidean convex set. His proofs are based on the properties of additively weighted Voroni diagrams of these distances.

In vector spaces over finite fields, we can define the distance set for $E \subseteq \mathbb{F}_q^d$ as

$$\Delta(E) = \{||x-y||: x, y \in E\},$$

where

$$||x-y||= (x_1-y_1)^2 + (x_2-y_2)^2 + \cdots (x_d-y_d)^2.$$

In this paper, we prove Erdős-Anning type theorems in two and three-dimensional vector spaces over finite fields using the distance metric defined above. Although further exploration is needed, we conjecture that similar results will hold in higher dimensions. We shall investigate this in detail in the sequel. Lastly, we prove a result involving integer area sets. Recall that in the Euclidean plane, the area of the parallelogram determined by the vectors $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ is given by $x_1y_2 - x_2y_1$. Similarly, we can define the area set for $E \subseteq \mathbb{F}_q^2$ as

$$A(E) = \{x_1y_2 - x_2y_1 : (x_1, y_1), (x_2, y_2) \in E\}.$$

The integer lattice is an example of an infinite non-collinear set that has an integer area set. Thus, integer area sets are not as restrictive as integer distance sets. This observation led us to consider the analogous question concerning integer area sets in the finite field setting. Unlike the Euclidean case, we found that the size of integer area sets is restricted.

Throughout the paper, we work with the field $\mathbb{F}_q$ where $q = p^2$. We write $\mathbb{F}_q$ as

$$\mathbb{F}_p[\alpha] = \{a + b\alpha | a, b \in \mathbb{F}_p\},$$

by choosing $\alpha = \sqrt{\beta}$ where β is a non-square element of $\mathbb{F}_p$.

Note that we write $\frac{a}{b}$ to mean ab^{-1} throughout.

The following two theorems are our finite field analogs of the Erdős-Anning Theorem.

Theorem 1.1. *Let $F \subset \mathbb{F}_q^2, q = p^2$, where p is an odd prime, and suppose that $\Delta(F) = \mathbb{F}_p$. Then $|F| \leq p^2$. Moreover, there exists an orthogonal matrix $M \in O_2(\mathbb{F}_q)$ such that $MF = E$ where E satisfies*

$$E \subseteq \mathbb{F}_p \times \mathbb{F}_p \text{ or } E \subseteq \{(x, b\alpha) | x, b \in \mathbb{F}_p\}.$$

Theorem 1.2. *Let $F \subset \mathbb{F}_q^3, q = p^2$, where p is an odd prime, and suppose that $\Delta(F) = \mathbb{F}_p$. Then $|F| \leq p^3$.*

Note that the proofs of Theorem 1.1 and Theorem 1.2 still hold if $\Delta(F) \subseteq \mathbb{F}_p$ as long as $\Delta(F)$ contains a perfect square. As you will see in the proofs below, we use the fact that $1 \in \Delta(F)$, but the same argument is valid with 1 replaced by any perfect square in $\mathbb{F}_p$.

The theorem involving area sets is as follows.

Theorem 1.3. *Let $E \subset \mathbb{F}_q^2, q = p^2$, where p is an odd prime, and suppose that $A(E) \subseteq \mathbb{F}_p$. Then $|E| \leq p^2$.*

Proof of Theorem 1.1. Suppose $F \subset \mathbb{F}_q^2, q = p^2$, where p is an odd prime and that $\Delta(F) = \mathbb{F}_p$. Then $1 \in \Delta(F)$. Therefore, there exists an orthogonal matrix $M \in O_2(\mathbb{F}_q)$ such that $MF = E$ where $(0, 0)$ and $(1, 0)$ are in E. Let $(x_1, y_1) \in E$ where (x_1, y_1) is not equal to $(0, 0)$ or $(1, 0)$. By assumption we have that

$$|(x_1, y_1) - (0, 0)| \in \mathbb{F}_p,$$

which implies that $x_1^2 + y_1^2 \in \mathbb{F}_p$. Similarly,

$$|(x_1, y_1) - (1, 0)| \in \mathbb{F}_p,$$

which implies

$$x_1^2 - 2x_1 + 1 + y_1^2 \in \mathbb{F}_p.$$

Since $x_1^2 + y_1^2 \in \mathbb{F}_p$ and $\mathbb{F}_p$ is closed, it follows that $x_1 \in \mathbb{F}_p$. We may conclude that $y_1^2 \in \mathbb{F}_p$. It follows immediately that either $y_1 \in \mathbb{F}_p$ or $y_1 = b\alpha$ where $b \in \mathbb{F}_p$ since

$$(a + b\alpha)^2 = (a^2 + b^2\alpha^2) + 2ab\alpha \in \mathbb{F}_p,$$

only if $a = 0$ or $b = 0$.

Let $(x_2, y_2) \in E$ where (x_2, y_2) is not equal to $(0, 0)$ or $(1, 0)$. Then

$$|(x_1, y_1) - (x_2, y_2)| \in \mathbb{F}_p,$$

which implies

$$x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2 \in \mathbb{F}_p.$$

Using the information above, it follows that $y_1y_2 \in \mathbb{F}_p$.

We have previously established that the y-coordinates of E are either elements of $\mathbb{F}_p$ or of the form $b\alpha$ where $b \in \mathbb{F}_p$. Since the product of two y-coordinates are in $\mathbb{F}_p$, it follows that either both y-coordinates are in $\mathbb{F}_p$ or both are of the form $b\alpha$ where $b \in \mathbb{F}_p$. Thus, either

$$E \subseteq \{(x,y)|x \in \mathbb{F}_p, y \in \mathbb{F}_p\} \text{ or } E \subseteq \{(x,b\alpha)|x,b \in \mathbb{F}_p\}.$$

In either case, $|E|\leq p^2$.

Example 1.4. If $p \equiv 3(mod4)$, then -1 is a non-square element of $\mathbb{F}_p$, so we can take $\mathbb{F}_q$ to be $\mathbb{F}_p[i]$. Up to transformations, there are precisely three subsets of $\mathbb{F}_p^2[i]$ of size p^2 where the distance set is $\mathbb{F}_p$, namely

$$E_1 = \{(x,y)|x \in \mathbb{F}_p, y \in \mathbb{F}_p\},$$

$$E_2 = \{(x,yi)|x \in \mathbb{F}_p, y \in \mathbb{F}_p\},$$

and

$$E_3 = \{(xi,y)|x \in \mathbb{F}_p, y \in \mathbb{F}_p\}.$$

□

Proof of Theorem 1.2. Suppose $F \subset \mathbb{F}_q^3, q = p^2$, where p is an odd prime and that $\Delta(F) = \mathbb{F}_p$. Then $1 \in \Delta(F)$. Therefore, there exists an orthogonal matrix $M \in O_3(\mathbb{F}_q)$ such that $MF = E$ so that $(0,0,0)$ and $(1,0,0)$ are in E. Let $(x_1,y_1,z_1) \in E$ where (x_1,y_1,z_1) is not equal to $(0,0,0)$ or $(1,0,0)$. By assumption we have that

$$|(x_1,y_1,z_1) - (0,0,0)|\in \mathbb{F}_p,$$

which implies that $x_1^2 + y_1^2 + z_1^2 \in \mathbb{F}_p$. Similarly,

$$|(x_1,y_1,z_1) - (1,0,0)|\in \mathbb{F}_p,$$

which implies

$$x_1^2 - 2x_1 + 1 + y_1^2 + z_1^2 \in \mathbb{F}_p.$$

Since $x_1^2 + y_1^2 + z_1^2 \in \mathbb{F}_p$ and $\mathbb{F}_p$ is closed, it follows that $x_1 \in \mathbb{F}_p$. Hence, $y_1^2 + z_1^2 \in \mathbb{F}_p$.

Let $(x_2,y_2,z_2) \in E$ where (x_2,y_2,z_2) is not equal to $(0,0,0)$, $(1,0,0)$, or (x_1,y_1,z_1). Then

$$|(x_1,y_1,z_1) - (x_2,y_2,z_2)|\in \mathbb{F}_p,$$

which implies

$$x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2 + z_1^2 - 2z_1z_2 + z_2^2 \in \mathbb{F}_p.$$

Using the information above, it follows that

$$y_1y_2 + z_1z_2 \in \mathbb{F}_p. \tag{1}$$

We know that $E \subseteq \mathbb{F}_p \times \mathbb{F}_p[\alpha] \times \mathbb{F}_p[\alpha]$. Note that if

$$E = \mathbb{F}_p \times \mathbb{F}_p \times \mathbb{F}_p \text{ or } E = \mathbb{F}_p \times \{(x\alpha, y\alpha) : x, y \in \mathbb{F}_p\},$$

then the theorem is satisfied. So, assume that E is not a subset of either of these sets. Write

$$y_1 = a_1 + b_1\alpha, y_2 = a_2 + b_2\alpha, z_1 = c_1 + d_1\alpha, \text{ and } z_2 = c_2 + d_2\alpha.$$

Since we are assuming that E is not a subset of the sets above, without loss of generality we may assume $b_1 \neq 0$. Since Condition (1) must be satisfied, it follows that

$$a_1b_2 + a_2b_1 + c_1d_2 + c_2d_1 = 0.$$

Choose another point $(x_3, y_3, z_3) \in E$ not equal to the previously chosen points. As before, write $y_3 = a_3 + b_3\alpha$ and $z_3 = c_3 + d_3\alpha$. By symmetry, Condition (1) must be satisfied using the points (x_2, y_2, z_2) and (x_3, y_3, z_3) which yields the equation

$$a_3b_2 + a_2b_3 + c_3d_2 + c_2d_3 = 0.$$

Similarly, Condition (1) must be satisfied using the points (x_1, y_1, z_1) and (x_3, y_3, z_3) which yields the equation

$$a_3b_1 + a_1b_3 + c_3d_1 + c_1d_3 = 0.$$

We may view these two equations as a system of linear homogeneous equations in $\mathbb{F}_p$ having $w_1 = a_3$, $w_2 = b_3$, $w_3 = c_3$, and $w_4 = d_3$ as solutions:

$$\begin{cases} b_1w_1 + a_1w_2 + d_1w_3 + c_1w_4 = 0, \\ b_2w_1 + a_2w_2 + d_2w_3 + c_2w_4 = 0. \end{cases}$$

Note that any point $(x, a + b\alpha, c + d\alpha)$ in E must satisfy this system when setting $w_1 = a$, $w_2 = b$, $w_3 = c$, and $w_4 = d$. This system yields the augmented matrix

$$\left(\begin{array}{cccc|c} b_1 & a_1 & d_1 & c_1 & 0 \\ b_2 & a_2 & d_2 & c_2 & 0 \end{array}\right).$$

Since $b_1 \neq 0$, we may perform the elementary row operations $\frac{1}{b_1}R_1 \to R_1$ followed by $-b_2R_1 + R_2 \to R_2$ to obtain

$$\left(\begin{array}{cccc|c} 1 & \frac{a_1}{b_1} & \frac{d_1}{b_1} & \frac{c_1}{b_1} & 0 \\ 0 & -\frac{b_2a_1}{b_1} + a_2 & -\frac{b_2d_1}{b_1} + d_2 & -\frac{b_2c_1}{b_1} + c_2 & 0 \end{array}\right).$$

If the second row is a row of zeros, then it follows that the second row from the original matrix is a scalar multiple of the first row. That is,

$$b_2 = \lambda b_1, a_2 = \lambda a_1, d_2 = \lambda d_1, \text{ and } c_2 = \lambda c_1,$$

for some $\lambda \in \mathbb{F}_p$. Since $(x_2, y_2, z_2) \in E$ is arbitrary, we have that if $(x_1, y_1, z_1) \in E$, then (x_2, y_2, z_2) must be of the form

$$(x_2, \lambda(a_1 + b_1\alpha), \lambda(c_1 + d_1\alpha)).$$

Thus, $|E|\leq p^2$ in this case as there are p choices for x_2 and p choices for λ.

Now, assume the second row does not consist entirely of zeros. Then no matter where the leading 1 appears in row two, the system has at most two free variables. This means that the system has at most p^2 solutions. Thus, $|E|\leq p^3$, and the proof is complete. □

Proof of Theorem 1.3. Suppose $E \subset \mathbb{F}_q^2, q = p^2$, where p is an odd prime and that $A(E) \subseteq \mathbb{F}_p$. We consider four cases, the first two of which are trivial. Case 3 investigates when one of the coordinates is a member of $\mathbb{F}_p$ and the other is a member of $\mathbb{F}_p[\alpha]$. In the fourth case, we consider $E \subseteq \mathbb{F}_p[\alpha] \times \mathbb{F}_p[\alpha] = \mathbb{F}_q^2$ where E is not contained in the previous cases, which establishes the result. Further, the assumption that each case differs from the previous cases guarantees the existence of certain nonzero elements, which play an important role in the proof. We begin with the following.

Case 1. $E \subseteq \mathbb{F}_p \times \mathbb{F}_p$.

It follows immediately that $A(E) \subseteq \mathbb{F}_p$ and $|E|\leq p^2$. Thus, the conclusion of the theorem is satisfied.

Case 2. $E \subseteq \{(x\alpha, y\alpha) : x, y \in \mathbb{F}_p\}$.

Again, the conclusion of the theorem is clearly satisfied.

Case 3. $E \subseteq \mathbb{F}_p \times \mathbb{F}_p[\alpha]$ or $E \subseteq \mathbb{F}_p[\alpha] \times \mathbb{F}_p$.

Without loss of generality, suppose that $E \subseteq \mathbb{F}_p \times \mathbb{F}_p[\alpha]$ and choose $(x_1, y_1) \in E$ where $x_1 \in \mathbb{F}_p$ and $y_1 \in \mathbb{F}_p[\alpha] - \mathbb{F}_p$. We know such a y_1 exists as otherwise we would be in Case 1. Let (x_2, y_2) be an arbitrary point in E. Since $A(E) \subseteq \mathbb{F}_p$, we have $x_1y_2 - x_2y_1 \in \mathbb{F}_p$. It follows that either both terms are in $\mathbb{F}_p$, or both terms are in $\mathbb{F}_p[\alpha] - \mathbb{F}_p$.

If both terms are in $\mathbb{F}_p$, then we have $x_2 = 0$ since y_1 is in $\mathbb{F}_p[\alpha] - \mathbb{F}_p$. In this case, we can take $E \subseteq \{(0, y) : y \in \mathbb{F}_p[\alpha]\}$ and $|E|\leq p^2$.

If both terms are in $\mathbb{F}_p[\alpha] - \mathbb{F}_p$, then let $y_1 = a_1 + b_1\alpha$ and $y_2 = a_2 + b_2\alpha$ where $a_1, a_2, b_1, b_2 \in \mathbb{F}_p$ and $b_1 \neq 0$. We have $x_1y_2 - x_2y_1 \in \mathbb{F}_p$ which implies

$$(x_1a_2 - x_2a_1) + (x_1b_2 - x_2b_1)\alpha \in \mathbb{F}_p.$$

It follows that $x_1b_2 - x_2b_1 = 0$. Since $b_1 \neq 0$, we can solve for x_2 to obtain $x_2 = \frac{x_1b_2}{b_1}$. That is, if $(x_1, a_1 + b_1\alpha)$ is an arbitrary element of E, then any other point in E must be of the form $\left(\frac{x_1b_2}{b_1}, a_2 + b_2\alpha\right)$. Thus, we can construct a set E such that $A(E) \subseteq \mathbb{F}_p$ as follows:

Choose an arbitrary point of the form $(x_1, a_1 + b_1\alpha) \in \mathbb{F}_q^2$ with $x_1, a_1, b_1 \in \mathbb{F}_p$ and $x_1, b_1 \neq 0$. (The situation where the first coordinate is zero is mentioned above.) Then

$$E = \left\{\left(\frac{x_1b}{b_1}, a + b\alpha\right) : a, b \in \mathbb{F}_p, b \neq 0\right\},$$

satisfies $A(E) \subseteq \mathbb{F}_p$ with $|E|= p(p-1)$ as there are p choices for a and $p-1$ choices for b.

Note that the first coordinate must be 0 in order for the second coordinate to be in $\mathbb{F}_p$ and satisfy $A(E) \subseteq \mathbb{F}_p$. To see this, suppose $(x, y) \in E$ where $x, y \in \mathbb{F}_p$ and $(x_1, a_1 + b_1\alpha) \in E$ where $x_1, a_1, b_1 \in \mathbb{F}_p$ and $b_1 \neq 0$. Then

$$x_1y - x(a_1 + b_1\alpha) = (x_1y - xa_1) - xb_1\alpha \notin \mathbb{F}_p.$$

unless $x = 0$.

Lastly, we may add the elements of the set $\{(0, y) : y \in \mathbb{F}_p\}$ to E to obtain a set of size p^2.

Example 1.5. When $p \equiv 3 \pmod 4$, then we can take $\alpha = i$ where $i = \sqrt{-1}$ since -1 is a nonsquare element. Consider $\mathbb{Z}_7[i]$. Choose any point whose first coordinate is in $\mathbb{Z}_7$ and whose second coordinate is in $\mathbb{Z}_7[i] - \mathbb{Z}_7$, say $(2, 1 + 3i)$. Then E would be given by

$$E = \left\{ \left(\frac{2b}{3}, a + bi \right) : a, b \in \mathbb{Z}_7, b \neq 0 \right\} = \{(3b, a + bi) : a, b \in \mathbb{Z}_7, b \neq 0\}.$$

This set has $6 \cdot 7 = 42$ elements, and we may union the set $\{(0, y) : y \in \mathbb{Z}_7\}$ to obtain a set of size 7^2. The reader can check that $A(E) \subseteq \mathbb{Z}_7$.

Case 4. $E \subseteq \mathbb{F}_p[\alpha] \times \mathbb{F}_p[\alpha]$ where E is not contained in the previous cases.

Choose $(x_1, y_1) \in E$ where $x_1 \in \mathbb{F}_p[\alpha] - \mathbb{F}_p$ and $y_1 \in \mathbb{F}_p[\alpha] - \mathbb{F}_p$. We know such a point (x_1, y_1) exists; otherwise, we would be in Case 3. Write $x_1 = a_1 + b_1\alpha$ and $y_1 = c_1 + d_1\alpha$ where $a_1, b_1, c_1, d_1 \in \mathbb{F}_p$ and $b_1, d_1 \neq 0$. Let (x_2, y_2) be an arbitrary point in E and write $x_2 = a_2 + b_2\alpha$ and $y_2 = c_2 + d_2\alpha$ where $a_2, b_2, c_2, d_2 \in \mathbb{F}_p$. By assumption, we have $x_1y_2 - x_2y_1 \in \mathbb{F}_p$ which implies

$$(-c_1a_2 + d_1b_2 + a_1c_2 - b_1d_2) + (-c_1b_2 - d_1a_2 + a_1d_2 + b_1c_2)\alpha \in \mathbb{F}_p.$$

It follows that

$$c_1b_2 + d_1a_2 - a_1d_2 - b_1c_2 = 0.$$

Now, suppose there is a third point (x_3, y_3) in E and write $x_3 = a_3 + b_3\alpha$ and $y_3 = c_3 + d_3\alpha$ where $a_3, b_3, c_3, d_3 \in \mathbb{F}_p$. Since $x_3y_1 - x_1y_3 \in \mathbb{F}_p$ and $x_3y_2 - x_2y_3 \in \mathbb{F}_p$, using symmetry we have that

$$c_1b_3 + d_1a_3 - a_1d_3 - b_1c_3 = 0 \text{ and } c_3b_2 + d_3a_2 - a_3d_2 - b_3c_2 = 0.$$

Using a similar approach as that in the proof of Theorem 1.2, we can view these two equations as the following system of linear homogeneous equations in $\mathbb{F}_p$ having $w_1 = a_3$, $w_2 = b_3$, $w_3 = c_3$, and $w_4 = d_3$ as solutions:

$$\begin{cases} d_1w_1 + c_1w_2 - b_1w_3 - a_1w_4 = 0, \\ -d_2w_1 - c_2w_2 + b_2w_3 + a_2w_4 = 0. \end{cases}$$

Note that any point in E must satisfy this system. This system yields the augmented matrix

$$\left(\begin{array}{cccc|c} d_1 & c_1 & -b_1 & -a_1 & 0 \\ -d_2 & -c_2 & b_2 & a_2 & 0 \end{array} \right).$$

Since $d_1 \neq 0$, we may perform the elementary row operations $\frac{1}{d_1}R_1 \to R_1$ followed by $d_2R_1 + R_2 \to R_2$ to obtain

$$\left(\begin{array}{cccc|c} 1 & \frac{c_1}{d_1} & \frac{-b_1}{d_1} & \frac{-a_1}{d_1} & 0 \\ 0 & \frac{d_2c_1}{d_1} - c_2 & \frac{-d_2b_1}{d_1} + b_2 & \frac{-d_2a_1}{d_1} + a_2 & 0 \end{array} \right).$$

If the second row is a row of zeros, then it follows that the second row from the original matrix is a scalar multiple of the first row. That is, $d_2 = \lambda d_1$, $c_2 = \lambda c_1$, $b_2 = \lambda b_1$, and $a_2 = \lambda a_1$ for some $\lambda \in \mathbb{F}_p$. Since $(x_2, y_2) \in E$ is arbitrary, we have that if $(x_1, y_1) \in E$ where $x_1 = a_1 + b_1\alpha$ and $y_1 = c_1 + d_1\alpha$ as described above, then (x_2, y_2) must be of the form $\lambda(a_1 + b_1\alpha, c_1 + d_1\alpha)$. Thus, $|E| \leq p$ in this case.

Now, assume the second row does not consist entirely of zeros. Then no matter where the leading 1 appears in row two, the system has at most two free variables. Thus, $|E| \leq p^2$. □

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