Real analytic methods in the formulations of some combinatorial inequalities

Hailu Bikila Yadeta*

ABSTRACT

In this paper, we derive some new combinatorial inequalities by applying well known real analytic results like Hölder's inequality, Young's inequality, and Minkowiski's inequality to the recursively defined sequence f_n of functions

$$f_0(x) = \chi_{(-1/2,1/2)}(x),$$
$$f_{n+1}(x) = f_n(x + 1/2) + f_n(x - 1/2), n \in \mathbb{N} \cup \{0\}.$$

Towards this goal, we derive the closed form of the aforementioned sequence $(f_n)_{n \in \mathbb{N} \cup \{0\}}$ of functions and show that it is a sequence of simple functions that are linear combinations of characteristic functions of some unit intervals $I_{n,i}$, $i = 0, 1, ..., n$, with values the binomial coefficients $\binom{n}{i}$ on each unit interval $I_{n,i}$. We show that $f_n \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$. Besides applying real analytic methods to formulate some combinatorial inequalities, we also illustrate the application of some combinatorial identities. For example, we use the Vandermonde convolution (or Vandermonde identity), in the study of some properties of the sequence of functions $(f_n)_{n \in \mathbb{N} \cup \{0\}}$. We show how the L^2 norm of f_n is related to the Catalan numbers.

Keywords: recurrence relation, binomial coefficient, shift operator, simple function, Vandermonde convolution, L^p space, Hölder's inequality, Young's inequality, Minkowiski's inequality

2020 Mathematics Subject Classification: Primary 05A10, 05A20; Secondary 26A18, 26A42, 26D07.

* Corresponding author.

1. Introduction

The binomial coefficients denoted by $\binom{n}{i}$ are integers that are the numerical coefficients in the expansion for the polynomial $B_n(x) = (x+1)^n$, so that

$$B_n(x) = \sum_{i=0}^{n} \binom{n}{i} x^i = (x+1)^n.$$

The binomial coefficient

$$\binom{n}{i} := \frac{n!}{i!\,(n-i)!} = \frac{n(n-1)(n-2)...(n-i+1)}{i(i-1)(i-2)...1}, \tag{1}$$

signify the number of ways of selecting i objects out of n without replacement. The binomial coefficient $\binom{n}{i} := 0$, if $i > n$. For the discussion of the historical introduction of the binomial theorem and binomial coefficients, one may refer to Cooldige [2]. Several results in combinatorial identities and inequalities are available. To mention one, a complete book of collections of combinatorial identities by H. W. Gould [3] is remarkable. Horst and Prodinger [1] present identities and inequalities involving binomial coefficients.

In this paper, we apply some combinatorial identities to the study some properties of a recursively defined sequence of functions $(f_n)_{n \in \mathbb{N}\{0\}}$. For example, in calculating the integral of the product $f_n(x)f_m(x)$, we use the Vandermonde convolution (or Vandermonde identity). For $p = 1$ and $p = 2$, we use some combinatorial identities to calculate the L^p norms of f_n. In the recursive definition of the sequence, an initial function f_0 and an operator that is composed of a sum of two shift operators is applied to recursively generate the elements of the sequence. By writing the closed form of f_n, we show that f_n is a sequence of simple functions that are linear combinations of characteristic functions of some unit intervals $I_{n,i}$, $i = 0, 1, ..., n$. The values of f_n on $I_{n,i}$ are the binomial coefficients $\binom{n}{i}$. We show this fact by setting a recurrence relation whose solutions are binomial coefficients. Other than using established combinatorial identities, we derive some combinatorial inequalities by using some real analytic methods. For example, we apply Hölder inequality to two elements f_m and f_n of the sequence to derive some new combinatorial inequalities. We also apply Young's inequality to the convolution $f_m * f_n$ to derive different combinatorial inequalities than the ones we derive using Hölder's inequality . Similar work is done by applying Minkowiski's inequality. To the best understanding of the author, these inequalities are new results. In the conclusion section, the question of whether well-known integer sequence other than the binomial coefficients can be generated by a recursive definition of a sequence of functions is raised.

2. Preliminaries

In this section, we consider recursive sequences of functions that are generated by the sum of two shift operators $E^{1/2}$ and $E^{-1/2}$ that are defined as follows:

$$E^{\frac{1}{2}}u(x) := u(x+1/2), \quad E^{-\frac{1}{2}}u(x) := u(x-1/2). \tag{2}$$

Therefore, the sum of the two shift operators are defined as

$$\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right) u(x) = u(x + 1/2) + u(x - 1/2).$$

Definition 2.1 (Characteristic Function). Let $A \subset \mathbb{R}$. The characteristic function of the set A is denoted by χ_A is the function with values 1 on A and is equal to 0 on the complement $\mathbb{R} \setminus A$.

Characteristic functions satisfy the following properties:

$$\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x), \tag{3}$$

in particular, if $A \cap B = \emptyset$ then $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x)$,

$$\chi_{A \cap B}(x) = \chi_A(x) \chi_B(x). \tag{4}$$

Definition 2.2 (Simple Function). A simple function is a finite linear combination of characteristic functions of measurable sets. A simple function $f : \mathbb{R} \to \mathbb{R}$, can be written in the form:

$$f(x) = \sum_{j=1}^{n} c_j \chi_{J_j}(x), \tag{5}$$

where $c_j \in \mathbb{R}$ and $J_j \subset \mathbb{R}, j = 1, 2, ...n$ are measurable sets.

A simple function always takes finite distinct values. The representation of a simple function in form (5) is not unique. A simple function is nonnegative if its range is a finite subset of $[0, \infty)$. A simple function f in (5) is said to be in *standard form* if the sets $J_j, j = 1, 2, ..., n$ are pairwise disjoint and the distinct values of f are $c_j, j = 1, 2, ..., n$ and 0. Any simple function can be arranged to be written in standard form. If simple function f given in (5) is in standard form, then k-th power of f is given by

$$f^k(x) = \sum_{j=1}^{n} c_j^k \chi_{J_j}(x). \tag{6}$$

Suppose that a function $\phi(x)$ exhibits a convergent Taylor series expansion

$$\phi(x) = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(0)}{k!} x^k,$$

for all x in some open interval I about 0 and suppose that the range of f is contained in I. Then we have

$$\phi((f(x)) = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(0)}{k!} (f(x))^k = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(0)}{k!} \sum_{j=1}^{n} c_j^k \chi_{J_i}(x)$$

$$= \sum_{j=1}^{n} \left(\sum_{k=0}^{\infty} \frac{\phi^{(k)}(0)}{k!} c_j^k\right) \chi_{J_j}(x) = \sum_{j=1}^{n} \phi(c_j) \chi_{J_i}(x).$$

Definition 2.3 (Integrals of Non-negative Simple Functions)**.** The Lebesgue integral over $\mathbb{R}$ of a nonnegative simple function f written in the form (5) is

$$\int_{\mathbb{R}} f = \int_{\mathbb{R}} f(x)dx = \sum_{j=1}^{n} c_j |J_i|, \tag{7}$$

where $|J_i|$ is the measure of the set J_i.

Definition 2.4 (Convolution)**.** [4] Given functions f and g on the real line, we say that their convolution is defined if for almost every $t \in \mathbb{R}$ the function $f(\tau)g(t-\tau)$ is Lebesgue integrable over $\mathbb{R}$ as a function of the variable t. In this case then the convolution of the functions f and g is the function $f * g$ defined for almost all $t \in \mathbb{R}$ by the formula

$$\int_{\mathbb{R}} f(\tau)g(t - \tau)d\tau. \tag{8}$$

3. Recursive sequences defined by sum of two shift operators

Consider the sequence of real functions $(f_n)_{n \in \mathbb{N} \cup \{0\}}$

$$f_0(x) = \chi_{(-1/2,1/2)}(x),$$
$$f_{n+1}(x) = f_n(x + 1/2) + f_n(x - 1/2) = \left(E^{\frac{1}{2}} + E^{-\frac{1}{2}} \right) f_n(x). \tag{9}$$

In the upcoming subsections, we study the sequence of functions f_n defined in (9) including its closed form, the L^p norm of f_n, and the integrals of the form $\int_{-\infty}^{\infty} f_n(x)f_m(x)dx$.

3.1. Closed form of f_n

We derive the closed form expression of f_n and then discuss some properties that are satisfied by f_n. Since $f_n(x - 1/2)$ is a shift of f_n half units to the right and $f_n(x + 1/2)$ is a shift of f_n half units to the left, f_{n+1} is the superposition of the two shifted functions. Manual calculation of the first three elements of the sequence yields,

$$f_1(x) = \chi_{(-1,0)}(x) + \chi_{(0,1)}(x), \tag{10}$$

$$f_2(x) = \chi_{(-3/2,-1/2)}(x) + 2\chi_{(-1/2,1/2)}(x) + \chi_{(1/2,3/2)}(x), \tag{11}$$

$$f_3(x) = \chi_{(-2,-1)}(x) + 3\chi_{(-1,0)}(x) + 3\chi_{(0,1)}(x) + \chi_{(1,2)}(x). \tag{12}$$

In the next theorem, we prove that the general closed form of f_n is the linear combinations of characteristic functions of unit intervals with the coefficients being the binomial numbers $\binom{n}{i}$.

Theorem 3.1. *The general closed form of the sequence (9) is given by*

$$f_n(x) = \sum_{i=0}^{n} \binom{n}{i} \chi_{\left(\frac{-n-1}{2} + i, \frac{-n+1}{2} + i \right)}(x). \tag{13}$$

Proof. Solving the first order iteration Eq. (9) yields

$$f_n(x) = \left(E^{\frac{1}{2}} + E^{-\frac{1}{2}} \right)^n f_0(x).$$

Then using the binomial expansion of $\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}} \right)^n$, we have

$$
\begin{aligned}
f_n(x) &= \left(E^{\frac{1}{2}} + E^{-\frac{1}{2}} \right)^n f_0(x) \\
&= \sum_{i=0}^{n} \binom{n}{i} E^{\frac{n-i}{2}} E^{-\frac{i}{2}} f_0(x) = \sum_{i=0}^{n} \binom{n}{i} E^{\frac{n}{2}-i} f_0(x) \\
&= \sum_{i=0}^{n} \binom{n}{i} f_0 \left(x + \frac{n}{2} - i \right) \\
&= \sum_{i=0}^{n} \binom{n}{i} \chi_{\left(\frac{-n-1}{2}+i, \frac{-n+1}{2}+i \right)}(x).
\end{aligned}
$$

This proves the Theorem. $\qquad\square$

The sequence f_n defined according to (9) is a sequence of simple functions which are the linear combination of characteristic functions of the $n+1$ unit intervals:

$$I_{n,i} := \left(\frac{-n-1}{2}+i, \frac{-n+1}{2}+i \right), \quad n \in \mathbb{N} \cup \{0\}, \ i = 0, 1, 2, ..., n. \tag{14}$$

The values of f_n on $I_{n,i}$, $i = 0, 1, ..n$, are $\binom{n}{i}$. These are the $n+1$ numbers on the $n+1$-th row of the Pascal's Triangle that is depicted partially in Table 1 below.

Table 1. Pascal Triangle of numbers $\binom{n}{i}$ for $0 \le n \le 6$ and $0 \le i \le n$

$n=0$						1						
$n=1$					1		1					
$n=2$				1		2		1				
$n=3$			1		3		3		1			
$n=4$		1		4		6		4		1		
$n=5$	1		5		10		10		5		1	
$n=6$	1	6		15		20		15		6		1

3.2.   Properties of f_n drawn from closed-form of the sequence f_n

In this subsection, we discuss some observable properties of the sequence f_n that we will use in the upcoming sections.

Definition 3.2. The support $\mathrm{supp}(f)$ of a function f is the smallest closed set outside of which the function f vanishes identically. It is the closure of the set, $\{x | f(x) \ne 0\}$.

Theorem 3.3. *For each $n \in \mathbb{N}_0 \cup \{0\}$, the support $\mathrm{supp}(f_n) = [-(n+1)/2, (n+1)/2]$.*

Theorem 3.4. *Each f_n is even and non-negative valued.*

Proof. From the closed form of f_n given in (13) we see that the values of f_n are $\binom{n}{i}$ on $I_{n,i}, i = 0, 1, ..., n$ and 0 elsewhere. Hence f_n is nonnegative. We prove that f_n is even function by induction on n. For $n = 0$, f_0 is even. Suppose that f_n is even for some $n \in \mathbb{N}_0 \cup \{0\}$. Then

$$f_{n+1}(-x) = f_n(-x + 1/2) + f_n(-x - 1/2) = f_n(x - 1/2) + f_n(x + 1/2) = f_{n+1}(x).$$

So f_{n+1} is even. Thus f_n is even for all $n \in \mathbb{N}_0 \cup \{0\}$. $\qquad\qquad\square$

Unlike integral operators, shift operators do not increase the order of smoothness of the functions they operate on. For shift operators $E^{\frac{1}{2}}$ and $E^{-\frac{1}{2}}$, the jump discontinuities are shifted half units to the left and the right respectively. The operator $\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$ when operated on f_n to yield f_{n+1}, expands the support of f_n by one unit and adds the number of jump discontinuities by one. For example, the jump discontinuities of f_0 are the points $x_{0,0} = -1/2$ and $x_{0,1} = 1/2$. That of f_1 are $x_{1,0} = -1$, $x_{1,1} = 0$, and $x_{1,2} = 1$. That of f_2 are $x_{2,0} = -3/2$, $x_{2,1} = -1/2$, $x_{2,2} = 1/2$, and $x_{2,3} = 3/2$. We generalize the form of the jump discontinuities of f_n in the next theorem.

Theorem 3.5 (Jump discontinuities of f_n). *For $n \in \mathbb{N} \cup \{0\}$, the points $x_{n,i} = i - \frac{n+1}{2}, i = 0, 1, 2, ..., n + 1$, are the points of jump discontinuities of f_n.*

Proof. We use induction on n. For $n = 0$ the hypothesis holds true as the points of jump discontinuities of f_0 are $x = \pm 1/2$. Suppose that the assumption is true for some arbitrary $n \in \mathbb{N} \cup \{0\}$. Then the jump discontinuities of $f_{n+1}(x) = f_n(x - 1/2) + f_n(x - 1/2)$ are those points x such that either $x - 1/2$ or $x + 1/2$ are the jump discontinuities of f_n. That is, the points where $x = i - \frac{n+1}{2}$ or $x = i - \frac{n-1}{2}$ for some $i \in \{0, 1, 2, ..., n + 1\}$. These are the points $x_{n+1,i} = i - \frac{n+2}{2}, i = 0, 1, ..., n + 2$. These are the jump discontinuities of f_{n+1} induced from the induction hypothesis. Thus the theorem is proved. $\qquad\square$

3.3.   The powers, and the L^p norms of f_n

Definition 3.6 (The Lebesgue space $L^p(\mathbb{R})$). Let $1 \leq p < \infty$. The function space, $\{f \mid f : \mathbb{R} \to \mathbb{R}\}$, of measurable functions satisfying $\int_{\mathbb{R}} |f(x)|^p dx < \infty$ is denoted by $L^p(\mathbb{R})$. For $f \in L^p(\mathbb{R})$ we define the corresponding norm as

$$\|f\|_p := \left(\int_{-\infty}^{\infty} |f(x)|^p dx\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty. \tag{15}$$

For $p = \infty$, then $L^\infty(\mathbb{R})$ is the set $f : \mathbb{R} \to \mathbb{R}$ of essentially bounded measurable functions. For $f \in L^\infty(\mathbb{R})$ the norm is defined as:

$$\|f\|_\infty := \operatorname{esssup}_{x \in \mathbb{R}} |f(x)|.$$

As f_n are simple functions in standard form, the p-power f_n^p of f_n, are calculated according to (6), are also simple functions given by

$$f_n^p(x) = \sum_{i=0}^n \binom{n}{i}^p \chi_{I_{n,i}}(x), \ 1 \le p < \infty, \ n \in \mathbb{N} \cup \{0\}. \tag{16}$$

From (15), (16), and (7) we have

$$\|f_n\|_p = \left(\sum_{i=0}^n \binom{n}{i}^p \right)^{\frac{1}{p}}, \quad 1 \le p < \infty, \ n \in \mathbb{N} \cup \{0\}. \tag{17}$$

In particular, for $p = 1$ and $p = 2$

$$\|f_n\|_1 = \int_{-\infty}^\infty |f_n(x)| dx = \sum_{i=0}^n \binom{n}{i} = 2^n, \tag{18}$$

and

$$\|f_n\|_2 = \left(\int_{-\infty}^\infty |f_n(x)|^2 dx \right)^{\frac{1}{2}} = \left(\sum_{i=0}^n \binom{n}{i}^2 \right)^{\frac{1}{2}} = \sqrt{\binom{2n}{n}}. \tag{19}$$

The mean integral of $|f_n|^2$ over the support $[-\frac{n+1}{2}, \frac{n+1}{2}]$ of f_n yields

$$\frac{1}{n+1} \int_{-\frac{n+1}{2}}^{\frac{n+1}{2}} |f_n(x)|^2 dx = \frac{1}{n+1} \int_{-\infty}^\infty |f_n(x)|^2 dx = \frac{1}{n+1} \sum_{i=0}^n \binom{n}{i}^2 = \frac{1}{n+1} \binom{2n}{n}, \tag{20}$$

the *Catalan numbers*. The L^∞ norm of f_n is defined depending whether n is even or odd. If n is even, the L^∞ norms of f_n is

$$\|f_n\|_\infty = \max_{0 \le i \le n} \binom{n}{i} = \binom{n}{\frac{n}{2}}, \tag{21}$$

whereas if n is odd,

$$\|f_n\|_\infty = \max_{0 \le i \le n} \binom{n}{i} = \binom{n}{\frac{n-1}{2}} = \binom{n}{\frac{n+1}{2}}. \tag{22}$$

For $n \in \mathbb{N} \cup \{0\}$, $\|f_n\|_\infty$ is the greatest numerical coefficient appearing in the expansion of $(1 + x)^n$. That is the same as the largest number that appear on the n-th row of the Pascal triangle that is partly displayed in Table 1. From the above results, we conclude that $f_n \in L^p(\mathbb{R})$, $1 \le p \le \infty$, $n \in \mathbb{N} \cup \{0\}$.

From property (4), for $I_{n,i}$ given in (14)

$$\chi_{I_{n,i} \cap I_{n,j}}(x) = \chi_{I_{n,i}}(x) \chi_{I_{n,j}}(x) = \delta_{i,j} \chi_{I_{n,i}}(x), \quad x \in \mathbb{R}, i, j \in \{0, 1, 2, ..., n\}, \tag{23}$$

where $\delta_{i,j}$ is the Kronecker delta defined as $\delta_{i,j} = 0$, $i \ne j$ and $\delta_{i,i} = 1$.

In the next Lemma, we apply the real analytic method to derive Vandermonde's identity, one of the established combinatorial identities.

Lemma 3.7 (Vandermonde's Identity (or Vandermonde's Convolution)). *For any non-negative integers* r, m, n

$$\binom{n+m}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}. \tag{24}$$

Proof. First, let us evaluate the convolution of the characteristic functions of the i-th unit interval in the $\mathrm{supp}(f_m)$ and the j-th unit interval in $\mathrm{supp}(f_n)$, $\chi_{I_{m,i}}(x) * \chi_{I_{n,j}}(x)$.

$$x \in I_{m,i} \Leftrightarrow x \in \left(-\frac{m+1}{2} + i, -\frac{m+1}{2} + i + 1\right)$$

$$\Leftrightarrow -\frac{1}{2} < x + \frac{m}{2} - i < \frac{1}{2}.$$

Therefore

$$\chi_{I_{m,i}}(x) = f_0(x + \frac{m}{2} - i) = E^{\frac{m}{2}-i} f_0(x). \tag{25}$$

By using (25), the convolution of the characteristic function of two unit intervals $I_{m,i}$ and $I_{n,j}$ is calculated as

$$\chi_{I_{m,i}}(x) * \chi_{I_{n,j}}(x) = E^{\frac{m}{2}-i} f_0(x) * E^{\frac{n}{2}-j} f_0(x) = E^{\frac{m+n}{2}-i-j} \left(f_0 * f_0\right)(x). \tag{26}$$

The convolution of two elements f_m and f_n of the sequence yield

$$\begin{aligned}
f_m(x) * f_n(x) &= \left(\sum_{i=0}^{m} \binom{m}{i} \chi_{I_{m,i}}(x)\right) * \left(\sum_{j=0}^{n} \binom{n}{j} \chi_{I_{n,j}}(x)\right) \\
&= \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i}\binom{n}{j} \chi_{I_{m,i}}(x) * \chi_{I_{n,j}}(x) \\
&= \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i}\binom{n}{j} E^{\frac{m+n}{2}-i-j} f_0(x) * f_0(x).
\end{aligned} \tag{27}$$

On the other hand,

$$\begin{aligned}
f_m(x) * f_n(x) &= \left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)^m f_0(x) * \left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)^n f_0(x) \\
&= \left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)^{m+n} f_0(x) * f_0(x) \\
&= \sum_{r=0}^{m+n} \binom{m+n}{r} E^{\frac{m+n-r}{2}} E^{\frac{-r}{2}} f_0(x) * f_0(x) \\
&= \sum_{r=0}^{m+n} \binom{m+n}{r} E^{\frac{m+n}{2}-r} f_0(x) * f_0(x).
\end{aligned} \tag{28}$$

Comparing the coefficients of $E^{\frac{m+n}{2}-r} f_0(x) * f_0(x)$ in (27) and (28), by setting $i+j = r$ in (27), we obtain

$$\sum_{0 \le i+j=r \le m+n} \binom{m}{i}\binom{n}{j} = \binom{m+n}{r}.$$

This sum when re-indexed and written yields the Lemma. □

In the prove the Vandermonde's identity we calculating $f_0(x) * f_0(x)$ as it is. However, we may need the explicit result for latter use. We can proceed as follows.

$$
\begin{aligned}
(f_0 * f_0)(x) &= \int_{-\infty}^{\infty} f_0(x - y) f_0(y) dy \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} f_0(x - y) dy \\
&= \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} f_0(y) dy.
\end{aligned}
\tag{29}
$$

Now differentiating the result in (29) we get

$$
(f_0 * f_0)'(x) = f_0(x + 1/2) - f_0(x - 1/2) = \chi_{(-1,0)}(x) - \chi_{(0,1)}(x),
\tag{30}
$$

and from (30) we get

$$
(f_0 * f_0)(x) = \int_{-\infty}^{x} (f_0 * f_0)'(s) ds = \int_{-\infty}^{x} (\chi_{(-1,0)}(s) - \chi_{(0,1)}(s)) ds.
\tag{31}
$$

Integrating out we get

$$
(f_0 * f_0)(x) = \begin{cases} 0, -\infty < x \le -1, \\ 1 + x, -1 < x \le 0, \\ 1 - x, 0 < x < 1, \\ 0, 1 \le x < \infty. \end{cases}
\tag{32}
$$

3.4. The evaluation of the integrals of the form $\int_{-\infty}^{\infty} f_n(x) f_m(x) dx$

3.4.1. The case where $n \ge m$ and $n - m \equiv 0 (\text{mod } 2)$. We use Vandermonde's identity in the proof of the two theorems that follow.

Theorem 3.8. *Let $n \ge m$ and $n \equiv m(\text{mod } 2)$. Then we have*

$$
\int_{-\infty}^{\infty} f_n(x) f_m(x) dx = \binom{n + m}{\frac{n+m}{2}}.
$$

Proof. Since $\text{supp}(f_n) = \left[-\frac{n+1}{2}, \frac{n+1}{2}\right]$ has $n+1$ unit intervals and $\text{supp}(f_m) = \left[-\frac{m+1}{2}, \frac{m+1}{2}\right]$ has $m+1$ unit intervals. According to an assumption in the theorem, $(n+1) - (m+1) = n - m$ is even. The middle $m + 1$ unit intervals in the $\text{supp}(f_n)$ coincide with the $m + 1$ unit intervals of $\text{supp}(f_m)$. We will observe these with the following steps. The first $\frac{n-m}{2}$ unit intervals in $\text{supp}(f_n)$ are

$$
I_{n,0}, \ I_{n,1}, \ ..., \ I_{n,\frac{n-m-2}{2}}.
$$

The next $m + 1$ unit intervals in the $\text{supp}(f_n)$ are

$$
I_{n,\frac{n-m}{2}}, \ I_{n,\frac{n-m+2}{2}}, \ ...., I_{n,\frac{n+m}{2}}.
$$

The last $\frac{n-m}{2}$ unit intervals in the supp(f_n) are

$$I_{n,\frac{n+m+2}{2}},\ I_{n,\frac{n+m+4}{2}},\ ....,\ I_{n,n}.$$

We have,

$$I_{n,\frac{n-m}{2}} = \left(-\frac{n+1}{2}+\frac{n-m}{2},-\frac{n+1}{2}+\frac{n-m}{2}+1\right) = \left(-\frac{m+1}{2},-\frac{m+1}{2}+1\right) = I_{m,0}.$$

Consequently,

$$I_{n,\frac{n-m}{2}} = I_{m,0},\quad I_{n,\frac{n-m+2}{2}} = I_{m,1},\quad I_{n,\frac{n-m+4}{2}} = I_{m,2},...,I_{n,\frac{n-m+2m}{2}} = I_{n,\frac{n+m}{2}} = I_{m,m}.$$

Therefore the middle $m+1$ unit intervals in supp(f_n) coincides with that of the unit intervals in supp(f_m). Therefore the product $f_n f_m$ is a simple function given by

$$f_n(x)f_m(x) = \sum_{i=0}^{m}\binom{m}{i}\binom{n}{\frac{n-m+2i}{2}}\chi_{I_{m,i}}(x). \tag{33}$$

Now integrating the product $f_n(x)f_m(x)$ that is given in (33) and using Vandermonde's identity and using the fact that $\binom{m}{i}=0, i>m$ and that $n \geq \frac{n+m}{2} \geq m$, we get

$$\int_{-\infty}^{\infty} f_n(x)f_m(x)dx = \sum_{i=0}^{m}\binom{n}{\frac{n-m}{2}+i}\binom{m}{i} = \sum_{i=0}^{m}\binom{n}{\frac{n+m}{2}-i}\binom{m}{i} = \binom{n+m}{\frac{n+m}{2}}.$$

$$\square$$

3.4.2.   The case where $n>m$ and $n-m \equiv 1(\text{mod } 2)$.

Theorem 3.9. *Let $n \geq m$ and $n-m \equiv 1(\text{mod 2})$. Then we have*

$$\int_{-\infty}^{\infty} f_n(x)f_m(x)dx = \frac{1}{2}\binom{n+m+1}{\frac{n+m+1}{2}}.$$

Proof. In this case, no unit interval $I_{m,i}$ in supp(f_m) fit onto a unit interval $I_{n,j}$ in supp(f_n). If we assume that $\left(-\frac{m+1}{2}+i,-\frac{m+1}{2}+i+1\right) = \left(-\frac{n+1}{2}+j,-\frac{n+i}{2}+j+1\right)$ for some $i \in \{0,1,2,...,m\}$ and for some $j \in \{0,1,2,...,n\}$, then $n-m=2(j-i)$. This contradicts the assumption that $n-m \equiv 1(\text{mod 2})$. Define the half-unit intervals in supp(f_n) as

$$J_{n,i} := \left(-\frac{n+1}{2}+\frac{i}{2},-\frac{n+1}{2}+\frac{i+1}{2}\right),\quad i=0,1,2,...,2n+1, n \in \mathbb{N}. \tag{34}$$

We divide the $2n+2$ half unit intervals in supp(f_n) given in (34) into three classes: The first $n-m$, the next $2m+2$ and the last $n-m$ half-unit intervals. Note that $(n-m)+(2m+2)+(n-m) = 2n+2$. The first $n-m$ half-unit intervals in supp(f_n) are

$$J_{n,0},\ J_{n,1},\ ...,\ J_{n,n-m-1}.$$

The next $2m + 2$ half-unit intervals in the supp(f_n) are

$$J_{n,n-m}, \; J_{n,n-m+1}, \; ..., \; J_{n,n+m+1}.$$

The last $n - m$ half-unit intervals in the supp(f_n) are

$$J_{n,n+m+2}, \; J_{n,n+m+3}, \; ..., \; J_{n,2n+1}.$$

We claim that the middle $2m+2$ half-unit intervals of supp(f_n) coincides with the $2m+2$ half-unit intervals in supp(f_m). In fact, according to (34) we have

$$J_{n,n-m} = \left(-\frac{n+1}{2} + \frac{n-m}{2}, -\frac{n+1}{2} + \frac{n-m}{2} + \frac{1}{2} \right) = \left(-\frac{m+1}{2}, -\frac{m+1}{2} + \frac{1}{2} \right) = J_{m,0}.$$

Consequently,

$$J_{n,n-m} = J_{m,0} \subset I_{m,0}, \qquad\qquad J_{n,n-m} \subset I_{n,\frac{n-m+1}{2}},$$
$$J_{n,n-m+1} = J_{m,1} \subset I_{m,0}, \qquad\qquad J_{n,n-m+1} \subset I_{n,\frac{n-m+3}{2}},$$
$$J_{n,n-m+2} = J_{m,2} \subset I_{m,1}, \qquad\qquad J_{n,n-m+2} \subset I_{n,\frac{n-m+3}{2}},$$
$$J_{n,n-m+3} = J_{m,3} \subset I_{m,1}, \qquad\qquad J_{n,n-m+3} \subset I_{n,\frac{n-m+5}{2}},$$
$$\cdots\cdots\cdots \quad \cdots\cdots\cdots \qquad\qquad \cdots\cdots\cdots$$
$$J_{n,n+m} = J_{m,2m} \subset I_{m,m}, \qquad\qquad J_{n,n+m} \subset I_{n,\frac{n+m-1}{2}},$$
$$J_{n,n+m+1} = J_{m,2m+1} \subset I_{m,m}. \qquad\qquad J_{n,n+m+1} \subset I_{n,\frac{n+m+1}{2}}.$$

f_m has values $\binom{m}{j}$, $j = 0, 1, 2, ..., m$ on the unit intervals $I_{m,j}$, that are defined in (14). On the other hand, f_n has no uniform value on such unit intervals. For example, consider the first unit interval $I_{m,0} = \left(-\frac{m+1}{2}, -\frac{m+1}{2} + 1 \right)$. In this unit interval, f_n has value $\binom{n}{\frac{n-m-1}{2}}$ in the first half unit interval $\left(-\frac{m+1}{2}, -\frac{m+1}{2} + \frac{1}{2} \right)$ and value $\binom{n}{\frac{n-m+1}{2}}$ in the remaining half-unit interval $\left(-\frac{m+1}{2} + \frac{1}{2}, -\frac{m+1}{2} + 1 \right)$. Therefore the product $f_n(x)f_m(x)$ is a simple function given by

$$f_n(x)f_m(x) = \sum_{j=0}^{m} \binom{m}{j} \binom{n}{\frac{n-m+2j-1}{2}} \chi_{J_{m,2j}}(x) + \sum_{j=0}^{m} \binom{m}{j} \binom{n}{\frac{n-m+2j+1}{2}} \chi_{J_{m,2j+1}}(x). \quad (35)$$

Integrating the simple function $f_n(x)f_m(x)$ given in (35), using Vandermonde's identity, we get

$$\int_{-\infty}^{\infty} f_n(x)f_m(x)dx = \frac{1}{2} \sum_{j=0}^{m} \binom{m}{j} \left[\binom{n}{\frac{n-m+2j-1}{2}} + \binom{n}{\frac{n-m+2j+1}{2}} \right]$$
$$= \frac{1}{2} \sum_{j=0}^{m} \binom{m}{j} \binom{n+1}{\frac{n-m+2j+1}{2}}$$
$$= \frac{1}{2} \binom{n+m+1}{\frac{n+m+1}{2}}.$$

$\square$

4. Formulation of some combinatorial inequalities by using some real analytic results

In this section, we apply some known results from real analysis on the sequences f_n to formulate new combinatorial inequalities. Specifically we apply Hölder's inequality, Young's inequality, and Minkowiski's inequality from real analysis.

4.1. Application of Hölder's inequality in the formulation of some combinatorial inequalities

Lemma 4.1 (Hölder's inequality). *For $1 < p < \infty$, let $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1(\mathbb{R})$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$.*

Theorem 4.2. *For $1 < p < \infty$, let $\frac{1}{p} + \frac{1}{q} = 1$, and $0 \leq n, m$ are integers. We have the following combinatorial inequalities:*

$$\binom{n+m}{\frac{n+m}{2}} \leq \left(\sum_{i=0}^{m} \binom{m}{i}^p \right)^{\frac{1}{p}} \left(\sum_{j=0}^{n} \binom{n}{j}^q \right)^{\frac{1}{q}}, \quad n \equiv m \, (mod \, 2), \tag{36}$$

$$\frac{1}{2} \binom{n+m+1}{\frac{n+m+1}{2}} \leq \left(\sum_{i=0}^{m} \binom{m}{i}^p \right)^{\frac{1}{p}} \left(\sum_{j=0}^{n} \binom{n}{j}^q \right)^{\frac{1}{q}}, \quad n - m \equiv 1 \, (mod \, 2). \tag{37}$$

Proof. Proof of inequalities given in (36) and (37) are obtained by applying Holder inequality to f_m and f_m and by using the results of Theorem 3.8 and Theorem 3.9. □

Corollary 4.3.
$$\binom{n+m}{\frac{n+m}{2}}^2 \leq \binom{2m}{m} \binom{2n}{n}, \quad n \equiv m \, (mod \, 2). \tag{38}$$

$$\frac{1}{4} \binom{n+m+1}{\frac{n+m+1}{2}}^2 \leq \binom{2m}{m} \binom{2n}{n}, \quad n - m \equiv 1 \, (mod \, 2). \tag{39}$$

Proof. The proofs of the inequalities in (38) and (39) follow from (36) and (37) by setting $p = q = 2$ and using (19). □

4.2. Application of Young's inequality in the formulation of some combinatorial inequalities

Lemma 4.4 (Young's inequality). *Assume that $1 \leq p, q, r \leq \infty$ satisfies*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1. \tag{40}$$

Let $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ then $f * g \in L^r(\mathbb{R})$ and $\|f * g\|_r \leq \|f\|_p \|g\|_q$.

Theorem 4.5. *Let $m, n \in \mathbb{N} \cup \{0\}$ with $m - n \equiv 0 \pmod 2$. Then for any $1 < p, q, r < \infty$ satisfying the condition in Young's inequality given (40), we have*

$$\left(\frac{2}{r+1} + \sum_{j=0}^{m+n-1} \frac{\binom{m+n}{j+1}^{r+1} - \binom{m+n}{j}^{r+1}}{(r+1)[\binom{m+n}{j+1} - \binom{m+n}{j}]} \right)^{\frac{1}{r}} \leq \left(\sum_{j=0}^{m} \binom{m}{j}^{q} \right)^{\frac{1}{q}} \left(\sum_{j=0}^{n} \binom{n}{j}^{p} \right)^{\frac{1}{p}}.$$

Proof. We apply young inequality to the convolution $f_m * f_n$. By (28) we have

$$f_m(x) * f_n(x) = \sum_{j=0}^{m+n} \binom{m+n}{j} E^{\frac{m+n}{2}-j} f_0(x) * f_0(x).$$

But by (32)

$$f_0(x) * f_0(x) = (1+x)\chi_{[-1,0)}(x) + (1-x)\chi_{[0,1)(x)}.$$

Then

$$E^{\frac{m+n}{2}-j}[(1+x)\chi_{[-1,0)}(x)] = (x + \frac{m+n}{2} + 1 - j)\chi_{[-\frac{m+n}{2}-1+j, -\frac{m+n}{2}+j)}(x), \qquad (41)$$

$$E^{\frac{m+n}{2}-j}\left[(1-x)\chi_{[0,1)}(x)\right] = (1 - x - \frac{m+n}{2} + j)\chi_{[-\frac{m+n}{2}+j, -\frac{m+n}{2}+j+1)}(x). \qquad (42)$$

Therefore,

$$f_m(x) * f_n(x) = \sum_{j=0}^{m+n} \binom{m+n}{j}(x + \frac{m+n}{2} + 1 - j)\chi_{I_j}(x) + (1 - x - \frac{m+n}{2} + j)\chi_{J_j}(x).$$

where

$$I_j := [-\frac{m+n}{2} - 1 + j, -\frac{m+n}{2} + j),$$
$$J_j := [-\frac{m+n}{2} + j, -\frac{m+n}{2} + j + 1), \quad j = 0, 1, 2, 3, ..., m+n. \qquad (43)$$

From (43) we note that the last $m+n$ intervals in the first collection I_j of unit intervals and the first $m + n$ intervals in the second collection J_j of unit intervals coincide. The first unit interval I_0 from the first collection and the last unit interval J_{m+n} remain. This is given by

$$I_{j+1} = J_j, \quad j = 0, 1, 2, ..., m+n-1. \qquad (44)$$

$$f_m(x) * f_n(x) = (1 + \frac{m+n}{2} + x)\chi_{I_0}(x)$$
$$+ \sum_{j=0}^{m+n-1} \left[\binom{m+n}{j}(1 - x - \frac{m+n}{2} + j) + \binom{m+n}{j+1}(x + \frac{m+n}{2} - j) \right] \chi_{J_j}(x)$$
$$+ (1 - x - \frac{m+n}{2} + j)\chi_{J_{m+n}}(x). \qquad (45)$$

Now (45) can be rearranged and rewritten as

$$f_m(x) * f_n(x) = (1 + \frac{m+n}{2} + x)\chi_{I_0}(x)$$

$$+ \sum_{j=0}^{m+n-1} \left[\binom{m+n}{j} + \left(\binom{m+n}{j+1} - \binom{m+n}{j} \right) \left(x + \frac{m+n}{2} - j \right) \right] \chi_{J_j}(x)$$

$$+ (1 - x - \frac{m+n}{2} + j) \chi_{J_{m+n}}(x). \tag{46}$$

Therefore,

$$|f_m(x) * f_n(x)|^r = (1 + \frac{m+n}{2} + x)^r \chi_{I_0}(x)$$

$$+ \sum_{j=0}^{m+n-1} \left[\binom{m+n}{j} + \left(\binom{m+n}{j+1} - \binom{m+n}{j} \right) \left(x + \frac{m+n}{2} - j \right) \right]^r \chi_{J_j}(x)$$

$$+ (1 - x - \frac{m+n}{2} + j)^r \chi_{J_{m+n}}(x). \tag{47}$$

Let us integrate the two separate terms as follows:

$$\int_{I_0} (1 + \frac{m+n}{2} + x)^r dx = \int_{-\mu-1}^{-\frac{m+n}{2}} (1 + \frac{m+n}{2} + x)^r dx = \int_0^1 (1 - x)^r dx = \frac{1}{1+r}, \tag{48}$$

and,

$$\int_{J_{m+n}} (1 + \mu - x)^r dx = \int_\mu^{\mu+1} (1 + \mu - x)^r dx = \int_0^1 x^r dx = \frac{1}{1+r}. \tag{49}$$

For notational convenience let

$$\beta_j := \binom{m+n}{j}, \quad \alpha_j := \binom{m+n}{j+1} - \binom{m+n}{j}. \tag{50}$$

For each j the integral of each term in the summation given in (47) over the unit interval J_j is

$$\int_{J_j} (\beta_j + \alpha_j (x + \frac{m+n}{2} - j))^r dx = \int_{-\frac{m+n}{2}+j}^{-\frac{m+n}{2}+j+1} (\beta_j + \alpha_j (x + \frac{m+n}{2} - j))^r dx$$

$$= \frac{(\alpha_j + \beta_j)^{r+1} - \beta_j^{r+1}}{\alpha_j(r+1)} = \frac{\binom{m+n}{j+1}^{r+1} - \binom{m+n}{j}^{r+1}}{(r+1) \left[\binom{m+n}{j+1} - \binom{m+n}{j} \right]}. \tag{51}$$

Then collecting the results in (48), (49), and (51) we get

$$\|f_m * f_n\|_r^r = \frac{2}{1+r} + \sum_{j=0}^{m+n-1} \frac{\binom{m+n}{j+1}^{r+1} - \binom{m+n}{j}^{r+1}}{(r+1) \left[\binom{m+n}{j+1} - \binom{m+n}{j} \right]}. \tag{52}$$

Now by considering that $f_n \in L^p(\mathbb{R})$, $f_m \in L^q(\mathbb{R})$, by using definition of the L^p norm of f_n, $n \in \mathbb{N} \cup \{0\}$ given in (17), and the result given the in (52), the Theorem follows by applying Young's inequality. □.

Corollary 4.6. *Let $m, n \in \mathbb{N} \cup \{0\}$ with $m - n \equiv 0 (mod\ 2)$. Then for any $1 < p = r < \infty$, we have*

$$\left(\frac{2}{r+1} + \sum_{j=0}^{m+n-1} \frac{\binom{m+n}{j+1}^{r+1} - \binom{m+n}{j}^{r+1}}{(r+1)[\binom{m+n}{j+1} - \binom{m+n}{j}]} \right)^{\frac{1}{r}} \leq 2^m \left(\sum_{j=0}^{n} \binom{n}{j}^r \right)^{\frac{1}{r}}$$

Proof. The proof follows from Theorem 4.5 by fixing $1 < r = p < \infty$, and $q = 1$. $\qquad\square$

Corollary 4.7. *Let $m, n \in \mathbb{N} \cup \{0\}$ with $m - n \equiv 0 (mod\ 2)$, we have*

$$1 + \frac{1}{2} \sum_{j=0}^{m+n-1} \binom{m+n+1}{j+1} \leq 2^{m+n} \tag{53}$$

Proof. The proof follows from Theorem 4.5 by fixing $r = p = q = 1$. Indeed,

$$1 + \frac{1}{2} \sum_{j=0}^{m+n-1} \binom{m+n+1}{j+1} = 1 + \frac{1}{2} \left(\sum_{j=0}^{m+n+1} \binom{m+n+1}{j} - 2 \right)$$

$$= \frac{1}{2} \sum_{j=0}^{m+n+1} \binom{m+n+1}{j} = 2^{m+n},$$

showing that equality holds for all $m, n \in \mathbb{N} \cup \{0\}$ with $m - n \equiv 0 (mod\ 2)$. $\qquad\square$

Note that the necessity of the inclusion of the condition that $n = m (mod\ 2)$ in Theorem 4.5 is that the denominator of the expression in the summation will never be zero in this case. The next theorem will treat the case where $n - m \equiv 1 (mod\ 2)$.

Theorem 4.8. *Let $m, n \in \mathbb{N} \cup \{0\}$ with $m - n \equiv 1 (mod\ 2)$. Then for any p, q, r satisfying the condition in Young's inequality given by (40), we have*

$$\left(\frac{2}{r+1} + \left(\frac{m+n}{\frac{m+n-1}{2}} \right)^r + \sum_{\substack{j=0 \\ j \neq \frac{m+n-1}{2}}}^{m+n-1} \frac{\binom{m+n}{j+1}^{r+1} - \binom{m+n}{j}^{r+1}}{(r+1)[\binom{m+n}{j+1} - \binom{m+n}{j}]} \right)^{\frac{1}{r}}$$

$$\leq \left(\sum_{j=0}^{m} \binom{m}{j}^q \right)^{\frac{1}{q}} \left(\sum_{j=0}^{n} \binom{n}{j}^p \right)^{\frac{1}{p}}.$$

Proof. For $m - n \equiv 1 (mod\ 2)$, $\binom{m+n}{j+1} = \binom{m+n}{j}$ when $j = \frac{m+n-1}{2}$. So, by the definition of α_j given in (50), $\alpha_{\frac{m+n-1}{2}} = 0$. The integral for the term corresponding to the index $j = \frac{m+n-1}{2}$ is the integral of the constant $(\beta_{\frac{m+n-1}{2}})^r = \left(\frac{m+n}{\frac{m+n-1}{2}} \right)^r$ over the unit interval $J_{\frac{m+n-1}{2}}$. This yields $\left(\frac{m+n}{\frac{m+n-1}{2}} \right)^r$. The integrals of other terms are similar to that of Theorem 4.5. This proves the Theorem. $\qquad\square$

Corollary 4.9. *Let $m, n \in \mathbb{N} \cup \{0\}$ with $m - n \equiv 1 (mod\ 2)$. Then for any $1 < p = r < \infty$ satisfying the condition in Young's inequality given by (40), we have*

$$\left(\frac{2}{r+1} + \binom{m+n}{\frac{m+n-1}{2}}^r + \sum_{\substack{j=0 \\ j \neq \frac{m+n-1}{2}}}^{m+n-1} \frac{\binom{m+n}{j+1}^{r+1} - \binom{m+n}{j}^{r+1}}{(r+1)[\binom{m+n}{j+1} - \binom{m+n}{j}]} \right)^{\frac{1}{r}} \leq 2^m \left(\sum_{j=0}^n \binom{n}{j}^r \right)^{\frac{1}{r}}.$$

Proof. The proof follows from Theorem 4.8 by fixing $1 < r = p < \infty$, and $q = 1$. □

Corollary 4.10. *Let $m, n \in \mathbb{N} \cup \{0\}$ with $m - n \equiv 1 (mod\ 2)$, we have*

$$1 + \binom{m+n}{\frac{n+m-1}{2}} + \frac{1}{2} \sum_{\substack{j=0 \\ j \neq \frac{m+n-1}{2}}}^{m+n-1} \binom{m+n+1}{j+1} \leq 2^{m+n}. \tag{54}$$

Proof. The proof follows from Theorem 4.8 by setting $r = p = q = 1$. In fact, for every $m, n \in \mathbb{N} \cup \{0\}$ with $m - n \equiv 1 (mod\ 2)$,

$$1 + \binom{m+n}{\frac{n+m-1}{2}} + \frac{1}{2} \sum_{\substack{j=0 \\ j \neq \frac{m+n-1}{2}}}^{m+n-1} \binom{m+n+1}{j+1}$$

$$= 1 + \binom{m+n}{\frac{n+m-1}{2}} - \binom{m+n+1}{\frac{n+m+1}{2}} + \frac{1}{2} \sum_{j=0}^{m+n-1} \binom{m+n+1}{j+1}$$

$$= \binom{m+n}{\frac{n+m-1}{2}} - \binom{m+n+1}{\frac{n+m+1}{2}} + \frac{1}{2} \sum_{j=0}^{m+n+1} \binom{m+n+1}{j}$$

$$= \binom{m+n}{\frac{n+m-1}{2}} - \binom{m+n+1}{\frac{n+m+1}{2}} + 2^{m+n}$$

$$= - \binom{m+n}{\frac{n+m-1}{2}} + 2^{m+n} \leq 2^{m+n}.$$

□

Lemma 4.11.
$$\|f_n * f_m\|_\infty = \max_{0 \leq j \leq m+n} \binom{m+n}{j} = \|f_{n+m}\|_\infty. \tag{55}$$

Proof. According to (46), $f_n * f_m$ is piecewise linear. We calculate the maximum value on each unit interval and then take the maximum over all the unit intervals. The maximum value on each unit interval appears at either end points of interval, as linear functions have no interior critical points. Accordingly,

$$\max_{x \in I_0} (1 + \frac{m+n}{2} + x) \chi_{I_0}(x) = \max\{0, 1\} = 1,$$

$$\max_{x \in J_j} \beta_j + \alpha_j(1 + \frac{m+n}{2} + x)\chi_{I_0}(x) = \max_{j=0,1,\dots,m+n-1} \left\{ \binom{m+n}{j}, \binom{m+n}{j+1} \right\},$$

$$\max_{x \in I_{m+n}} (1 - x - \frac{m+n}{2} + j)\chi_{I_{m+n}}(x) = \max\{1,0\} = 1.$$

Summarizing all the above results and taking overall maximum value we get

$$\|f_n * f_m\|_\infty = \max_{0 \le j \le m+n} \binom{m+n}{j}.$$

The second equality in (55) follows from (21) and (22). □

Theorem 4.12.

$$\max_{0 \le j \le m+n} \left\{ \binom{m+n}{j} \right\} \le 2^m \left[\max_{0 \le j \le n} \left\{ \binom{n}{j} \right\} \right].$$

Proof. The proof follows from the Young's inequality with $r = p = \infty$, and $q = 1$ so that

$$\|f_m * f_n\|_\infty \le \|f_n\|_\infty \|f_m\|_1.$$

We have applied Lemma 4.11 to find $\|f_m * f_n\|_\infty$ and used the results in (18), and (21) or (22). □

Corollary 4.13. *If m and n are both odd then*

$$\binom{m+n}{\frac{m+n}{2}} \le \binom{m}{\frac{m-1}{2}} 2^n.$$

If m and n are both even then

$$\binom{m+n}{\frac{m+n}{2}} \le \binom{m}{\frac{m}{2}} 2^n.$$

Corollary 4.14. *Let $n \in \mathbb{N}$. If n is odd then $\binom{2n}{n} \le \binom{n}{\frac{n-1}{2}} 2^n$. If n is even then $\binom{2n}{n} \le \binom{n}{\frac{n}{2}} 2^n$.*

4.3.   Application of Minkowiski's inequality for formulation of some combinatorial inequalities

Lemma 4.15 (Minkowski's Inequality)**.** *Let $1 \le p \le \infty$. For all $f, g \in L^p(\mathbb{R})$. Then*

$$\|f + g\|_p \le \|f\|_p + \|g\|_p. \tag{56}$$

Theorem 4.16. *Let $1 \le p < \infty$. Let $n, m \in \mathbb{N} \cup \{0\}$, with $n \ge m$ and $n - m \equiv 0 (mod\ 2)$. Then*

$$\left[\sum_{i=0}^{\frac{n-m-2}{2}} 2\binom{n}{i}^p + \sum_{i=0}^{m} \left[\binom{m}{i} + \binom{n}{\frac{n-m}{2} + i} \right]^p \right]^{\frac{1}{p}} \le \left[\sum_{i=0}^{n} \binom{n}{i}^p \right]^{\frac{1}{p}} + \left[\sum_{i=0}^{n} \binom{m}{i}^p \right]^{\frac{1}{p}}. \tag{57}$$

Proof. For the proof, we apply Minkowiski's inequality given in (56) to the elements f_m and f_n of the sequence defined in (9) so that we have $\|f_m + f_n\|_p \le \|f_m\|_p + \|f_n\|_p$. If $n > m$ and $n - m \equiv 0 \pmod 2$, then the middle $m + 1$ unit intervals in the support of $\operatorname{supp}(f_n)$ coincide with the $m + 1$ unit intervals in $\operatorname{supp}(f_m)$. Therefore,

$$f_n(x) + f_m(x) = \sum_{i=0}^{\frac{n-m-2}{2}} \binom{n}{i} \chi_{I_{n,i}}(x) + \sum_{i=0}^{m} \left[\binom{m}{i} + \binom{n}{\frac{n-m}{2} + i} \right] \chi_{I_{m,i}}(x)$$
$$+ \sum_{i=0}^{\frac{n-m-2}{2}} \binom{n}{\frac{n+m+2}{2} + i} \chi_{I_{n, \frac{n+m+2}{2} + i}}(x), \tag{58}$$

Consideration of the supports of f_m and f_m with the given condition $n > m$,

$$\int_{-\infty}^{\infty} |f_n(x) + f_m(x)|^p dx = \int_{-\frac{n+1}{2}}^{-\frac{m+1}{2}} |f_n(x)|^p dx + \int_{-\frac{m+1}{2}}^{\frac{m+1}{2}} |f_n(x) + f_m(x)|^p dx + \int_{\frac{m+1}{2}}^{\frac{n+1}{2}} |f_n(x)|^p dx, \tag{59}$$

and by the fact that f_n is even function we have

$$\int_{-\frac{n+1}{2}}^{-\frac{m+1}{2}} |f_n(x)|^p dx = \int_{\frac{m+1}{2}}^{\frac{n+1}{2}} |f_n(x)|^p dx = \sum_{i=0}^{\frac{n-m-2}{2}} \binom{n}{i}^p, \tag{60}$$

$$\int_{-\frac{m+1}{2}}^{\frac{m+1}{2}} |f_n(x) + f_m(x)|^p dx = \sum_{i=0}^{m} \left[\binom{m}{i} + \binom{n}{\frac{n-m}{2} + i} \right]^p. \tag{61}$$

Plugging the results in (60) and (61) into (59) and then taking the pth root yields the left hand side of the inequality in (57). The right hand of (57) follows normally from the result in (17). $\qquad \square$

Corollary 4.17.

$$\left(2 + \sum_{i=0}^{m} \left[\binom{m}{i} + \binom{m+1}{i+1} \right]^p \right)^{\frac{1}{p}} \le \left[\sum_{i=0}^{m+2} \binom{m+2}{i}^p \right]^{\frac{1}{p}} + \left[\sum_{i=0}^{m} \binom{m}{i}^p \right]^{\frac{1}{p}}.$$

Proof. If $n = m + 2$ then $n - m =$ mod 2. The proof the Corollary follows by setting $n = m + 2$ in Theorem 4.16. $\qquad \square$

Theorem 4.18. *Let $1 \le p < \infty$. Let $m \in \mathbb{N} \cup \{0\}$. Then*

$$\left[2 + \frac{1}{2} \sum_{j=0}^{m} \left[\binom{m}{j} + \binom{m+1}{j} \right]^p + \left[\binom{m}{j} + \binom{m+1}{j+1} \right]^p \right]^{\frac{1}{p}}$$
$$\le \left[\sum_{j=0}^{m} \binom{m}{j}^p \right]^{\frac{1}{p}} + \left[\sum_{j=0}^{m+1} \binom{m+1}{j}^p \right]^{\frac{1}{p}}. \tag{62}$$

Let $1 \leq p < \infty$. Let $n, m \in \mathbb{N} \cup \{0\}$, with $n - m \geq 3$ and $n - m \equiv 1 (mod \ 2)$. Then

$$\left[\sum_{i=0}^{\frac{n-m-3}{2}} \binom{n}{i}^p + \binom{n}{\frac{n-m-1}{2}}^p + \frac{1}{2} \sum_{j=0}^{m} \left(\left[\binom{m}{j} + \binom{n}{\frac{n-m+2j-1}{2}} \right]^p + \left[\binom{m}{j} + \binom{n}{\frac{n-m+2j+1}{2}} \right]^p \right) \right]^{\frac{1}{p}}$$

$$\leq \left[\sum_{j=0}^{m} \binom{m}{j}^p \right]^{\frac{1}{p}} + \left[\sum_{j=0}^{n} \binom{n}{j}^p \right]^{\frac{1}{p}} \tag{63}$$

Proof. For $n = m + 1$ we have

$$f_n(x) + f_m(x) = \chi_{J_{n,0}}(x)$$
$$+ \sum_{j=0}^{m} \left[\binom{m}{j} + \binom{m+1}{j} \right] \chi_{J_{m,2j}}(x) + \left[\binom{m}{j} + \binom{m+1}{j+1} \right] \chi_{J_{m,2j+1}}(x)$$
$$+ \chi_{J_{n,2n+1}}(x), \tag{64}$$

whereas for $n \geq m + 3$ and $n - m \equiv 1 (mod \ 2)$ we have

$$f_n(x) + f_m(x) = \sum_{i=0}^{\frac{n-m-3}{2}} \binom{n}{i} \chi_{I_{n,i}}(x) + \binom{n}{\frac{n-m-1}{2}} \chi_{J_{n,n-m-1}}(x)$$
$$+ \sum_{j=0}^{m} \left[\binom{m}{j} + \binom{n}{\frac{n-m+2j-1}{2}} \right] \chi_{J_{m,2j}}(x) + \left[\binom{m}{j} + \binom{n}{\frac{n-m+2j+1}{2}} \right] \chi_{J_{m,2j+1}}(x)$$
$$+ \binom{n}{\frac{n+m+1}{2}} \chi_{J_{n,n+m+1}}(x) + \sum_{i=0}^{\frac{n-m-3}{2}} \binom{n}{\frac{m+n+3}{2}+i} \chi_{I_{n,\frac{m+n+3}{2}+i}}(x), \tag{65}$$

where I_i and J_j are as defined in (14) and (34) respectively. Now (62) follows by applying Minkowski's inequality to (64) and (63) follows by applying Minkowski's inequality to (65). □

5. Conclusions and Remarks

In this paper, we derived some new combinatorial inequalities by applying some well known real analytic inequalities. Towards this goal, we calculated the closed form of expression of some recursively-defined sequence of functions. We could have began our task from the closed form of the sequence. However the author believes that the task of calculation of the closed form expression of the sequence from the recursive definition of the starting sequence has its own beauty and adds some value to the readers of this work. Different combinatorial inequalities may be derived by using different classes of sequences and some real analytic techniques.

The key point in the study of recursively defined sequences similar to the ones given in (9) is based on the choice of the pair $(\mathcal{A}, f_0)$, where $\mathcal{A}$ an operator, and f_0 the initial function. Here in this paper, we used the pair $(\mathcal{A}, f_0(x)) = \left(\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}} \right), \chi_{(-1/2,1/2)}(x) \right)$.

Table 2. Table for some combinatorial inequalities derived by using real-analytic inequalities

Real-analytic inequality used	Combinatorial inequality derived
H ölder's Inequality	$\binom{n+m}{\frac{n+m}{2}} \leq \left(\sum_{i=0}^{m} \binom{m}{i}^p\right)^{\frac{1}{p}} \left(\sum_{j=0}^{n} \binom{n}{j}^q\right)^{\frac{1}{q}},$ $n \equiv m (\text{mod } 2)$ $\frac{1}{2}\binom{n+m+1}{\frac{n+m+1}{2}} \leq \left(\sum_{i=0}^{m} \binom{m}{i}^p\right)^{\frac{1}{p}} \left(\sum_{j=0}^{n} \binom{n}{j}^q\right)^{\frac{1}{q}}, \quad n-m \equiv 1(\text{mod } 2)$ $\binom{n+m}{\frac{n+m}{2}}^2 \leq \binom{2m}{m}\binom{2n}{n}, \quad n \equiv m(\text{mod } 2)$ $\frac{1}{4}\binom{n+m+1}{\frac{n+m+1}{2}}^2 \leq \binom{2m}{m}\binom{2n}{n}, \quad n-m \equiv 1(\text{mod } 2)$
Young's Inequality	$\left(\frac{2}{r+1} + \sum_{j=0}^{m+n-1} \frac{\binom{m+n}{j+1}^{r+1}-\binom{m+n}{j}^{r+1}}{(r+1)[\binom{m+n}{j+1}-\binom{m+n}{j}]}\right)^{\frac{1}{r}} \leq \left(\sum_{j=0}^{m}\binom{m}{j}^q\right)^{\frac{1}{q}} \left(\sum_{j=0}^{n}\binom{n}{j}^p\right)^{\frac{1}{p}}.$ $n \equiv m(\text{mod } 2)$ $\left(\frac{2}{r+1} + \sum_{j=0}^{m+n-1} \frac{\binom{m+n}{j+1}^{r+1}-\binom{m+n}{j}^{r+1}}{(r+1)[\binom{m+n}{j+1}-\binom{m+n}{j}]}\right)^{\frac{1}{r}} \leq 2^m \left(\sum_{j=0}^{n}\binom{n}{j}^r\right)^{\frac{1}{r}}, \quad n-m \equiv 0(\text{mod } 2).$ $1 + \frac{1}{2}\sum_{j=0}^{m+n-1} \binom{m+n+1}{j+1} \leq 2^{m+n},$ $n = m(\text{mod } 2).$ $\left(\frac{2}{r+1} + \binom{m+n}{\frac{m+n-1}{2}}^r + \sum_{\substack{j=0 \\ j\neq\frac{m+n-1}{2}}}^{m+n-1} \frac{\binom{m+n}{j+1}^{r+1}-\binom{m+n}{j}^{r+1}}{(r+1)[\binom{m+n}{j+1}-\binom{m+n}{j}]}\right)^{\frac{1}{r}} \leq \left(\sum_{j=0}^{m}\binom{m}{j}^q\right)^{\frac{1}{q}} \left(\sum_{j=0}^{n}\binom{n}{j}^p\right)^{\frac{1}{p}}$ $n-m \equiv 1(\text{mod } 2).$ $\left(\frac{2}{r+1} + \binom{m+n}{\frac{m+n-1}{2}}^r + \sum_{\substack{j=0 \\ j\neq\frac{m+n-1}{2}}}^{m+n-1} \frac{\binom{m+n}{j+1}^{r+1}-\binom{m+n}{j}^{r+1}}{(r+1)[\binom{m+n}{j+1}-\binom{m+n}{j}]}\right)^{\frac{1}{r}} \leq 2^m \left(\sum_{j=0}^{n}\binom{n}{j}^r\right)^{\frac{1}{r}}$ $n-m \equiv 1(\text{mod } 2).$ $\binom{m+n}{\frac{m+n}{2}} \leq \binom{m}{\frac{m-1}{2}}2^n,$ if m and n are both odd. $\binom{m+n}{\frac{m+n}{2}} \leq \binom{m}{\frac{m}{2}}2^n,$ if m and n are both even.
Minkowiski's Inequality:	$\left[\sum_{i=0}^{\frac{n-m-2}{2}} 2\binom{n}{i}^p + \sum_{i=0}^{m}\left[\binom{m}{i}+\binom{n}{\frac{n}{2}+i}\right]^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=0}^{n}\binom{n}{i}^p\right]^{\frac{1}{p}} + \left[\sum_{i=0}^{m}\binom{m}{i}^p\right]^{\frac{1}{p}},$ $\left(2 + \sum_{i=0}^{m}\left[\binom{m}{i}+\binom{m+1}{i+1}\right]^p\right)^{\frac{1}{p}} \leq \left[\sum_{i=0}^{m+2}\binom{m+2}{i}^p\right]^{\frac{1}{p}} + \left[\sum_{i=0}^{m}\binom{m}{i}^p\right]^{\frac{1}{p}},$ $\left[2 + \frac{1}{2}\sum_{j=0}^{m}\left[\binom{m}{j}+\binom{m+1}{j}\right]^p + \left[\binom{m}{j}+\binom{m+1}{j+1}\right]^p\right]^{\frac{1}{p}} \leq \left[\sum_{j=0}^{m}\binom{m}{j}^p\right]^{\frac{1}{p}} + \left[\sum_{j=0}^{m+1}\binom{m+1}{j}^p\right]^{\frac{1}{p}},$ $\left[\sum_{i=0}^{\frac{n-m-3}{2}}\binom{n}{i}^p + \binom{n}{\frac{n-m-1}{2}}^p + \frac{1}{2}\sum_{j=0}^{m}\left(\left[\binom{m}{j}+\binom{n}{\frac{n-m+2j-1}{2}}\right]^p + \left[\binom{m}{j}+\binom{n}{\frac{n-m+2j+1}{2}}\right]^p\right)\right]^{\frac{1}{p}}$ $\leq \left[\sum_{j=0}^{m}\binom{m}{j}^p\right]^{\frac{1}{p}} + \left[\sum_{j=0}^{n}\binom{n}{j}^p\right]^{\frac{1}{p}}.$

The author technically choose these operator and initial function f_0 to generate a sequence of simple function whose distinct values are binomial coefficients. Some important properties of the sequence are studied. Well known combinatorial identities were used in the study of the sequence. For example, we have used Vandermonde's identity in the evaluation of the integrals of the product $f_n * f_m$ over $\mathbb{R}$.

Conflict of interests

The author declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The author is thankful to the anonymous reviewers for their constructive and valuable suggestions.

Data availability

The author assures the publishers to provide the data or sources of data that are used in this paper upon request.

Funding

This Research work is not funded by any institution or person.

References

[1] H. Alzer and H. Prodinger. Identities and inequalities for sums involving binomial coefficents. *Intgers*, 20:#A9, 2020.

[2] J. L. Coolidge. The story of binomial theorem. *The American Mathematical Monthly*, 56(3):147–157.

[3] H. W. Gould. *Combinatorial Identities for Stirling Numbers*. World Scientific Publishing Company, 2016.

[4] V. Kadets. *A Course in Functional Analysis and Measure Theory*. Springer International Publishing, 2018.

Hailu Bikila Yadeta
Salale University, College of Natural Sciences, Department of Mathematics
Fiche, Oromia Region, Ethiopia
E-mail haybik@gmail.com