

Inverse binomial coefficients in Egorychev method

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ABSTRACT

A certain residue representation of the inverse binomial coefficients makes them amenable to Egorychev method for the reduction of sums by analytic methods, wherein the main idea is to identify parts of the summands as residues of analytic functions. We illustrate the use of such residue representation on some instances varying in complexity, including a generalization of an identity by Sung Sik U and Kyu Song Chae in [13].

Keywords: Egorychev method, combinatorial identity, formal power series, functions of complex variables, residues

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1. Introduction

In [6], G. P. Egorychev introduces a technique to solve combinatorial identities by using generating functions on the one hand as formal power series and on the other hand as analytic functions. Whence, contour integrals are calculated to solve problems of this kind. Different levels of complexity occur during the calculation, which can be categorized into three, as outlined in [12].

In this manuscript, we treat certain sums involving inverse binomial coefficients by appealing to a residue form of these coefficients. The idea is to replace some factors in the summand with residue operators, which is classified as intermediate level *Egorychev II* in [12]. For background on the theory see [1, 5] For more applications of the Egorychev Method see [7, 10, 11].

Throughout, we use the notation $[z^n]$ for the extractor of the coefficient of z^n in a generating function expanded in z . For a function f of z with a formal power series, the power shift $[z^{n-k}] f(z) = [z^n] z^k f(z)$ is handy and is used repeatedly in the sequel. We

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write the residue of f at s as $\underset{z=s}{res} f$. Following Egorychev's notation in [6], we simply write $\underset{z}{res} f$ for $\underset{z=0}{res} f$.

2. A representation for $\frac{1}{k} \binom{n}{k}^{-1}$

Before we prove some identities involving inverse binomial coefficients, we need a specific representation for them.

Lemma 2.1. *Let n be a positive integer and $1 \leq k \leq n$. We have*

$$\frac{1}{k} \binom{n}{k}^{-1} = [z^n] \log \left(\frac{1}{1-z} \right) (z-1)^{n-k}.$$

Proof. Let $k \geq 1$. We consider a meromorphic function f which has simple poles at the nonnegative integers $0, \dots, n$ and a simple pole at $z = -k$. Namely, we choose

$$f(z) = n! (-1)^n \frac{1}{z+k} \prod_{p=0}^n \frac{1}{z-p}.$$

We calculate the residues of f at $0 \leq r \leq n$, and obtain

$$\begin{aligned} \underset{z=r}{res} f(z) &= n! (-1)^n \frac{1}{r+k} \prod_{p=0}^{r-1} \frac{1}{r-p} \prod_{p=r+1}^n \frac{1}{r-p} \\ &= n! (-1)^n \frac{(-1)^{n-r}}{r! (n-r)! (r+k)} \\ &= \binom{n}{r} \frac{(-1)^r}{r+k}. \end{aligned}$$

According to Cauchy's residue theorem (cf. Theorem 17, Section 4.5 in [1]), the residues of f , including the residue at infinity, sum up to zero. Since the residue of f at infinity¹ is zero by inspection, we obtain

$$\begin{aligned} \sum_{r=0}^n \binom{n}{r} \frac{(-1)^r}{r+k} &= -\underset{z=-k}{res} f(z) \\ &= -n! (-1)^n \prod_{p=0}^n \frac{1}{-k-p} \\ &= \frac{1}{k} \binom{n+k}{k}^{-1}. \end{aligned}$$

Replacing n by $n-k$, for $1 \leq k \leq n$, we get

$$\frac{1}{k} \binom{n}{k}^{-1} = \sum_{r=0}^{n-k} \binom{n-k}{r} \frac{(-1)^r}{r+k}$$

¹ We remind the reader that for a function $g(w)$, the residue at infinity $\underset{w=\infty}{res} g(w)$ is $-\underset{w}{res} w^{-2} g(1/w)$.

$$\begin{aligned}
 &= (-1)^{n-k} \sum_{s=0}^{n-k} \binom{n-k}{n-s-k} \frac{(-1)^s}{n-s}, \quad (\text{setting } r = n-s-k) \\
 &= [z^n] \left(\sum_{p=1}^{\infty} \frac{z^p}{p} \right) (-1)^{n-k} \sum_{s=0}^{n-k} \binom{n-k}{s} (-z)^s \\
 &= [z^n] \log \left(\frac{1}{1-z} \right) (z-1)^{n-k}.
 \end{aligned}$$

□

There is a useful alternative to the form in Lemma 2.1 which is used in the forthcoming examples.

Corollary 2.2. *For $1 \leq k \leq n$, we express the inverse binomial coefficient in Lemma (2.1) as*

$$\frac{1}{k} \binom{n}{k}^{-1} = [z^n] \log \left(\frac{1}{1-z} \right) (z-1)^{n-k} = \frac{1}{n} \binom{n-1}{k-1}^{-1}.$$

In conjunction with Egorychev method, Lemma 2.1 can be conveniently used to reduce certain sums with inverse binomial coefficients. We give a few examples.

3. Two warm-up examples

As a warm up, we deal with two examples. In the first example, we prove the identity

$$S_n := \sum_{k=0}^n \binom{n}{k} \frac{k}{k+1} \binom{2n}{k+1}^{-1} = \frac{1}{n+1}.$$

We start with the Left-hand side, use the identity $k \binom{n}{k} = n \binom{n-1}{k-1}$ and apply Lemma 2.1 to get

$$\begin{aligned}
 S_n &:= n \sum_{k=1}^n \binom{n-1}{k-1} [z^{2n}] \log \left(\frac{1}{1-z} \right) (z-1)^{2n-k-1} \\
 &= n [z^{2n}] \log \left(\frac{1}{1-z} \right) (z-1)^{2n-2} \sum_{k=1}^n \binom{n-1}{k-1} \frac{1}{(z-1)^{k-1}} \\
 &= n [z^{2n}] \log \left(\frac{1}{1-z} \right) (z-1)^{2n-2} \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{1}{(z-1)^j} \\
 &= n [z^{2n}] \log \left(\frac{1}{1-z} \right) (z-1)^{2n-2} \left(1 + \frac{1}{z-1} \right)^{n-1} \quad (\text{the binomial theorem}) \\
 &= n [z^{n+1}] \log \left(\frac{1}{1-z} \right) (z-1)^{n-1} \\
 &= \frac{n}{2} \binom{n+1}{2}^{-1} = \frac{1}{n+1},
 \end{aligned}$$

showing the validity of the claimed identity. Note that the lemma is in action a second time at the end of the proof.

As a second example, we prove the identity

$$S_n := \sum_{k=0}^n \binom{n}{k} \binom{2n}{k}^{-1} = \frac{2n+1}{n+1}.$$

One strategy is to subsume the factor $2n+1$ in the sum, so that Corollary 2.2 is readily applicable. Toward this end, we write

$$\begin{aligned} \frac{S_n}{2n+1} &= \sum_{k=0}^n \binom{n}{k} \frac{1}{2n+1} \binom{2n}{k}^{-1} \\ &= \sum_{k=0}^n \binom{n}{k} [z^{2n+1}] \log\left(\frac{1}{1-z}\right) (z-1)^{2n-k} \\ &= [z^{2n+1}] \log\left(\frac{1}{1-z}\right) (z-1)^{2n} \sum_{k=0}^n \binom{n}{k} (z-1)^{-k} \\ &= [z^{2n+1}] \log\left(\frac{1}{1-z}\right) (z-1)^{2n} \left(1 + \frac{1}{z-1}\right)^n \\ &= [z^{n+1}] \log\left(\frac{1}{1-z}\right) (z-1)^{(n+1)-1}. \end{aligned}$$

Using Lemma 2.1 again, the extracted coefficient is $\binom{n+1}{1}^{-1} = 1/(n+1)$, and the claimed identity follows.

4. An inverse binomial identity for weighted reciprocals

In this section, we prove the identity

$$\sum_{k=1}^n \frac{1}{k} \binom{n}{k}^{-1} = \sum_{k=1}^n \frac{1}{k \times 2^{n-k}}.$$

The identity appears as a question in [4]. Note the sheer beauty in that different weights for the reciprocals (one is inverse binomial coefficients, the other is inverse powers of 2) yield equal sums!

Using Lemma 2.1 changes the sum to be

$$\begin{aligned} \sum_{k=1}^n \frac{1}{k} \binom{n}{k}^{-1} &= [v^n] \sum_{k=1}^n \log\left(\frac{1}{v-1}\right) (v-1)^{n-k} \\ &= [v^n] \log\left(\frac{1}{v-1}\right) (v-1)^{n-1} \sum_{k=1}^n (v-1)^{-(k-1)} \\ &= [v^n] \log\left(\frac{1}{v-1}\right) (v-1)^{n-1} \sum_{j=0}^{n-1} (v-1)^{-j} \\ &= [v^n] \log\left(\frac{1}{v-1}\right) (v-1)^{n-1} \frac{1 - 1/(v-1)^n}{1 - 1/(v-1)} \end{aligned}$$

$$\begin{aligned}
 &= [v^n] \log \left(\frac{1}{v-1} \right) \left(\frac{(v-1)^n}{v-2} - \frac{1}{v-2} \right) \\
 &= \operatorname{res}_v \frac{1}{v^{n+1}} \log \left(\frac{1}{v-1} \right) \frac{(v-1)^n}{v-2} + \frac{1}{2} \sum_{k=1}^n \frac{1}{k \times 2^{n-k}}. \tag{1}
 \end{aligned}$$

To handle the remaining residue, we affect a transformation of variables (this is Rule 5 in Egorychev [6, p. 16]). Put $w = v/(v-1)$, so that $v = w/(w-1)$ and $dv = -dw/(w-1)^2$ to get

$$\begin{aligned}
 &\operatorname{res}_w \frac{1}{w^{n+1}} (w-1) \log \left(\frac{1}{1-w/(w-1)} \right) \left(\frac{1}{w/(w-1)-2} \right) \left(-\frac{1}{(w-1)^2} \right) \\
 &= -\operatorname{res}_w \frac{1}{w^{n+1}} \log(1-w) \times \frac{1}{2-w} \\
 &= \frac{1}{2} \operatorname{res}_w \frac{1}{w^{n+1}} \left(\sum_{k=1}^{\infty} \frac{w^k}{k} \right) \left(\sum_{j=0}^{\infty} \frac{w^j}{2^j} \right) \\
 &= \frac{1}{2} \sum_{k=1}^n \frac{1}{k \times 2^{n-k}}.
 \end{aligned}$$

Inserting this residue into (1), we get the claimed identity.

5. An inverse binomial identity for Bernoulli numbers

Bernoulli numbers, B_n , are fundamental combinatorial numbers that have deep connections to several fields of mathematics, such as number theory. For example, the zeta function can be written in terms of Bernoulli numbers (see Chapter 12 in [3]).

There are no explicit closed forms for Bernoulli numbers. The standard definitions are either recursive or taken from expansions in series. The form we need in this example is

$$B_n = n! [z^n] \frac{z}{e^z - 1}. \tag{2}$$

The instance we consider here is the identity

$$S_n := \frac{1}{n+1} \sum_{k=1}^n \sum_{j=1}^k (-1)^j j^n \binom{n+1}{k-j} \binom{n}{k}^{-1} = B_n,$$

for $n \geq 1$, reducing a double sum to a single Bernoulli number. This identity appears in Gould [8].

Toward this end, we write S_n in the form

$$\begin{aligned}
 S_n &= \frac{1}{n+1} \sum_{k=1}^n \binom{n}{k}^{-1} \sum_{j=1}^k (-1)^j j^n \binom{n+1}{k-j} \\
 &= \frac{n!}{n+1} [z^n] \sum_{k=1}^n \binom{n}{k}^{-1} \sum_{j=0}^{\infty} (-1)^j e^{jz} \binom{n+1}{k-j}.
 \end{aligned}$$

Note that we extended the inner sum to $j \geq 0$, as the value at $j = 0$ adds constants that do not make a contribution in the coefficient extraction of z^n , for $n \geq 1$, and the binomial coefficients are 0 past $j = k$. We continue with

$$\begin{aligned} S_n &= \frac{n!}{n+1} [z^n] \sum_{k=1}^n \binom{n}{k}^{-1} \sum_{j=0}^{\infty} (-1)^j e^{jz} [w^{k-j}] (1+w)^{n+1} \\ &= \frac{n!}{n+1} [z^n] \sum_{k=1}^n \binom{n}{k}^{-1} [w^k] (1+w)^{n+1} \sum_{j=0}^{\infty} (-1)^j (we^z)^j \\ &= \frac{n!}{n+1} [z^n] \sum_{k=1}^n \binom{n}{n-k}^{-1} [w^k] (1+w)^{n+1} \frac{1}{1+we^z}. \end{aligned}$$

The time has come to replace the inverse binomial coefficient by a residue. Reversing the sum and using Lemma 2.1, we get

$$S_n = n! [z^n] [w^n] \frac{(1+w)^{n+1}}{1+we^z} \sum_{k=0}^{n-1} [v^{n+1}] \log\left(\frac{1}{1-v}\right) (v-1)^k w^k.$$

Observe that we may include the term for $k = n$ as the first two extractors combined give a zero value. Moreover, for $k > n$, the extractor in w gives 0, too. Writing

$$S_n = n! [z^n] [w^n] \frac{(1+w)^{n+1}}{1+we^z} [v^{n+1}] \log\left(\frac{1}{1-v}\right) \sum_{k=0}^{\infty} (v-1)^k w^k,$$

the contribution from w is

$$res_w \frac{1}{w^{n+1}} \left(\frac{(1+w)^{n+1}}{1+we^z} \right) \left(\frac{1}{1-(v-1)w} \right).$$

We appeal to the Substitution Rule (Rule 5 in Egorychev [6, p. 16]). Put $u = w/(1+w)$, so that $w = u/(1-u)$ and $dw = du/(1-u)^2$ to transform the residue to be

$$\begin{aligned} &res_u \frac{1}{u^{n+1}} \left(\frac{1}{1+ue^z/(1-u)} \right) \left(\frac{1}{1-(v-1)u/(1-u)} \right) \frac{1}{(1-u)^2} \\ &= res_u \frac{1}{u^{n+1}} \left(\frac{1}{1-u(1-e^z)} \right) \left(\frac{1}{1-uv} \right) \\ &= res_u \frac{1}{u^{n+1}} \left(\sum_{q=0}^{\infty} u^q (1-e^z)^q \right) \left(\sum_{r=0}^{\infty} u^r v^r \right) \\ &= \sum_{q=0}^n (1-e^z)^q v^{n-q}. \end{aligned}$$

Taking the expansion of the logarithmic term into account and activating the extractor $[v^{n+1}]$ via Lemma 2.1, we get

$$S_n = n! [z^n] \sum_{q=0}^n \frac{1}{1+q} (1-e^z)^q.$$

Note that the expansion of the summand in z begins at $(-1)^q z^q$, and we may extend the sum to infinity, to get

$$\begin{aligned} S_n &= n! [z^n] \frac{1}{1-ez} \sum_{q=0}^{\infty} \frac{1}{1+q} (1-ez)^{q+1} \\ &= n! [z^n] \frac{1}{1-ez} \log \frac{1}{1-(1-ez)} \\ &= n! [z^n] \frac{z}{e^z - 1}. \end{aligned}$$

According to (2), S_n is the n th Bernoulli number B_n .

6. A generalization of an identity by U and Chae

We are interested in proving a generalization of two identities by Sung Sik U and Kyu Song Chae in [13]. We consider this generalization the main result of this paper. It motivated us to try the logarithm identity of Lemma 2.1.

$$\begin{aligned} S_{a,k,p} &:= \sum_{m=0}^{\lfloor \frac{p+a}{2} \rfloor} (-1)^m \binom{p}{2m-a} \binom{p}{k+m} \binom{2p}{2k+2m}^{-1} \\ &= (-1)^{k+a} 2^p \binom{2p}{p}^{-1}. \end{aligned} \quad (3)$$

Here, we assume that a is an integer $a \geq -1$ and p and k are nonnegative integers so that $a+k \geq 0$ and $p \geq 2k+a$. For the inverse binomial coefficient in (3) not to be zero, we need $2p \geq 2k+2m$. The latter is at most $p+a+2k$ owing to the upper limit of the sum. The first binomial coefficient enforces the range with $\binom{p}{p+a-2m}$ and the given boundary conditions respect these aspects.

We use the residue form in Corollary 2.2 to write the inverse binomial coefficient in (3) as

$$\binom{2p}{2k+2m}^{-1} = (2p+1) [v^{2p+1}] \log \left(\frac{1}{1-z} \right) (v-1)^{2p-(2k+2m)}. \quad (4)$$

Note that the boundary conditions stated after (3) ensure that $0 \leq 2k+2m \leq 2p$, which allows the representation via Lemma 2.1.

We use the extraction operator twice in (3) to write

$$\begin{aligned} S_{a,k,p} &= [z^{p+a}] (1+z)^p \sum_{m \geq 0} (-1)^m z^{2m} \binom{p}{k+m} \binom{2p}{2k+2m}^{-1} \\ &= [z^{p+a}] (1+z)^p [w^p] w^k (1+w)^p \sum_{m \geq 0} (-1)^m z^{2m} w^m \binom{2p}{2k+2m}^{-1}. \end{aligned}$$

Thanks to the extraction operator in z , we could safely extend the upper limit of the sum to infinity. Appealing to (4), we proceed with

$$S_{a,k,p} = (2p+1) [v^{2p+1}] \log \left(\frac{1}{1-v} \right) (v-1)^{2p-2k}$$

$$\begin{aligned}
& \times [z^{p+a}] (1+z)^p [w^p] w^k (1+w)^p \sum_{m \geq 0} (-1)^m \frac{z^{2m} w^m}{(v-1)^{2m}} \\
& = (2p+1) [v^{2p+1}] \log \left(\frac{1}{1-v} \right) (v-1)^{2p-2k+2} \\
& \times [z^{p+a+2}] (1+z)^p [w^p] w^k \frac{(1+w)^p}{w + (v-1)^2/z^2}. \tag{5}
\end{aligned}$$

We consider in (5) the contribution from w written in residue form:

$$res_w \frac{1}{w^{p-k+1}} \left(\frac{(1+w)^p}{w + (v-1)^2/z^2} \right).$$

It is advantageous to use the fact that all residues sum to zero including the residue at infinity, so that we may evaluate using a simple pole, which is straightforward.

6.1. Residues at poles away from zero

We start with calculating the residue at infinity and get

$$\begin{aligned}
& -res_{w=\infty} \frac{1}{w^{p-k+1}} \left(\frac{(1+w)^p}{w + (v-1)^2/z^2} \right) \\
& = res_w \frac{1}{w^2} w^{p-k+1} \left(\frac{(1+\frac{1}{w})^p}{\frac{1}{w} + (v-1)^2/z^2} \right) \\
& = res_w \frac{1}{w^k} \left(\frac{(1+w)^p}{1+w(v-1)^2/z^2} \right) \\
& = res_w \frac{1}{w^k} \left(\sum_{j=0}^p \binom{p}{j} w^j \right) \left(\sum_{q=0}^{\infty} (-1)^q \frac{w^q (v-1)^{2q}}{z^{2q}} \right) \\
& = \sum_{q=0}^{k-1} \binom{p}{k-q-1} (-1)^q \frac{(v-1)^{2q}}{z^{2q}}.
\end{aligned}$$

In (5), the part under the extractor $[z^{p+a+2}]$ coming from the residue at infinity is

$$\begin{aligned}
& res_z \frac{1}{z^{p+a+3}} (1+z)^p \sum_{q=0}^{k-1} \binom{p}{k-1-q} (-1)^q \frac{(v-1)^{2q}}{z^{2q}} \\
& = \sum_{q=0}^k \binom{p}{k-1-q} \binom{p}{p+a+2q+2} (-1)^q (v-1)^{2q}.
\end{aligned}$$

Reducing the residue to 0—the series in z is *finite*, with powers less than or equal to p , and with $a \geq -1$, the second binomial coefficient is zero.

Note that we are using the residue definition of the coefficient extractor here, as we are dealing with a Laurent series. Calculating the residue at the first order pole $w = -(v-1)^2/z^2$, we obtain

$$\begin{aligned}
& -res_{w=-(v-1)^2/z^2} \frac{1}{w^{p-k+1}} \left(\frac{(1+w)^p}{w + (v-1)^2/z^2} \right) = (-1)^{p-k} \frac{z^{2p-2k+2}}{(v-1)^{2p-2k+2}} \left(1 - \frac{(v-1)^2}{z^2} \right)^p \\
& = (-1)^{p-k} \frac{1}{z^{2k-2}} \frac{(z^2 - (v-1)^2)^p}{(v-1)^{2p-2k+2}}. \tag{6}
\end{aligned}$$

6.2. The rest of the residues

We use the residue representation from (6) to replace the residue in (5), which is admissible since we have already shown that the residue at infinity is 0. We obtain

$$\begin{aligned}
 S_{a,k,p} &= (-1)^{p-k}(2p+1)[v^{2p+1}] \log\left(\frac{1}{1-v}\right)(v-1)^{2p-2k+2} \\
 &\quad \times [z^{p+a+2}](1+z)^p \frac{1}{z^{2k-2}} \frac{(z^2 - (v-1)^2)^p}{(v-1)^{2p-2k+2}} \\
 &= (-1)^{p-k}(2p+1)[v^{2p+1}] \log\left(\frac{1}{1-v}\right) \\
 &\quad \times [z^{p+a+2k}](1+z)^p (z^2 - (v-1)^2)^p.
 \end{aligned} \tag{7}$$

Working with the extractor in z now yields

$$\sum_{q=0}^p \binom{p}{q} [z^{p+a+2k-q}](z - (v-1))^p (z + (v-1))^p.$$

Introducing $b = q - a - 2k$ we find

$$\begin{aligned}
 &\sum_{q=0}^p \binom{p}{q} [z^{p-b}](z - (v-1))^p (z + (v-1))^p \\
 &= \sum_{q=0}^p \binom{p}{q} [z^{p-b}](z - (v-1))^p \sum_{r=0}^p \binom{p}{r} z^r (v-1)^{p-r} \\
 &= \sum_{q=0}^p \binom{p}{q} \sum_{r=0}^p \binom{p}{r} [z^{p-b-r}](v-1)^{p-r} (z - (v-1))^p \\
 &= \sum_{q=0}^p \binom{p}{q} \sum_{r=0}^p \binom{p}{r} (v-1)^{p-r} \binom{p}{p-b-r} (-1)^{b+r} (v-1)^{b+r} \\
 &= \sum_{q=0}^p \binom{p}{q} (v-1)^{p+b} \sum_{r=0}^p \binom{p}{r} \binom{p}{p-b-r} (-1)^{b+r}.
 \end{aligned}$$

Manipulating the inner sum

$$\begin{aligned}
 [z^{p-b}](1+z)^p \sum_{r=0}^p \binom{p}{r} z^r (-1)^{b+r} &= (-1)^b [z^{p-b}](1+z)^p (1-z)^p \quad (\text{the binomial theorem}) \\
 &= (-1)^b [z^{p-b}](1-z)^p \sum_{r=0}^p \binom{p}{r} (1-z)^{p-r} 2^r z^r \\
 &= (-1)^b \sum_{r=0}^p \binom{p}{r} [z^{p-b-r}](1-z)^{2p-r} 2^r \\
 &= (-1)^b \sum_{r=0}^p \binom{p}{r} (-1)^{p-b-r} \binom{2p-r}{p-b-r} 2^r \\
 &= \sum_{r=0}^p \binom{p}{r} (-1)^{p-r} \binom{2p-r}{p-b-r} 2^r.
 \end{aligned} \tag{8}$$

6.3. Transformation of inverse binomial coefficients

We take (7), combine it with and (8) and apply Corollary 2.2 again. We continue to use the variable $b = q - a - 2k$, until we have to work with q again. We obtain

$$\begin{aligned} S_{a,k,p} &= (-1)^{p-k}(2p+1)[v^{2p+1}] \log\left(\frac{1}{1-v}\right) \sum_{q=0}^p \binom{p}{q} (v-1)^{p+b} \\ &\quad \times \sum_{r=0}^{p-b} \binom{p}{r} \binom{2p-r}{p-b-r} (-1)^{p-r} 2^r \\ &= (-1)^{p-k} \sum_{q=0}^p \binom{p}{q} \binom{2p}{p-b}^{-1} \sum_{r=0}^{p-b} \binom{p}{r} \binom{2p-r}{p-b-r} (-1)^{p-r} 2^r. \end{aligned}$$

Since

$$\binom{2p}{p-b}^{-1} \binom{p}{r} \binom{2p-r}{p-b-r} = \binom{2p}{p}^{-1} \binom{p-b}{r} \binom{2p-r}{p},$$

we finally get

$$\begin{aligned} S_{a,k,p} &= (-1)^{p-k} \binom{2p}{p}^{-1} \sum_{q=0}^p \binom{p}{q} \sum_{r=0}^{p-b} \binom{p-b}{r} \binom{2p-r}{p} (-1)^{p-r} 2^r \\ &= (-1)^k \binom{2p}{p}^{-1} \sum_{q=0}^p \binom{p}{q} \sum_{r=0}^{p-b} \binom{p-b}{r} (-1)^r 2^r [z^p] (1+z)^{2p-r} \\ &= (-1)^k \binom{2p}{p}^{-1} [z^p] (1+z)^{2p} \sum_{q=0}^p \binom{p}{q} \sum_{r=0}^{p-b} \binom{p-b}{r} \left(-\frac{2}{1+z}\right)^r \\ &= (-1)^k \binom{2p}{p}^{-1} [z^p] (1+z)^{2p} \sum_{q=0}^p \binom{p}{q} \left(\frac{z-1}{1+z}\right)^{p-(q-a-2k)} \\ &\hspace{15em} (\text{recalling that } b = q - a - 2k) \\ &= (-1)^k \binom{2p}{p}^{-1} [z^p] (1+z)^{p-a-2k} (z-1)^{p+a+2k} \left(1 + \frac{1+z}{z-1}\right)^p \\ &= (-1)^k \binom{2p}{p}^{-1} [z^p] (1+z)^{p-a-2k} (z-1)^{a+2k} (2z)^p; \end{aligned}$$

note that the least power of z in the expansion is p , which we get from the part $(2z)^p$, and we must take the free term in the expansion of both $(1+z)^{p-a-2k}$ and $(z-1)^{a+2k}$. Thus, we have

$$S_{a,k,p} = (-1)^{k+a} 2^p \binom{2p}{p}^{-1}.$$

The claimed identity follows.

This is a generalization of Equations (4) and (5) in U and Chae's paper [13].

7. Concluding remarks

Some of the simpler examples involving inverse binomial coefficients are amenable to symbolic algebra systems. For instance, Maple © successfully gets the reduction in the two examples in Section 3. These mechanical devices use algorithms based on Wilf-Zeilberger theories [9] including Zeilberger algorithm and Almkvist-Zeilberger algorithm [2, 9, 14]. In fact Almkvist-Zeilberger algorithm [2] is implemented as a default method in some computer algebra systems. These methods do not deal with instances to reduce sums involving fundamental combinatorial numbers (such as Bernoulli numbers or Stirling numbers) other than binomial coefficients. Egorychev method has a wider range of applicability.

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