On a Balanced Property of Compositions

Miklós Bóna
Department of Mathematics
University of Florida
Gainesville FL 32611-8105
USA

Submitted: October 2, 2006; Accepted: January 24, 2007; Published: March 13, 2007

Abstract

Let S be a finite set of positive integers with largest element m. Let us randomly select a composition a of the integer n with parts in S, and let $m(a)$ be the multiplicity of m as a part of a. Let $0 \leq r < q$ be integers, with $q \geq 2$, and let $p_{n,r}$ be the probability that $m(a)$ is congruent to r modulo q. We show that if S satisfies a certain simple condition, then $\lim_{n\rightarrow\infty} p_{n,r} = 1/q$. In fact, we show that an obvious necessary condition on S turns out to be sufficient.

1 Introduction

A *composition* of the positive integer n is a sequence $(a_1, a_2, \cdots, a_k)$ of positive integers so that $\sum_{i=1}^{k} a_i = n$. The a_i are called the *parts* of the composition. It is well-known [1] that the number of compositions of n into k parts is $\binom{n-1}{k-1}$. From this fact, it is possible to prove the following. Let $0 \leq r < q$ be integers, with $q \geq 2$, and let $P_{n,r}$ be the probability of the event that the number of parts of a randomly selected composition of n is congruent to r modulo q. Then $\lim_{n\rightarrow\infty} P_{n,r} = 1/q$. In other words,

$$\lim_{n\rightarrow\infty} \frac{\sum_{i=0}^{\lfloor (n-1)/q \rfloor} \binom{n-1}{iq+r}}{2^{n-1}} = \frac{1}{q}.$$

For $q = 2$, this follows from the well-known fact that the number of even-sized subsets of a non-empty finite set is equal to the number of odd-sized subsets of that set. For $q > 2$, the statement can be proved, for example, by the method we will use in this paper. The special cases of $q = 3$ and $q = 4$ appear as Exercises 4.41 and 4.42 in [1]. In other words, all residue classes are equally likely to occur. We will refer to this phenomenon by saying that the number of part sizes of a randomly selected composition of n is *balanced*.

Now let us impose a restriction on the *part sizes* of the compositions of n that form our sample space by requiring that all part sizes come from a finite set S. Is it still true that the number of part sizes of a randomly selected composition of n is balanced? That will certainly depend on the restriction we impose on the part sizes. For instance, if the set S of allowed parts consists of odd numbers only, then the number of part sizes will not be balanced. Indeed, if $q = 2$, then $P_{n,0} = 1$ if n is even, and $P_{n,0} = 0$ if n is odd. Let m be the largest element of the set S of allowed parts. It turns out that it is easier to (directly) work with the multiplicity $m(a)$ of m as a part of the randomly selected composition a than with its number of parts. For the special case when S has only two elements, the results obtained for $m(a)$ can then be translated back to results on the number of parts of a.

In this paper, we prove that if S satisfies a certain obviously necessary condition, then the parameter $m(a)$ is balanced, as described in the abstract. That is, the remainder of $m(a)$ modulo q is equally likely to take all possible values.

2 The Strategy

Let $S = \{s_1, s_2, \cdots, s_k = m\}$ be a finite set of positive integers with at least two elements. Let us assume without loss of generality that no integer larger than 1 divides all k elements of S. Clearly, if $s_1, s_2, \cdots, s_{k-1}$ are all divisible by a certain prime $h > 1$, then the multiplicity $m(a)$ of $m = s_k$ as a part of a composition a of n is restricted. Indeed, $n - m(a)m$ must be divisible by h. In particular, if n is divisible by h, then $m(a)m$ is divisible by h, and so $m(a)$ must be divisible by h. Therefore, the parameter $m(a)$ is not balanced. Indeed, if n is divisible by h, and we choose $q = h$, then $p_{n,0} = 1$, and $p_{n,r} = 0$ for $r \neq 0$, while if n is not divisible by h and $q = h$, then $p_{n,0} = 0$.

So for $m(a)$ to be a balanced parameter, it is *necessary* for S to satisfy the condition that its smallest $k-1$ elements do not have a proper common divisor. In the rest of this paper, we prove that this condition is at the same time *sufficient* for $m(a)$ to be balanced.

Unless otherwise stated, let S be a finite set of positive integers with at least two elements, and let $S = \{s_1, s_2, \cdots, s_k = m\}$, where the s_i are listed in increasing order. So m is the largest element of S. *Unless otherwise stated*, let us also assume that no integer larger than 1 is a divisor of all of $s_1, s_2, \cdots, s_{k-1}$. Note that this means that if $|S| = 2$, then $s_1 = 1$.

For a fixed positive integer n, let $A_{S,n}(x)$ be the ordinary generating function of all compositions of the integer n into parts in S according to their number of parts equal to m. In other words,

$$A_{S,n}(x) = \sum_a x^{m(a)} = \sum_d a_{S,n,d} x^d,$$

where a ranges over all compositions of n into parts in S, and $m(a)$ is the multiplicity of m as a part in a. On the far right, $a_{S,n,d}$ is the number of compositions a of n into parts in S so that $m(a) = d$.

Example 1 *Let $S = \{1, 3\}$. Then the first few polynomials $A_n(x) = A_{S,n}(x)$ are as follows.*

- $A_0(x) = A_1(x) = A_2(x) = 1,$
- $A_3(x) = x,\ A_4(x) = 2x + 1,\ A_5(x) = 3x + 1.$
- $A_6(x) = x^2 + 4x + 1,\ A_7(x) = 3x^2 + 5x + 1,\ A_8(x) = 6x^2 + 6x + 1.$

Let $q \geq 2$ be a positive integer, and let $0 \leq r \leq q - 1$. Let $A_{S,n,r}$ be the number of compositions a of n with parts in S so that $m(a)$ is congruent to r modulo q. So

$$A_{S,n,r} = a_{S,n,r} + a_{S,n,q+r} + \cdots + a_{S,n,\lfloor n/q \rfloor q + r}.$$

In order to simplify the presentations of our results, we will first discuss the special case when n is divisible by q and $r = 0$. Let w be a primitive qth root of unity. Then

$$\sum_{t=0}^{q-1} A_{S,n}(w^t) = \sum_{t=0}^{q-1}\sum_{d=0}^{n/m} a_{S,n,d} w^{td} \tag{1}$$
$$= \sum_{d=0}^{n/m}\sum_{t=0}^{q-1} a_{S,n,d} w^{td} \tag{2}$$
$$= \sum_{d=0}^{n/m} a_{S,n,d} \sum_{t=0}^{q-1} w^{td}. \tag{3}$$

Using the summation formula of a geometric progression, we get that

$$\sum_{t=0}^{q-1} (w^d)^t = \begin{cases} 0 \text{ if } w^d \neq 1, \text{ that is, if } q \nmid d, \\ q \text{ if } w^d = 1, \text{ that is, if } q|d. \end{cases}$$

Therefore, (1) reduces to

$$\sum_{t=0}^{q-1} A_{S,n}(w^t) = q \cdot \sum_{j=1}^{n/q} a_{S,n,jq} = qA_{S,n,0},$$

$$\frac{1}{q}\sum_{t=0}^{q-1} A_{S,n}(w^t) = A_{S,n,0}. \tag{4}$$

So in order to find the approximate value of $A_{S,n,0}$, it suffices to find the approximate values of $A_n(w^t)$, for $0 \leq t \leq r-1$, and for a primitive root of unity w. The number $A_{S,n}$ of *all* compositions of n into parts in S is equal to $A_n(1)$, so we will need that value as well, in order to compute the ratio $A_{S,n,0}/A_{S,n}$.

Finally, note that if n is not divisible by q, but $r = 0$, then the same argument applies, and $\frac{1}{q}\sum_{t=0}^{q-1} A_{S,n}(w^t) = A_{S,n,0}$ still holds. If $r \neq 0$, then instead of computing $\sum_{t=0}^{q-1} A_{S,n}(w^t)$, we compute

$$T_n(w) = \sum_{t=0}^{q-1} A_{S,n}(w^t)w^{-rt} = \sum_{d=0}^{n/m} a_{S,n,d} \sum_{t=0}^{q-1} w^{t(d-r)}.$$

This shows that the coefficient of w^k in $T_n(w)$ is 0, unless $w^{d-r} = 1$, that is, unless $d - r$ is divisible by q. If $d - r$ is divisible by q, then this coefficient is q. This shows again that

$$\frac{1}{q}\sum_{t=0}^{q-1} A_{S,n}(w^t)w^{-tr} = A_{S,n,r}. \tag{5}$$

Therefore, computing $A_{S,n}(w^t)$ will be useful in the general case as well.

3 Linear Recurrence Relations

In order to compute the values of $A_{S,n}(x)$ for various values of x, we can keep x fixed, and let n grow. The connection among the polynomials $A_{S,n}(x)$ is explained by the following Proposition.

Proposition 1 *Let $S = \{s_1, s_2, \cdots, s_k = m\}$, with $k \geq 2$. Then for all positive integers $n \geq 3$, the polynomials $A_{S,n}(x)$ satisfy the recurrence relation*

$$A_{S,n}(x) = A_{S,n-s_1}(x) + A_{S,n-s_2}(x) + \cdots + A_{S,n-s_{k-1}}(x) + xA_{S,n-m}(x). \tag{6}$$

Proof: Let a be a composition of n with parts in S. If the first part of a is s_i, for some $i \in [1, k-1]$, then the rest of a forms a composition of $n - s_i$ with parts in S in which the multiplicity of m as a part is still $m(a)$. These compositions of n are counted by $A_{S,n-s_i}(x)$. If the first part of a is m, then the rest of a forms a composition of $n - m$ with parts in S in which the multiplicity of m as a part is $m(a) - 1$. These compositions of n are counted by $xA_{S,n-m}(x)$. $\diamond$

Example 2 *If $S = \{1, 3\}$, then (6) reduces to*

$$A_{S,n}(x) = A_{S,n-1}(x) + xA_{S,n-3}(x). \tag{7}$$

For a *fixed* real number x, the recurrence relation (6) becomes a recurrence relation on real numbers. The solutions of such recurrence relations are described by the following well-known theorem. (See, for instance, [4], Section 7.2.)

Theorem 1 *Let*

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} \tag{8}$$

be a recurrence relation, where the c_i are complex constants. Let $\alpha_1, \alpha_2, \cdots, \alpha_t$ be the distinct roots of the characteristic equation

$$z^k - c_1z^{k-1} - c_2z^{k-2} - \cdots - c_k = 0, \tag{9}$$

and let M_i be the multiplicity of α_i. Then the sequence $a_0, a_1, \cdots$ of complex numbers satisfies (8) if and only if there exist constants $b_1, b_2, \cdots, b_k$ so that for all $n \geq 0$, we have

$$a_n = b_1\alpha_1^n + b_2n\alpha_1^n + \cdots + b_{M_1}n^{M_1-1}\alpha_1^n + b_{M_1+1}\alpha_2^n, \cdots \tag{10}$$

$$\cdots + b_{M_1+M_2}n^{M_2-1}\alpha_2^n, \cdots, \cdots + b_kn^{M_k-1}\alpha_k^n. \tag{11}$$

In other words, the solutions of (8) form a k-dimensional vector space.

We will need the following consequence of Theorem 1.

Corollary 1 *Let us assume that the sequence $\{a_n\}$ is a solution of (8) and that there is no linear recurrence relation of a degree less than k that is satisfied by $\{a_n\}$. Let us further assume that the characteristic equation (9) of (8) has a unique root α_1 of largest modulus. Then there is a* nonzero *constant C so that*

$$a_n = C\alpha_1^n + o(\alpha_1^n).$$

Proof: As $\{a_n\}$ does not satisfy a recurrence relation of a degree less than k, we must have $c_1 \neq 0$. As $|\alpha_1| > |\alpha_i|$ for $i \neq 1$, the statement follows. $\diamond$

Let us now apply Theorem 1 to find the solution of (6) for a fixed x. The characteristic equation of (6) is

$$f(z) = z^m - \sum_{i=1}^{k-1} z^{m-s_i} - x = 0. \tag{12}$$

As explained in Section 2, we will need to compute $A_{S,n}(1)$ and also, $A_{S,n}(w^t)$ for the case when $w \neq 1$ is a qth primitive root of unity. To that end, we need to find the roots of the corresponding characteristic equations. That is, we will compare the root of largest modulus of the characteristic equation

$$f_1(z) = z^m - \sum_{i=1}^{k-1} z^{m-s_i} - 1 = 0 \tag{13}$$

and the root(s) of the largest modulus of the characteristic equation

$$f_w(z) = z^m - \sum_{i=1}^{k-1} z^{m-s_i} - w = 0 \tag{14}$$

The following lemma, helping to compute root of largest modulus of $f_1(z)$, is a special case of Exercise III.16 in [3].

Lemma 1 *The polynomial $f_1(z) = z^m - \sum_{i=1}^{k-1} z^{m-s_i} - 1$ has a unique positive real root α.*

Proof: Let α be the smallest positive real root of $f_1(z)$. We know such a root exists since $f_1(0) = -1$ and $\lim_{z\to\infty} f(z) = \infty$. If $r > 1$, then

$$\begin{aligned} f_1(r\alpha) + 1 &= (r\alpha)^m - \sum_{i=1}^{k-1} (r\alpha)^{m-s_i} \\ &> r^m \left(\alpha^m - \sum_{i=1}^{k-1} \alpha^{m-s_i} \right) \\ &= r^m \\ &> 1, \end{aligned}$$

and so $f_1(r\alpha) > 0$. $\diamond$

Now we address the problem of finding the roots of the characteristic equation (12) in the case when $w \neq 1$ is a root of unity. It turns out that it suffices to assume that $|w| = 1$. (The following Lemma is similar to Exercise III.17 in [3].)

Lemma 2 *Let α be defined as in Lemma 1. Let w be any complex number satisfying $w \neq 1$ and $|w| = 1$. Then all roots of the polynomial $f_w(z)$ are of smaller modulus than α.*

Proof: Let y be a root of f_w. Then

$$\begin{aligned} |y|^m &= \left| w + \sum_{i=1}^{k-1} y^{m-s_i} \right| \\ &\leq 1 + \sum_{i=1}^{k-1} \left| y^{m-s_i} \right|. \end{aligned}$$

Therefore, $f_1(|y|) \leq 0$. This implies that $|y| \leq \alpha$ since we have seen in the proof of Lemma 1 that $f_1(t) > 0$ if $t > \alpha$.

Furthermore, in the last displayed inequality, the inequality is strict unless for all i so that $1 \leq i \leq k-1$, the complex numbers y^{m-s_i} have the same argument as w, and that argument is the same as the argument of y^m. That happens only if the complex numbers $y^{s_1}, y^{s_2-s_1}, \cdots y^{s_{k-1}-s_{k-2}}$ all have argument 0, that is, when these numbers are positive real numbers. However, that happens precisely when $s_1, s_2 - s_1, \cdots, s_{k-1} - s_{k-2}$ are all multiples of the multiplicative order o_y of $y/|y|$ as a complex number. That implies that $s_1, s_2, \cdots, s_{k-1}$ are all divisible by o_y, contradicting our hypothesis on S. $\diamond$

The previous two lemmas show that the largest root of the characteristic equation for the sequence $\{A_{S,n}(1)\}_{n\geq 0}$ is larger than the largest root(s) of the characteristic equation for the sequence $\{A_{S,n}(w)\}_{n\geq 0}$ for any complex number $w \neq 1$ with absolute value 1. Given formula (10), in order to see that the first sequence indeed grows faster than the second, all we need to show is that the *coefficient* b_1 of α^n in (8) is not 0. (Here α, the largest root of $f_1(z)$, plays the role of α_1 in (10)). This is the content of the next lemma.

Lemma 3 *Let $S = \{s_1, s_2, \cdots, s_{k-1}\}$ be any finite set of positive integers (so for this Lemma, we do not require that $s_1, s_2, \cdots s_{k-1}$ do not have a proper common divisor). Then the sequence $\{A_{S,n}\}_{n\geq 0} = \{A_{S,n}(1)\}_{n\geq 0}$ does not satisfies a linear recurrence relation with constant coefficients and less than $|S|+1$ terms. In other words, if $|S| = k$, then there do not exist constants $c_2, c_3, \cdots, c_k$ and positive integers $j_1, j_2, \cdots, j_{k-1}$ so that for all $n \geq 0$,*

$$A_{S,n} = \sum_{i=1}^{k-1} c_i A_{S,n-j_i}.$$

Proof: Let us assume that S is a minimal counterexample. It is then straightforward to verify that $|S| > 2$. Let $S' = S - m$, that is, the set obtained from S by removing the largest element of S. Then

$$A_{S',n} = \sum_{i=1}^{k-1} c'_i A_{S',n-s_i}, \tag{15}$$

and there is no shorter recurrence satisfied by $\{A_{S',n}\}$.

Now crucially, $A_{S',n} = A_{S,n}$ for all n satisfying $0 \leq n < m$. So these sequences agree in $m - 1 \geq k - 1$ values. So if $\{A_{S,n}\}$ satisfied a linear recurrence relation of degree $k-1$, that would have to be the recurrence relation (15). Indeed, by Theorem 1, the solutions of (15) form a $(k-1)$-dimensional vector space, so knowing $(k-1)$ elements of a solution determines the whole solution. However, $\{A_{S,n}\}$ does not satisfy (15) since $A_{S,m} = A_{S',m} + 1 \neq A_{S',m}$, where the difference is caused by the one-part composition m. $\diamond$

Now we are in position to express the growth rate of $A_{S,n} = A_{S,n}(1)$.

Proposition 2 *Let α be defined as in Lemma 1. Then*

$$A_{S,n}(1) = C\alpha^n + o(\alpha^n),$$

for some nonzero constant C.

Proof: Immediate from Corollary 1 and Lemma 3. ◇

We can now compare the growth rates of $A_{S,n}(w)$ and $A_{S,n}(1)$.

Lemma 4 *Let $w \neq 1$ be any complex number so that $|w| = 1$. Then*

$$\lim_{n\to\infty} \frac{A_{S,n}(w)}{A_{S,n}(1)} = 0.$$

Proof: Lemma (1) shows that the unique positive root of the characteristic equation (13) is larger than the absolute value of all roots of the characteristic equation (14). Therefore, $A_{S,n}(w) = O(n^k\beta^k)$, with $\beta < \alpha$. ◇

Finally, we can use the results of this section to prove the balanced properties of the numbers $A_{S,n,r}$.

Theorem 2 *Let $q \geq 2$ be an integer, and let r be an integer satisfying $0 \leq r \leq q-1$. Then*

$$\lim_{n\to\infty} p_{n,r} = \lim_{n\to\infty} \frac{A_{S,n,r}}{A_{S,n}} = \frac{1}{q}.$$

Proof: Let us first address the case of $r = 0$. Dividing (4) by $A_{S,n}$, we get that

$$A_{S,n,0} = \frac{1}{q}\sum_{t=0}^{q-1} \frac{A_{S,n}(w^t)}{A_{S,n}}.$$

However, Lemma 4 shows that all but one of the q summands on the right-hand side converge to 0, and the remaining one (the first summand) is equal to 1.

For general r, the only change is that instead of dividing both sides of (4) by $A_{S,n}$, we divide both sides of (5) by $A_{S,n}$. As $|w| = 1$, the result follows in the same way. ◇

4 Further Directions

Let $S = \{1, 3\}$. Numerical evidence suggest that for all n, the polynomials $A_{S,n}(x)$ have real roots only. Furthermore, numerical evidence also suggests that the sequences of polynomials $\{A_{S,3n+r}(x)\}_{n\geq 0}$ form a Sturm sequence for each of $r = 0, 1, 2$. (See [6] for the definition and importance of Sturm sequences.) This raises the following intriguing questions.

Question 1 *For which sets S is it true that the polynomials $A_{S,n}(x)$ have real roots only?*

Question 2 *For which sets S is it true that the polynomials $A_{S,n}(x)$ can be partitioned into a few Sturm sequences?*

Herb Wilf [5] proved that the set $S = \{1, 2\}$ does have both of these properties.

If $A_{S,n}(x)$ has real roots only, then its coefficients form a log-concave (and therefore, unimodal) sequence. (See Chapter 8 of [2] for an introduction into into unimodal and log-concave sequences.) This raises the following questions.

Question 3 *Let us assume that $A_{S,n}(x)$ has real roots only. Is there a combinatorial proof for the log-concavity of its coefficients?*

Question 4 *Let us assume that $A_{S,n}(x)$ has real roots only. Where is the peak (or peaks) of the unimodal sequence of its coefficients?*

Another interesting question is the following.

Question 5 *For what S and n does the equality $A_{S,n}(-1) = 0$ hold? When it does, the number of compositions of n with parts in S and with $m(a)$ even equals the number of compositions of n with parts in S and with $m(a)$ odd. Is there a combinatorial proof of that fact?*

Finally, our methods rested on the finiteness of S, but we can still ask what can be said for *infinite* sets of allowed parts.

Acknowledgment

I am grateful to Herb Wilf for valuable discussions and advice.

References

[1] M. Bóna, A Walk Through Combinatorics, 2nd edition, *World Scientific*, 2006.

[2] M. Bóna, Introduction to Enumerative Combinatorics, McGraw-Hill, 2007.

[3] G. Pólya, G. Szegő, *Problems and Theorems in Analysis*, Springer, 1972.

[4] K. Rosen, Discrete Mathematics and Its Applications, McGraw-Hill, Sixth Edition, 2007.

[5] H. Wilf, *Personal Communication*, 2006.

[6] H. Wilf, *Mathematics for the Physical Sciences*, third edition, Dover, 2006.