Compact ultrametric spaces generated by labeled star graphs

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ABSTRACT

Let *US* be the class of all ultrametric spaces generated by labeled star graphs. We prove that compact *US*-spaces are the completions of totally bounded ultrametric spaces generated by decreasingly labeled rays. We characterize the ultrametric spaces which are weakly similar to finite *US*-spaces and describe these spaces by certain four-point conditions.

Keywords: compact ultrametric space, four-point condition, labeled star graph, weak similarity

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1. Introduction

As is known, any finite ultrametric space can be described up to isometry by a Gurvich–Vyalyi representing tree [31]. This representation and its geometric interpretation [36] allow us to find solutions of different extremal problems related to such spaces [17, 18, 26]. Recently, analogues of the Gurvich–Vyalyi representation and its geometric interpretation were also found for totally bounded ultrametric spaces in [20]. The above mentioned representing trees form a special class of trees endowed with vertex labelings. Some important properties of infinite trees endowed with positive edge labelings have been described in [2, 3, 4, 8, 9, 10, 11, 12, 13, 14, 16, 24].

The ultrametric spaces generated by arbitrary nonnegative vertex labelings on both finite and infinite trees were first considered in [19] and studied in [21, 23]. The simplest types of infinite trees are rays and star graphs. The totally bounded ultrametric spaces generated by labeled almost rays have been characterized in [28]. Furthermore, paper [27]

contains a purely metric characterization of ultrametric spaces generated by labeled star graphs.

The main goal of this paper is to give a metric description of compact ultrametric spaces generated by labeled star graphs.

The paper is organized as follows. In Section 2 we collect together some definitions and facts related to ultrametric spaces and trees.

Section 3 contains the formulations of two problems that initially motivate our study.

The finite ultrametric spaces generated by labeled stars are considered in Section 4. Theorem 4.3 shows that Conjecture 4.1 of [27] is true. A semimetric modification of Theorem 4.3 is given in Theorem 4.7.

Theorem 5.1, the first result of Section 5, describes up to isomorphism all labeled star graphs that generate compact ultrametric spaces. Compact ultrametric spaces generated by labeled star graphs are described up to isometry in Theorem 5.2. In Theorem 5.3 we prove that Conjecture 4.2 of paper [27] is true.

Our last result, Theorem 5.6, shows that the completions of totally bounded ultrametric spaces generated by decreasingly labeled rays coincide with compact ultrametric spaces generated by labeled star graphs.

The final Section 6 contains two new conjectures.

2.   Basic definitions and preliminary results

Let us start from the fundamental concept of semimetric spaces introduced by M. Fréchet in [30].

Definition 2.1. Let $\mathbb{R}^+ = [0, \infty)$. A *semimetric* on a non-empty set X is a symmetric function $d \colon X \times X \to \mathbb{R}^+$ such that $d(x, y) = 0$ if and only if $x = y$. A semimetric d is called a *metric* if the *triangle inequality*

$$d(x, y) \leq d(x, z) + d(z, y),$$

holds for all $x, y, z \in X$. If we have the *strong triangle inequality*,

$$d(x, y) \leq \max\{d(x, z), d(z, y)\},$$

for all $x, y, z \in X$, then d is an *ultrametric*.

The object of our study is a certain family of compact, and in particular finite, ultrametric spaces. A standard definition of compactness is usually formulated as: A subset S of a metric space is *compact* if each open cover of S has a finite subcover.

There exists a simple interdependence between the compactness of a set and the so-called convergent sequences. Let us remember that a sequence $(x_n)_{n \in \mathbb{N}}$ of points in a metric space (X, d) is said to converge to a point $a \in X$ if

$$\lim_{n \to \infty} d(x_n, a) = 0. \tag{1}$$

When (1) holds, we write

$$\lim_{n \to \infty} x_n = a.$$

Recall that a point $a \in X$ is a *limit point* (or equivalently, an *accumulation point*) of a set $A \subseteq X$ if there is a sequence $(a_n)_{n \in \mathbb{N}}$ of different points of A such that

$$\lim_{n \to \infty} a_n = a.$$

Proposition 2.2. *A subset A of a metric space is compact if and only if every infinite sequence of points of A contains a subsequence which converges to a point of A.*

For the proof of Proposition 2.2 see, for example, Theorem 12.1.3 in [38].

Let (X, d) be a semimetric space. An *open ball* with a *radius $r > 0$* and a *center $c \in X$* is the set

$$B_r(c) = \{x \in X : d(c, x) < r\}.$$

A subset O of a metric space (X, d) is called *open* if for every point $p \in O$ there is an open ball B such that

$$p \in B \subseteq O.$$

Proposition 2.3. *Let (X, d) be a compact metric space. Then, for every open subset O of X, the set $X \setminus O$ is also compact.*

The last proposition follows directly from Theorem 12.2.3 of [38].

Definition 2.4. A subset A of a metric space (X, d) is called *totally bounded* if for every $r > 0$ there is a finite set $\{x_1, \ldots, x_n\} \subseteq X$ such that

$$A \subseteq \bigcup_{i=1}^{n} B_r(x_i).$$

Recall also that a sequence $(x_n)_{n \in \mathbb{N}}$ of points in a metric space (X, d) is a *Cauchy sequence* iff

$$\lim_{\substack{n \to \infty \\ m \to \infty}} d(x_n, x_m) = 0.$$

Proposition 2.5. *A subset A of a metric space (X, d) is totally bounded if and only if every infinite sequence of points in A contains a Cauchy subsequence.*

See, for example, Theorem 7.8.2 in [38].

Let (X, d) and (Y, ρ) be semimetric spaces. Recall that a bijective mapping $\Phi : X \to Y$ is an *isometry* if the equality

$$d(x, y) = \rho(\Phi(x), \Phi(y)),$$

holds for all $x, y \in X$.

Let (X, d) be a metric space and let $A \subseteq X$. A subset S of A is said to be *dense* in A if for every $a \in A$ there is a sequence $(s_n)_{n \in \mathbb{N}}$ of points of S such that

$$a = \lim_{n \to \infty} s_n.$$

A metric space (X, d) is *complete* if every Cauchy sequence of points of X converges to a point of X.

Definition 2.6. Let (X, d) be a metric space. A metric space (Y, ρ) is called a *completion* of (X, d) if (Y, ρ) is complete and (X, d) is isometric to dense subspace of (Y, ρ).

Proposition 2.7. *The completion* (Y, ρ) *of a metric space* (X, d) *is compact if and only if* (X, d) *is totally bounded.*

For the proof, see, for example, Corollary 4.3.30 in [29].

Let (X, d) be a semimetric space. Below we denote by $D(X)$ the set of all distances between points of (X, d),

$$D(X) := \{d(x, y) : x, y \in X\}. \tag{2}$$

The next definition gives us a generalization of the concept of isometry.

Definition 2.8. Let (X, d) and (Y, ρ) be semimetric spaces. A bijective mapping $\Phi \colon X \to Y$ is a *weak similarity* if there is a strictly increasing bijection $f \colon D(Y) \to D(X)$ such that the equality

$$d(x, y) = f\left(\rho\left(\Phi(x), \Phi(y)\right)\right),$$

holds for all $x, y \in X$.

Remark 2.9. The notion of weak similarity was introduced in [25].

We will say that semimetric spaces (X, d) and (Y, ρ) are weakly similar if there is weak similarity $X \to Y$.

The next proposition follows directly from Definition 2.1 and Definition 2.8.

Proposition 2.10. *Let* (X, d) *be an ultrametric space and* (Y, ρ) *be a semimetric space. If* (X, d) *and* (Y, ρ) *are weakly similar, then* (Y, ρ) *is also an ultrametric space.*

Let us now recall some concepts from graph theory.

A *simple graph* is a pair (V, E) consisting of a non-empty set V and a set E whose elements are unordered pairs $\{u, v\}$ of different points $u, v \in V$. For a graph $G = (V, E)$, the sets $V = V(G)$ and $E = E(G)$ are called the *set of vertices* and the *set of edges*, respectively. A graph G is *finite* if $V(G)$ is a finite set. A graph H is, by definition, a *subgraph* of a graph G if the inclusions $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ hold. In this case, we simply write $H \subseteq G$.

In what follows, we will use the standard definitions of paths and cycles, see, for example, [7, Section 1.3]. A graph G is *connected* if for every two distinct $u, v \in V(G)$ there is

a path P joining u and v in G,

$$u, \, v \in V(P) \text{ and } P \subseteq G.$$

A connected graph without cycles is called a *tree*.

Definition 2.11. A tree T is a *star graph* if there is a vertex $c \in V(T)$, the *center* of T, such that c and v are adjacent for every $v \in V(T) \setminus \{c\}$ but for all $u, w \in T$ we have $\{u, w\} \notin E(T)$ whenever $u \neq c$ and $w \neq c$.

An infinite graph R of the form

$$V(R) = \{x_1, x_2, \ldots, x_n, x_{n+1}, \ldots\}, \tag{3}$$
$$E(R) = \{\{x_1, x_2\}, \ldots, \{x_n, x_{n+1}\}, \ldots\}, \tag{4}$$

where all x_n are assumed to be distinct, is called a *ray* [7]. If (3) and (4) hold, we will write $R = (x_1, x_2, \ldots, x_n, \ldots)$. It is clear that every ray is a tree.

In what follows, by a *labeled* tree $T(l)$ we will mean a tree T equipped with a labeling $l\colon V(T) \to \mathbb{R}^+$.

Let $T(l)$ be a labeled tree. As in [19] we define a mapping $d_l\colon V(T) \times V(T) \to \mathbb{R}^+$ by

$$d_l(u, v) = \begin{cases} 0, & \text{if } u = v, \\ \max_{w \in V(P)} l(w), & \text{otherwise,} \end{cases} \tag{5}$$

where P is the path joining u and v in T.

The following result is a direct corollary of Proposition 3.2 from [19].

Theorem 2.12. *Let $T = T(l)$ be a labeled tree. Then the function d_l is an ultrametric on $V(T)$ if and only if the inequality*

$$\max\{l(u), l(v)\} > 0,$$

holds for every edge $\{u, v\}$ of T.

In what follows, we also will say that an ultrametric space (X, d) is *generated by a labeled tree $T(l)$* if the equalities $X = V(T)$ and $d = d_l$ hold, where d_l is defined as in (5).

Definition 2.13. An ultrametric space (X, d) belongs to the class *US* if (X, d) is generated by some labeled star graph.

The following theorem was proved in [27].

Theorem 2.14. *Let (X, d) be an ultrametric space. Then the following statements are equivalent:*
(i) $(X, d) \in US$.

(ii) *There is $x_0 \in X$ such that*

$$d(x_0, x) \leq d(y, x), \tag{6}$$

whenever $x, y \in X$ and

$$x_0 \neq x \neq y. \tag{7}$$

The next proposition can be considered as a partial refinement of Theorem 2.14.

Proposition 2.15. *Let (X, d) be an ultrametric space and let $x_0 \in X$. If, for $x, y \in X$, inequality (6) holds whenever we have (7), then (X, d) is generated by a labeled star graph $S(l)$ with the center x_0 and the labeling $l : X \to \mathbb{R}^+$ defined as*

$$l(x) = d(x, x_0),$$

for each $x \in X$.

The proof of this proposition coincides with the second part of the proof of Theorem 2.1 in [27].

3.   Motivating problems

The four-point ultrametric spaces (X_4, d_4) and (Y_4, ρ_4) depicted in Figure 1 below do not belong to US.
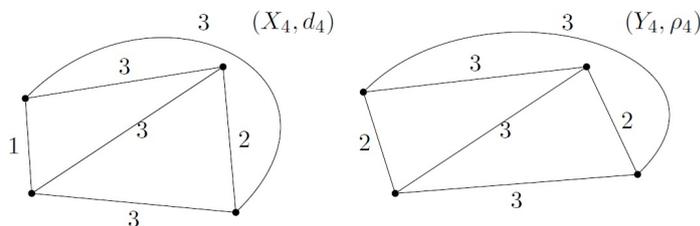

Fig. 1. (X_4, d_4) and (Y_4, ρ_4) are not US-spaces by Theorem 2.14

The next conjectures were formulated in [27].

Conjecture 3.1. *The following statements are equivalent for every finite ultrametric space (X^*, d^*):*
   (i) $(X^*, d^*) \notin US$.
   (ii) (X^*, d^*) *contains a four-point subspace which is weakly similar either to (X_4, d_4) or to (Y_4, ρ_4).*

Conjecture 3.2. *Let (X, d) be an infinite US-space generated by a labeled star graph with a center c, let $X_0 := X \setminus \{c\}$, and let d_0 be the restriction of d on the set $X_0 \times X_0$. Then the following statements are equivalent:*
   (i) (X, d) *is compact.*

(ii) (X_0, d_0) *is generated by a labeled ray* $R(l)$, $R = (x_1, x_2, \ldots, x_n, \ldots)$, *such that*

$$\lim_{n \to \infty} l(x_n) = 0,$$

and

$$l(x_n) \geq l(x_{n+1}) > 0,$$

for every $n \in \mathbb{N}$.

4. Finite US-spaces

The main goal of the present section is to describe finite US-spaces by the above four-point condition.

Let (X, d) be an ultrametric space generated by the labeled star graph $S(l)$ with a center c and let Y be a subset of X such that

$$c \in Y. \tag{8}$$

Then using Definition 2.11 and formula (5) with $T(l) = S(l)$ it is easy to see that

$$(Y, d|_{Y \times Y}) \in US, \tag{9}$$

where $d|_{Y \times Y}$ is the restriction of the ultrametric d on the set $Y \times Y$.

The following example shows that (9) can be false if $c \notin Y$.

Example 4.1. Let (X, d) be the ultrametric space generated by the labeled star graph $S(l)$ depicted in Figure 2 and let

$$Y := X \setminus \{c\}.$$

Then, using Theorem 2.14, we obtain

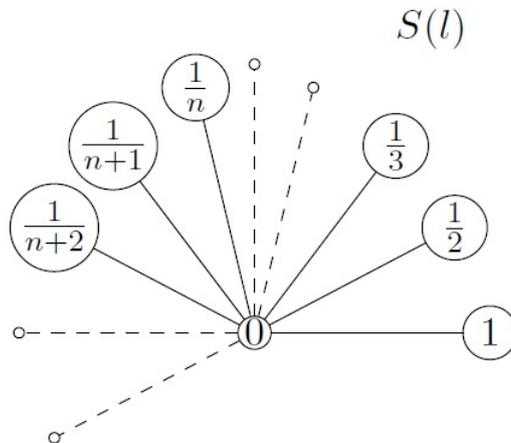$$(Y, d|_{Y \times Y}) \notin US.$$

Fig. 2. The unique point x_0 satisfying (6) for all $x, y \in Y$ which satisfy (7), is the center of the labeled star graph $S(l)$

In the case of finite Y, condition (8) is not necessary for membership (9).

Proposition 4.2. *Let (X,d) be an US-space. Then $(Y, d|_{Y \times Y})$ is also an US-space for every finite non-empty $Y \subseteq X$.*

Proof. Let $S(l)$ be a labeled star graph generating (X, d), let c be the center of S, and let Y be a finite non-empty subset of X. Since Y is finite, there is a point $y^* \in Y$ such that

$$l(y^*) \leq l(y), \tag{10}$$

for each $y \in Y$. Now, using Definition 2.11 and formula (5) with $T(l) = S(l)$ we obtain the equality

$$d(y_i, y_j) = \max\{l(c), l(y_i), l(y_j)\},$$

for all different $y_i, y_j \in Y$. The last equality and (10) give us

$$d(y^*, y_i) \leq d(y_j, y_i), \tag{11}$$

whenever $y^* \neq y_i \neq y_j$. Hence $(Y, d|_{Y \times Y}) \in US$ holds by Theorem 2.14.

$\square$

The following theorem shows, in particular, that Conjecture 3.1 is true.

Theorem 4.3. *Let (X, d) be a nonempty ultrametric space. Suppose that either X is finite, or X has a limit point. Then (X, d) is an US-space if and only if (X, d) contains no four-point subspace which is weakly similar to (X_4, d_4) or to (Y_4, ρ_4).*

Proof. The theorem is trivially true if

$$\operatorname{card} X \leq 3.$$

Let us consider the case when

$$\operatorname{card} X \geq 4.$$

Suppose that $(X, d) \in \mathbf{US}$. Then, by Theorem 2.14 there is a point $x_0 \in X$ such that $d(x, y) = \max\{d(x, x_0), d(y, x_0)\}$ for all $x, y \in X$. Let us define $l : X \to \mathbb{R}^+$ by $l(x) = d(x, x_0)$ for every $x \in X$. For all distinct $x, y, z, w \in X$ we have

$$d(x, y) = \max\{l(x), l(y)\} \leq \max\{l(x), l(y), l(z), l(w)\}$$
$$= \max\{d(x, z), d(y, w)\},$$

and, in particular, $\{x, y, z, w\}$ is not weakly similar to (X_4, d_4) or (Y_4, ρ_4).

Now, suppose that $(X, d) \notin \mathbf{US}$. Define $l : X \to \mathbb{R}^+$ by

$$l(x) = \inf_{y \neq x} d(x, y).$$

If $d(x, y) = \max\{l(x), l(y)\}$ for all distinct $x, y \in X$, we can choose $x_0 \in X$ such that $l(x_0)$ is minimal (if X has a limit point x_0, $l(x_0) = 0$ is minimal) and then $(X, d) \in \mathbf{US}$ by Theorem 2.14. Therefore, there are distinct $x, y \in X$ such that

$$d(x, y) \neq \max\{l(x), l(y)\}.$$

Clearly, $\max\{l(x), l(y)\} \leq d(x, y)$ by definition of l. Since $x \neq y$, we have $\max\{l(x), l(y)\} < d(x, y)$. Therefore, there are $z, w \in X$ such that $z \neq x$, $d(x, z) < d(x, y)$, $w \neq y$, and $d(y, w) < d(x, y)$. It is also clear that the relations $z \neq y$ and $w \neq x$ and $z \neq w$ hold, since otherwise we would have $d(x, y) > \max\{d(x, z), d(z, y)\}$. Hence, x, y, z, w are distinct. We have

$$d(x, y) = d(x, w) = d(z, y) = d(z, w),$$

by the strong triangle inequality. It follows that $\{x, y, z, w\}$ is weakly similar to either (X_4, d_4) (if $d(x, z) \neq d(y, w)$) or to (Y_4, ρ_4) (if $d(x, z) = d(y, w)$).

The proof is completed. $\qquad\square$

The following lemma easily follows from Theorem 4.1 of [37], see also Theorem 4.4 and Proposition 4.1 in [21].

Lemma 4.4. *Let (X, d) be a four-point ultrametric space. Then the following statements are equivalent:*
  (i) *(X, d) is generated by a labeled tree.*
  (ii) *There is a point $x_0 \in X$ such that*

$$d(x_0, x) = \max\{d(p, q) : p, q \in X\},$$

for each $x \in X \setminus \{x_0\}$.

Theorem 4.3 and Lemma 4.4 give us the following.

Corollary 4.5. *Let (X, d) be a finite ultrametric space with*

$$\operatorname{card} X \leq 4. \tag{12}$$

Then the following statements are equivalent:
  (i) *$(X, d) \in US$.*
  (ii) *There is a labeled tree $T(l)$ such that (X, d) is generated by $T(l)$.*

The next example shows that the number 4 is the best possible constant in inequality (12).

Example 4.6. The five-point ultrametric spaces generated by labeled trees $T_X(l_X)$ and $T_Y(l_Y)$ (see Figure 3 below) do not belong to US.

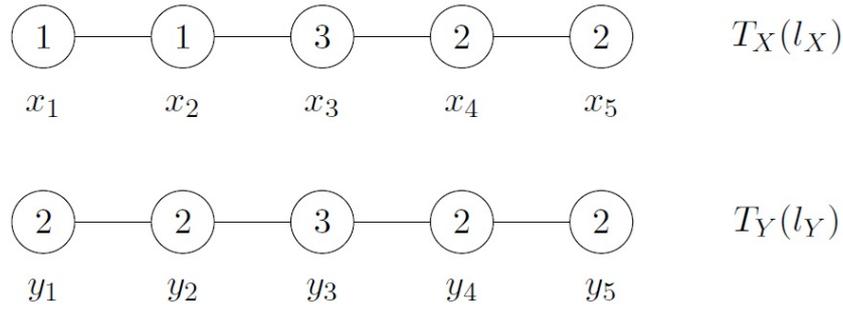The next result is a semimetric modification of Theorem 4.3.

Fig. 3. The ultrametric space $(V(T_X), d_{l_X})$ contains the four-point subspace $\{x_1, x_2, x_4, x_5\}$ which is isometric to (X_4, d_4). Similarly, $(V(T_Y), d_{l_Y})$ contains the subspace $\{y_1, y_2, y_4, y_5\}$ which is isometric to (Y_4, ρ_4)

Theorem 4.7. *Let (X, d) be a finite semimetric space with*

$$\operatorname{card} X \neq 3. \tag{13}$$

Then the following statements are equivalent:

(i) $(X, d) \in US$.

(ii) *Each four-point subspace of (X, d) is weakly similar to an US-space.*

(iii) *Each four-point subspace of (X, d) is weakly similar to an ultrametric space generated by a labeled tree.*

Proof. (i) $\Rightarrow$ (ii). The validity of this implication follows from Proposition 4.2.

(ii) $\Rightarrow$ (iii). Each labeled star graph is a labeled tree. Thus, implication (ii) $\Rightarrow$ (iii) is true.

(iii) $\Rightarrow$ (i). Let (iii) hold. It follows from Definition 2.1 and condition (13) that the semimetric space (X, d) is ultrametric iff every four-point subspace of (X, d) is ultrametric.

Therefore, by Proposition 2.10, each of statements (ii) and (iii) implies that (X, d) is ultrametric. Now, using Corollary 4.5, we see that statements (ii) and (iii) are logically equivalent.

To complete the proof, it suffices to note that, for the ultrametric space (X, d), statement (ii) implies statement (i) by Theorem 4.3.

$\square$

The following example shows that restriction (13) is really necessary.

Example 4.8. Let (X, d) be a three-point space consisting of the vertices of a right triangle on the Euclidean plane. Then statements (ii) and (iii) of Theorem 4.7 are vacuously true, but statement (i) of this theorem is false.

5. Compact US-spaces

Let us start with a characterization of infinite labeled star graphs $S(l)$ which generate compact *US*-spaces.

Theorem 5.1. *Let (X, d) be an infinite US-space generated by a labeled star graph $S(l)$ with a center c. Then the following statements are equivalent:*

(i) *(X, d) is compact.*

(ii) *The equality*

$$l(c) = 0, \tag{14}$$

holds and the set

$$A_\varepsilon := \{x \in X : l(x) \geq \varepsilon\}, \tag{15}$$

is finite for each $\varepsilon > 0$.

Proof. (i)$\Rightarrow$ (ii). Let (X, d) be compact. If (14) does not hold, then we have

$$l(c) = t_0,$$

for some $t_0 > 0$.

Definition 2.11 and formula (5) with $T(l) = S(l)$ give us the inequality

$$d_l(u, v) \geq t_0, \tag{16}$$

for all distinct $u, v \in X$.

Let $(u_n)_{n \in \mathbb{N}}$ be an arbitrary sequence of distinct points of X. Since (16) holds for all distinct $u, v \in X$, the sequence $(u_n)_{n \in \mathbb{N}}$ has no convergent subsequence. Thus, (X, d) is not compact by Proposition 2.2.

Suppose now that the set A_{ε_0} is infinite for some $\varepsilon_0 > 0$. Then there is a sequence $(a_n)_{n \in \mathbb{N}}$ of distinct points $a_n \in A$. As above, using formula (5) and Definition 2.11 we see that the inequality

$$d(u_n, u_m) \geq \varepsilon_0,$$

holds for all distinct $n, m \in \mathbb{N}$. Thus, $(u_n)_{n \in \mathbb{N}}$ has no convergent subsequence, and, consequently, (X, d) is not compact by Proposition 2.2.

(ii)$\Rightarrow$ (i). Let (ii) hold and let $X_0 := X \setminus \{c\}$. Then equality (14) and Theorem 2.12 imply the inequality

$$l(x) > 0, \tag{17}$$

for every $x \in X_0$. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a strictly decreasing sequence of positive real numbers such that

$$\lim_{n \to \infty} \varepsilon_n = 0. \tag{18}$$

Then from limit relation (18) and inequality (17) we have the equality

$$X_0 = \bigcup_{n \in \mathbb{N}} A_{\varepsilon_n}. \tag{19}$$

Now using the finiteness of A_ε and formulas (15), (19) we see that X_0 is countably infinite and there is a bijection between $\mathbb{N}$ and X_0 given by a sequence $(x_n)_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} l(x_n) = 0, \tag{20}$$

and

$$0 < l(x_{n+1}) \leq l(x_n), \tag{21}$$

for each $n \in \mathbb{N}$. Now, using (5) and (14) we obtain the equality

$$d(c, x_n) = l(x_n), \tag{22}$$

for every $n \in \mathbb{N}$. Equalities (20) and (22) give us

$$\lim_{n \to \infty} d(c, x_n) = 0.$$

Thus, $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence in (X, d).

Let us consider an arbitrary sequence $(y_m)_{m \in \mathbb{N}}$ of points of X. According to Proposition 2.2, to complete the proof it is sufficient to verify that $(y_m)_{m \in \mathbb{N}}$ contains a convergent subsequence. The existence of such subsequence is obvious if the range of $(y_m)_{m \in \mathbb{N}}$ is finite. Otherwise, using the finiteness of the set A_ε for all $\varepsilon > 0$, we can find a subsequence $(y_{m_k})_{k \in \mathbb{N}}$ of $(y_m)_{m \in \mathbb{N}}$ such that

$$d(c, y_{m_k}) = l(y_{m_k}) < \varepsilon_k,$$

for all $k \in \mathbb{N}$. Equality (18) then implies that

$$\lim_{n \to \infty} d(c, y_{m_k}) = 0.$$

Thus, $(y_{m_k})_{k \in \mathbb{N}}$ is a convergent subsequence of $(y_m)_{m \in \mathbb{N}}$.
The proof is completed.

$\square$

The following theorem is a corollary of Theorem 4.3.

Theorem 5.2. *Let (X, d) be a compact ultrametric space. Then (X, d) is an US-space if and only if (X, d) contains no four-point subspace which is weakly similar to (X_4, d_4) or to (Y_4, ρ_4).*

Proof. Theorem 4.3 holds for every nonempty ultrametric space, except the infinite discrete spaces. The compact space (X, d) cannot be infinite and discrete. Thus, Theorem 4.3 gives us the desired result.

$\square$

The next theorem shows that Conjecture 3.2 is true.

Theorem 5.3. *Let (X, d) be an infinite US-space generated by a labeled star graph $S(l)$ with a center c, let $X_0 := X \setminus \{c\}$, and let d_0 be the restriction of d on the set $X_0 \times X_0$. Then the following statements are equivalent:*
  (i) (X, d) is compact.
  (ii) (X_0, d_0) is generated by a labeled ray $R(l^*)$, $R = (x_1, x_2, \ldots, x_n, \ldots)$, such that

$$\lim_{n \to \infty} l^*(x_n) = 0,$$

and

$$l^*(x_n) \geq l^*(x_{n+1}) > 0,$$

for every $n \in \mathbb{N}$.

Proof. We will prove that conditions (i) and (ii) are both equivalent to a third condition (iii), which states that the set $\{x \in X : d(x, c) \geq \varepsilon\}$ is finite for every $\varepsilon > 0$.

Suppose that condition (iii) does not hold. Then there are a positive number $\varepsilon > 0$ and an infinite sequence $(x_n)_{n \in \mathbb{N}}$ of distinct points $x_n \in X$ such that $d(x_n, c) \geq \varepsilon$ for all $n \in \mathbb{N}$. It follows from formula (5) with $T(l) = S(l)$ that $d(x_n, y) \geq \varepsilon$ holds for all $n \in \mathbb{N}$ and $y \in X \setminus \{x_n\}$. Thus, the sequence $(x_n)_{n \in \mathbb{N}}$ has no convergent subsequence, so condition (i) does not hold by Proposition 2.2.

Suppose that condition (ii) holds. Let $\varepsilon > 0$. Then for all but finitely many $n \in \mathbb{N}$ we have $l^*(x_n) < \varepsilon$, since $\lim\limits_{n \to \infty} l^*(x_n) = 0$. It follows from Proposition 2.15 and (ii) that $d(x_n, c) \leq d(x_n, x_{n+1}) = l^*(x_n)$ for all $n \in \mathbb{N}$, so condition (iii) holds.

Suppose that condition (iii) holds. Every open ball around c is cofinite, so every open set containing c is cofinite. This easily implies condition (i). Every $Y \subseteq X$ also satisfies condition (iii), so for given $y_* \in Y \setminus \{c\}$ the set $\{y \in Y : d(y, c) \geq d(y_*, c)\}$ is finite. This implies that if $\emptyset \neq Y \subseteq X$, then there exists $y \in Y$ such that $d(y, c)$ is maximal. We define a sequence x_n of points in X_0 inductively. For all $n \in \mathbb{N}$, let x_n be a point $x \in X_0 \setminus \{x_1, \ldots, x_{n-1}\}$ such that $d(x, c)$ is maximal ($X_0 \setminus \{x_1, \ldots, x_{n-1}\} \neq \emptyset$, since X is infinite). This is a sequence of distinct points by construction, in particular, its range is infinite. Given $x \in X_0$, the set $\{y \in X : d(y, c) \geq d(x, c)\}$ is finite, so $d(x_n, c) < d(x, c)$ for some $n \in \mathbb{N}$. Hence $x \in \{x_1, \ldots, x_{n-1}\}$ by the definition of x_n. Therefore, the equality $X_0 = \{x_n : n \in \mathbb{N}\}$ holds. Let us define $l^* : X_0 \to \mathbb{R}^+$ by $l^*(x) = d(x, c)$. We have the inequality $l^*(x_n) \geq l^*(x_{n+1})$ by the definition of x_n, and $\lim\limits_{n \to \infty} l^*(x_n) = 0$ follows from condition (iii) since the points x_n are pairwise distinct. If $m < n$, then $d(x_m, x_n) = \max\{l^*(x_m), l^*(x_n)\}$ since $(X, d) \in \mathbf{US}$, so since $(l^*(x_n))_{n \in \mathbb{N}}$ is a decreasing sequence, we have $d(x_m, x_n) = \max\limits_{m \leq i \leq n} l^*(x_i)$. Therefore, the labeled ray $R(l^*)$ generates (X_0, d_0), so condition (ii) holds.

The proof is completed. $\square$

The following lemma is a reformulation of Proposition 3.6 from [28].

Lemma 5.4. *Let $R = (x_1, x_2, \ldots)$ be a ray with a labeling $l^* : V(R) \to \mathbb{R}^+$ and let $R(l^*)$ generate an ultrametric space $(V(R), d_{l^*})$. Then the following statements are equivalent:*
  (i) *The sequence $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(V(R), d_{l^*})$.*
  (ii) *Each infinite sequence of points of $(V(R), d_{l^*})$ contains a Cauchy subsequence.*

The next lemma is a direct corollary of Lemma 3.19 from [1].

Lemma 5.5. *Let (X, d) be a compact ultrametric space and let Y be a compact subset of X. If (X, d) is isometric to some subspace of the space $(Y, d|_{Y \times Y})$, then the equality*

$X = Y$ holds.

The next theorem claims that every compact US-space is, up to isometry, the completion of an ultrametric space generated by a labeled ray.

Let $R = (x_1, x_2, \dots)$ be a ray with labeling $l^* : V(R) \to \mathbb{R}^+$. We say that l^* is a *decreasing labeling* if the sequence $(l^*(x_n))_{n \in \mathbb{N}}$, is decreasing.

Theorem 5.6. *Let (X, d) be an infinite ultrametric space. Then the following statements are equivalent:*

(i) (X, d) is a compact US-space.

(ii) (X, d) is the completion of a totally bounded $X_0 \subsetneq X$ generated by a decreasingly labeled ray.

Proof. (i) $\Rightarrow$ (ii). Let (X, d) be a compact US-space, generated by labeled star graph $S(l)$ with the center c. As in Theorem 5.3 we write

$$X_0 := X \setminus \{c\} \text{ and } d_0 := d|_{X_0 \times X_0}. \tag{23}$$

By Theorem 5.3 the space (X_0, d_0) is generated by a labeled ray $R(l^*)$, $R = (x_1, x_2, \dots,)$ such that $l^* : V(R) \to \mathbb{R}^+$ is decreasing and

$$\lim_{n \to \infty} l^*(x_n) = 0, \tag{24}$$

where

$$l^*(x_n) = d(c, x_n), \tag{25}$$

for every $n \in \mathbb{N}$. Using (23)–(25) we see that X_0 is a dense subset of (X, d). Moreover, (X_0, d_0) is totally bounded as a subspace of compact space (X, d). Since every compact ultrametric space is complete, (X, d) is a completion of a totally bounded space $X_0 \subsetneq X$ generated by $R(l^*)$.

(ii) $\Rightarrow$ (i). Let (X, d) be the completion of a totally bounded $X_0 \subsetneq X$ generated by the labeled ray $R(l^*)$, $R = (x_1, x_2, \dots)$ with decreasing labeling $l^* : V(R) \to \mathbb{R}^+$. Then the space (X, d) is compact by Proposition 2.7. Hence, to complete the proof it suffices to show that

$$(X, d) \in US. \tag{26}$$

Proposition 2.5 and Lemma 5.4 imply that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in (X, d). Since every Cauchy sequence of points of compact ultrametric space is a convergent sequence in this space, $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence in (X, d).

Let $x_0 \in X$ be a limit point of the sequence $(x_n)_{n \in \mathbb{N}}$,

$$\lim_{n \to \infty} x_n = x_0. \tag{27}$$

Then the set $V(R) \cup \{x_0\}$ is a compact subset of (X, d). Using Lemma 5.5 we obtain the equality

$$X = V(R) \cup \{x_0\}, \tag{28}$$

because (X, d) is a compact space, and $V(R) \cup \{x_0\}$ is compact subset of (X, d), and, by Definition 2.6, (X, d) is isometric to compact subset of the compact set $V(R) \cup \{x_0\}$.

Since the labeling $l^* : V(R) \to \mathbb{R}^+$ is decreasing, formula (5) with $T(l) = R(l^*)$ gives us

$$d(x_n, x_m) = d_{l^*}(x_n, x_m) = \max\{l^*(x_n), l^*(x_m)\} = l^*(x_{\min\{m,n\}}), \qquad (29)$$

for all distinct $m, n \in \mathbb{N}$.

For every $p \in X$ the function

$$X \ni x \mapsto d(x, p) \in \mathbb{R}^+,$$

is a continuous mapping from (X, d) to $\mathbb{R}^+$. Hence, (27) implies

$$\lim_{n \to \infty} d(x_n, p) = d(x_0, p), \qquad (30)$$

for each $p \in X$. In particular, using (29) with $m = n_1$ and (30) with $p = x_{n_1}$ we obtain

$$\begin{aligned}
d(x_0, x_{n_1}) &= \lim_{n \to \infty} d(x_n, x_{n_1}) \\
&= \lim_{n \to \infty} l^*(x_{\min\{n_1,n\}}) = l^*(x_{n_1}) \\
&\leq \max\{l^*(x_{n_1}), l(x_n)\} = d(x_{n_1}, x_n),
\end{aligned}$$

wherever $n \in \mathbb{N}$ and $n \neq n_1$. Thus, the inequality

$$d(x_0, x) \leq d(y, x), \qquad (31)$$

holds whenever $x, y \in V(R) \cup \{x_0\}$ and $x_0 \neq x \neq y$.

Membership (26) follows from (28) and (31) by Theorem 2.14.

The proof is completed.

$\square$

Example 5.7. Let us define an ultrametric $d^+ \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ as

$$d^+(p, q) = \begin{cases} 0, & \text{if } p = q, \\ \max\{p, q\}, & \text{if } p \neq q. \end{cases}$$

In [27] it was noted that $(\mathbb{R}^+, d^+)$ is an US-space.

Using Theorem 5.3 we can prove that an infinite subset A of $\mathbb{R}^+$ is the compact subset of $(\mathbb{R}^+, d^+)$ iff $A = \{t_n : n \in \mathbb{N}\} \cup \{0\}$ where $(t_n)_{n \in \mathbb{N}}$ is a strictly decreasing sequence $(t_n)_{n \in \mathbb{N}}$ of points of $\mathbb{R}^+$ such that the limit relation

$$\lim_{n \to \infty} t_n = 0,$$

holds in the usual Euclidean topology.

Remark 5.8. The ultrametric d^+ on $\mathbb{R}^+$ was introduced by Delhommé, Laflamme, Pouzet, and Sauer in [6]. In [33] Yoshito Ishiki wrote: "The space $(\mathbb{R}^+, d^+)$ is as significant for ultrametric spaces as the space $\mathbb{R}^+$ or $\mathbb{R}$ with the Euclidean topology in the theory of usual metric spaces." Some results related to the ultrametric space $(\mathbb{R}^+, d^+)$ can be found in [15, 22, 32, 34].

6. Two conjectures

We believe that the following hypothesis is correct.

Conjecture 6.1. *Let (X, d) be an infinite ultrametric space. Then the following statements are equivalent:*

(i) There is $(X^*, d^*) \in US$ such that (X, d) is isometric to a subspace of (X^*, d^*).

(ii) (X, d) contains no four-point subspace which is weakly similar to (X_4, d_4) or to (Y_4, ρ_4).

We note that Conjecture 6.1 is true by Theorem 5.2 if (X, d) is compact. Moreover, using Proposition 4.2, it is easy to prove the validity of implication (i) $\Rightarrow$ (ii) for arbitrary infinite (X, d). Thus, to prove Conjecture 6.1 it suffices to show that (ii) $\Rightarrow$ (i) is valid for non-compact ultrametric spaces (X, d).

The next conjecture gives us a partial generalization of Theorem 5.6.

Recall that a tree T is *starlike* if it has exactly one vertex with degree greater than 2. (See, for example, [5, 35]).

Conjecture 6.2. *Let (X, d) be an infinite ultrametric space. Then the following statements are equivalent.*

(i) (X, d) is the completion of a totally bounded $X_0 \subsetneq X$ generated by a labeled ray.

(ii) There is a starlike rayless tree T with a labeling $l : V(T) \to \mathbb{R}^+$ such that (X, d) is a compact ultrametric space generated by $T(l)$.

If T is a star graph, then T is starlike and rayless, but not vice versa, see, for example, Figure 4 below.

Fig. 4. Trees T_1 and T_2 are starlike and rayless

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The authors declare no conflict of interest.

Data availability

All necessary data are included into the paper.

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