New combinatorial interpretations of some mock theta functions

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Abstract

In 1972, Bender and Knuth established a bijection between certain infinite matrices of non-negative integers and plane partitions and in [2] a bijection between Bender-Knuth matrices and n-color partitions was shown. Here we use this later bijection and translate the recently found n-color partition theoretic interpretations of four mock theta functions of S. Ramanujan in [1] to new combinatorial interpretations of the same mock theta functions involving Bender-Knuth matrices.

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1 Introduction, Definitions and the Main Results

In his last letter dated 12 January, 1920 to G.H. Hardy , S. Ramanujan listed 17 functions which he called mock theta functions. He separated these 17 functions into three classes. First containing 4 functions of order 3, second containing 10 functions of order 5 and the third containg 3 functions of order 7. Watson [12] found three more functions of order 3 and two more of order 5 appear in the *lost notebook* [11]. Mock theta functions of order 6 and 8 have also been studied in [5] and [8], respectively. For the definitions of mock theta functions and their order the reader is referred to [10].

A partition of a positive integer n is any non-increasing sequence of positive integers whose sum is n. 0 also has a partition called "empty partition". The rank of a partition is defined to be the largest part minus the number of its parts. Partition theoretic interpretations of some of the mock theta functions are found in the literature. For example, $\Psi(q)$, defined by (1.1) below , has been interpretated as generating function for partitions without gaps (cf.[9]). A survey of work done on mock theta functions is given in Andrews paper [4]. Very recently Bringmann and Ono [7] redefined mock theta functions as the holomorphic projection of weight 1/2 weak Maass forms and used their ideas in solving the classical problem of obtaining formulas for $N_e(n)$ (resp.$N_o(n)$), the number of partitions of n with even (resp.odd) rank by showing the equivalence of this problem and the problem of deriving exact formulas for the coefficients $\alpha(n)$ of the series

$$f(q) = 1 + \sum_{n=1}^{\infty} \alpha(n) q^n = \sum_{n=0}^{\infty} q^{n^2}/(-q;q)_n^2,$$

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where$f(q)$ is the first mock theta function of order 3 in Ramanujan's list of 17 mock theta functions (cf.[9]) and

$$(a;q)_n = \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})},$$

for any constant a. Here our objective is to translate the n-color partition theoretic interpretations of four mock theta functions obtained in [1] to matrix theoretic interpretations of the same mock theta functions by using the bijection of [2]. First we recall the definitions of n-color partitions and the weighted differences from [3]:

Definition 1.1. An $n-color$ partition (also called a partition with 'n copies of n') of a positive integer ν is a partition in which a part of size n can come in n different colors denoted by subscripts: $n_1, n_2, \cdots, n_n$ and the parts satisfy the order $1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < \cdots$. Thus, for example, the n-color partitions of 3 are

$$3_1, 3_2, 3_3, 2_1 1_1, 2_2 1_1, 1_1 1_1 1_1.$$

Definition 1.2. The weighted difference of two parts $m_i, n_j, m \geq n$ is defined by $m-n-i-j$ and denoted by $((m_i - n_j))$.

Recently in [1] we showed that the following four mock theta functions (first is of order 3 and the remaining three are of order 5) of S. Ramanujan:

$$\Psi(q) = \sum_{m=1}^{\infty} \frac{q^{m^2}}{(q;q^2)_m}, \tag{1.1}$$

$$F_0(q) = \sum_{m=0}^{\infty} \frac{q^{2m^2}}{(q;q^2)_m}, \tag{1.2}$$

$$\Phi_0(q) = \sum_{m=0}^{\infty} q^{m^2}(-q;q^2)_m, \tag{1.3}$$

and

$$\Phi_1(q) = \sum_{m=0}^{\infty} q^{(m+1)^2}(-q;q^2)_m, \tag{1.4}$$

have their n-color partition theoretic interpretations in the following theorems, respectively,

Theorem 1. For $\nu \geq 1$, let $A_1(\nu)$ denote the number of n-color partitions of ν such that even parts appear with even subscripts and odd with odd, for some k, k_k is a part, and the weighted difference of any two consecutive parts is 0. Then,

$$\sum_{\nu=1}^{\infty} A_1(\nu) q^\nu = \Psi(q). \tag{1.5}$$

Theorem 2. For $\nu \geq 0$, let $A_2(\nu)$ denote the number of n-color partitions of ν such that even parts appear with even subscripts and odd with odd greater than 1, for some k, k_k is a part, and the weighted difference of any two consecutive parts is 0. Then,

$$\sum_{\nu=0}^{\infty} A_2(\nu)q^\nu = F_0(q). \tag{1.6}$$

Theorem 3. For $\nu \geq 0$, let $A_3(\nu)$ denote the number of n-color partitions of ν such that only the first copy of the odd parts and the second copy of the even parts are used, that is, the parts are of the type $(2k-1)_1$ or $(2k)_2$, the minimum part is 1_1 or 2_2, and the weighted difference of any two consecutive parts is 0. Then,

$$\sum_{\nu=0}^{\infty} A_3(\nu)q^\nu = \Phi_0(q). \tag{1.7}$$

Theorem 4. For $\nu \geq 1$, let $A_4(\nu)$ denote the number of n-color partitions of ν such that only the first copy of the odd parts and the second copy of the even parts are used, the minimum part is 1_1, and the weighted difference of any two consecutive parts is 0. Then,

$$\sum_{\nu=1}^{\infty} A_4(\nu) = \Phi_1(q). \tag{1.8}$$

It was pointed out in [3] that the number of n-color partitions of a positive integer ν equals the number of plane partitions of ν. A plane partition π of n is an array

$$\begin{array}{|cccc} \hline n_{1,1} & n_{1,2} & n_{1,3} & \cdots \\ n_{2,1} & n_{2,2} & n_{2,3} & \cdots \\ . & . & . & \\ . & . & . & \\ . & . & . & \end{array}$$

of non-negative integers which is non increasing along each row and column and such that $\sum n_{i,j} = n$. The non-zero entries $n_{i,j} > 0$ are called the parts of π.
E.A. Bender and D.E. Knuth [6] proved the following:

Theorem [Bender and Knuth]. There is a one-to-one correspondence between plane partitions of ν, on the one hand, and infinite matrices a_{ij} $(i, j \geq 1)$ of non negative integer entries which satisfy,

$$\sum_{r\geq 1} r\{\sum_{i+j=r+1} a_{ij}\} = \nu,$$

on the other.

Note. For the definition and other details of the one-to-one correspondence of this theorem which is denoted by ϕ the reader is referred to [6].

Corresponding to every non negative integer ν we shall call the matrices of the above theorem $BK_\nu-$ matrices (BK for Bender and Knuth). These are infinite matrices but will be represented in the sequel by the largest possible square matrices whose last row (column) is non-zero. Thus, for example, we will represent six BK_3 -matrices by

$$(3), \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We give here three more definitions:

Definition 3. We define a matrix $E_{i,j}$ as an infinite matrix whose $(i, j)-$th entry is 1 and the other entries are all zeros. We call $E_{i,j}$ distinct units of a $BK_\nu-$matrix.

Definition 4. In the set of all units the order is defined as follows:
If $r + s < p + q$ then $E_{r,s} < E_{p,q}$ and if $r + s = p + q$ then $E_{r,s} < E_{p,q}$ when $r < p$. Thus the units satisfy the order:
$E_{1,1} < E_{1,2} < E_{2,1} < E_{1,3} < E_{2,2} < E_{3,1} < E_{1,4} < E_{2,3} < E_{3,2} < \cdots$.

Definition 5. The order difference of two units $E_{p,q},\ E_{r,s}(p + q \geq r + s)$ is defined by $q - s - 2r$ and is denoted by $[[E_{p,q} - E_{r,s}]]$.

In [2] we established the following bijection between n-color partitions and BK_ν-matrices:

Let $\Delta = a_{1,1}E_{1,1} + a_{1,2}E_{1,2} + ... + a_{2,1}E_{2,1} + a_{2,2}E_{2,2} + ... + a_{3,1}E_{3,1} + a_{3,2}E_{3,2} + ...$
be *a* $BK_\nu-$ matrix, where $a_{i,j}$ are non negative integers which denote the multiplicities of $E_{i,j}$. We map each part $E_{p,q}$ of Δ to a single part m_i of an n-color partition of ν. The mapping ψ is

$$\psi : E_{p,q} \longrightarrow (p + q - 1)_p \tag{1.9}$$

and the inverse mapping ψ^{-1} is easily seen to be

$$\psi^{-1} : m_i \longrightarrow E_{i,m-i+1}. \tag{1.10}$$

Remark. Clearly for a non-negative integer 'a', we have

$$\psi(aE_{p,q}) = a\psi(E_{p,q}),$$

and

$$\psi^{-1}(am_i) = a\psi^{-1}(m_i).$$

Under this mapping we see that each $BK_\nu-$matrix uniquely corresponds to an n-colour partition of ν and vice-versa.

We note that the composite of the two mappings ϕ and ψ denoted by $\psi.\phi$ is a bijection between plane partitions of ν, on the one hand, and the n-color partitions of ν on the other.

Using the above bijection we propose here to prove the following new combinatorial interpretations of (1.1)-(1.4), respectively,:

Theorem 5. For $\nu \geq 1$, let $C_1(\nu)$ denote the number of BK_ν-matrices such that even columns are zero, 1 is an entry in the first column and the order difference between any two consecutive units $E_{p,q}$ and $E_{r,s}$, $(p+q \geq r+s)$ is zero. Then

$$\sum_{\nu=0}^{\infty} C_1(\nu) q^\nu = \Psi(q). \tag{1.11}$$

Example. $C_1(6) = 2$. The relevant matrices are

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Theorem 6. For $\nu \geq 0$, let $C_2(\nu)$ denote the number of BK_ν-matrices such that even columns are zero, first row is zero, 1 is an entry in the first column and the order difference between any two consecutive units is zero. Then

$$\sum_{\nu=0}^{\infty} C_2(\nu) q^\nu = F_0(q). \tag{1.12}$$

Theorem 7. For $\nu \geq 0$, let $C_3(\nu)$ denote the number of BK_ν-matrices such that even columns are zero, all rows after the second row are zero, first column is non zero, and the order difference between any two consecutive units is zero. Then

$$\sum_{\nu=0}^{\infty} C_3(\nu) q^\nu = \Phi_0(q). \tag{1.13}$$

Theorem 8. For $\nu \geq 1$, let $C_4(\nu)$ denote the number of BK_ν-matrices such that even columns are zero, all rows after the second row are zero, first element in the first row is 1 and in the second row it is 0, and the order difference between any two consecutive units is zero. Then

$$\sum_{\nu=1}^{\infty} C_4(\nu) q^\nu = \Phi_1(q). \tag{1.14}$$

2 Proofs of Theorems 5-8

For proving Theorem 5 we use Theorem 1. In view of the bijection ψ given by (1.9) above, we have only to prove that if a BK_ν- matrix Δ is enumerated by $C_1(\nu)$, then in the n-color partition $\psi(\Delta)$ the even parts appear with even subscripts and odd with odd, for some k, k_k is a part, and the weighted difference of any two consecutive parts is 0, and conversely, if an n-color partition π is enumerated by $A_1(\nu)$ then in the BK_ν-matrix $\psi^{-1}(\pi)$ even columns are zero, 1 is an entry in the first column and the order difference of any two consecutive units is zero.
Let $E_{p,q}$ and $E_{r,s}$, $(p+q \geq r+s)$ be any two consecutive units of a BK_ν-matrix Δ enumerated

by $C_1(\nu)$ which correspond to two consecutive parts m_i, n_j of $\psi(\Delta)$. Then

$m_i = (p+q-1)_p$ and $n_j = (r+s-1)_r$.

Since $p+q \geq r+s$, therefore $m \geq n$ and

$$\begin{aligned} ((m_i - n_j)) &= (p+q-1) - p - (r+s-1) - r \\ &= q - s - 2r \\ &= [[E_{p,q} - E_{r,s}]] \\ &= 0. \end{aligned}$$

Since in $E_{p,q}, q$ is odd, we see that $p+q-1$ and p have the same parity which implies that in $\psi(\Delta)$ even parts appear with even subscripts and odd with odd. Furthemore, since for some p, $E_{p,1}$ is non-zero, this implies that p_p is a part of $\psi(\Delta)$, that is, for some k, k_k is a part of $\psi(\Delta)$.
Next, suppose π is an n-color partition enumerated by $A_1(\nu)$ and $m_i, n_j (m \geq n)$ are its two consecutive parts which correspond to two consecutive units $E_{p,q}$ and $E_{r,s}$ of $\psi^{-1}(\pi)$. Then

$E_{p,q} = E_{i,m-i+1}$ and $E_{r,s} = E_{j,n-j+1}$.

Since $m \geq n$, we have

$$p+q = m+1 \geq n+1 = r+s,$$

and

$$\begin{aligned} [[E_{p,q} - E_{r,s}]] &= [[E_{i,m-i+1} - E_{j,n-j+1}]] \\ &= (m-i+1) - (n-j+1) - 2j \\ &= m - n - i - j \\ &= ((m_i - n_j)) \\ &= 0. \end{aligned}$$

Since, in π even parts appear with even subscripts and odd with odd, we see that $m-i+1$ and $n-j+1$ are both odd. This implies that in $\psi^{-1}(\pi)$ all even columns are zero. Furthermore, the part k_k of π corresponds to the unit $E_{k,1}$ which implies that the first column of $\psi^{-1}(\pi)$ does contain 1. This completes the proof of Theorem 5.
Theorems 6-8 can be proved similarly hence their proofs are omitted.

Remark

It would be of interest to provide combinatorial interpretations for other mock theta functions also by using the technique of this paper.

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References

1. A.K. Agarwal, n-color partition theoretic interpretations of some mock theta functions, Electron. J. Combin. 11(2004), no. 1, Note 14, 6pp (electronic)
2. A.K. Agarwal, n-Color analogues of Gaussian polynomials, ARS Combinatoria, 61(2001), 97-117.
3. A.K. Agarwal and G.E. Andrews, Rogers-Ramanujan identities for partitions with "N copies of N", J. Combin. Theory Ser. A, 45(1),(1987), 40-49.
4. G.E. Andrews, Mock theta functions in Theta Functions Bowdoin, Part2, Proceedings of Symposia in Pure Mathematics, Vol.49, American Mathematical Society, Providence, Rhode Island, 1989, pp. 283-296.
5. G.E. Andrews and D. Hickerson, Ramanujan's "Lost" Notebook VII: The sixth order mock theta functions, Adv. Math.,89(1991), 60-105.
6. E.A. Bender and D.E. Knuth, Enumeration of plane partitions, J. Combin. Theory(A), 13(1972), 40-54.
7. K. Bringmann and K. Ono, The $f(q)$ mock theta function conjecture and partition ranks, Invent. math. 165(2006), 243-266.
8. B. Gordon and R.J. McIntosh, Some eight order mock theta functions, J. London Math. Soc.(2)62(2000), 321-335.
9. N.J. Fine, Basic Hypergeometric Series and Applications, Mathematical Surveys and Monographs,No. 27, AMS,(1988)
10. G.H. Hardy, P.V. SeshuAiyar and B.M. Wilson, Collected papers of Srinivasa Ramanujan, Cambridge Univ. Press, 1927.
11. S. Ramanujan, The Lost Notebook and other Unpublished Papers, Narosa Publishing House, New Delhi, 1988
12. G.N. Watson, The final problem: an account of the mock theta functions, J. London Math. Soc.,11(1936), 55-80.