An Elementary Proof of Thomae's Formulae

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1 Introduction

Thomae's formulae go back to two papers $\{T_1\}$,$\{T_2\}$ and express the proportionalities between the 4th powers of the non-vanishing theta constants, and polynomials in the variables $\lambda_1, \cdots, \lambda_{2g-1}$, which are the algebraic parameters of the hyperelliptic surface. They take the form

$$\frac{\theta^4 \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0, \Pi)}{P_{\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}} (\lambda_1, \cdots, \lambda_{2g-1})} = K,$$

where K is independent of $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$. Since there are precisely $\begin{pmatrix} 2g+1 \\ g \end{pmatrix}$ non-vanishing theta constants, this is the number of terms on the left hand side.

Thomae's formulae can be viewed as a generalization of the Λ- function of elliptic curves, a subject studied for example in {F},{G}. In those papers one does not obtain Thomae but rather express the λ_i, the algebraic parameters of the hyperelliptic curve in terms of the elements of the period matrix $\pi_{i,j}$ i,j=1,...g which are of course transcendental parameters. There are however instances where Thomae's formulae are actually essential {S} . Thomae's formulae have been generalized to the case of Z_n curves in {N} .

In this note we give a proof of Thomae's formulae for hyperelliptic surfaces using only extremely elementary tools. In addition we shall also prove Thomae type formulae for Z_3 curves, that is, curves with an algebraic equation of the form $w^3 = \prod_{i=0}^{3r-2}(z - \lambda_i)$ using similar techniques.

Our method is based on two stages. First, we compare certain pairs of theta constants attached to certain characteristics, obtaining equalities by means of elementary theta functions theory. Then we show that all pairs may be compared to each other. The second stage uses a simple combinatorial argument for the Z_3 case. In the hyperelliptic case we use an inductive argument for the second stage , but here also lies implicitly a combinatorial argument of a similar kind.

We begin with the algebraic equation defining a hyperelliptic surface of genus g.

$$w^2 = z(z-1)\Pi_{i=1}^{2g-1}(z - \lambda_i)$$

This is a branched two sheeted cover of the Riemann sphere, branched over the points $0, 1, \infty, \lambda_1, \cdots, \lambda_{2g-1}$

For a construction of a standard set of generators for the first homology group we refer to {FK} pp. 102-103. Denote this set by $\{\gamma_1, \ldots, \gamma_g, \delta_1, \ldots, \delta_g\}$.

A basis for the holomorphic differentials dual to the chosen homology basis is given by $\theta_1, \ldots, \theta_g$, where $\int_{\gamma_j} \theta_i = \delta_{ij}$, $\int_{\delta_j} \theta_i = \pi_{ij}$ for $i, j = 1, \cdots, g$.

We shall now recall some general notions regarding the Jacobi variety of a Riemann surface and theta functions.

The Jacobi variety of the surface is $J(S) = C^g/G$ where G is the group of translations of C^g generated by

$$< z \to z + e^{(i)}, z \to z + \pi^{(i)} >,$$

and the points of order two in $J(S)$,

$$\sum_{i=1}^{g} \frac{\epsilon_i'}{2} e^i + \sum_{i=1}^{g} \frac{\epsilon_i}{2} \pi^i = I\frac{\epsilon'}{2} + \Pi\frac{\epsilon}{2}$$

will be denoted by $\begin{pmatrix} \epsilon_1, \cdots, \epsilon_g \\ \epsilon_1', \cdots, \epsilon_g' \end{pmatrix} = \begin{pmatrix} \epsilon \\ \epsilon' \end{pmatrix}$.

If one chooses a base point $P_0 \in S$ one has a mapping

$$S \to J(S)$$

given by

$$P \to \int_{P_0}^{P} {}^t(\theta_1, \cdots, \theta_g)$$

which we shall denote by ϕ_{P_0}. One then may extend this mapping to the set of integral divisors of degree n by setting $\phi_{P_0}(P_1 \cdots P_n) = \sum_{i=1}^{n} \phi_{P_0}(P_i)$

Recall the definition of a theta function with characteristics:

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta, \Pi) = \sum_{N \in \mathbb{Z}^g} exp 2\pi i [\frac{1}{2} {}^t(N + \frac{\epsilon}{2}) \Pi (N + \frac{\epsilon}{2}) + {}^t(N + \frac{\epsilon}{2})(\zeta + \frac{\epsilon'}{2})]$$

where $\begin{bmatrix} \epsilon_1, \cdots, \epsilon_g \\ \epsilon_1', \cdots, \epsilon_g' \end{bmatrix} = \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} \in \mathbb{Z}^{2g}$,$\zeta \in \mathbb{C}^g, \Pi \in \mathcal{S}_g$. We shall in fact restrict ϵ_i, ϵ_i' to be 0 or 1 in our following discussion, as up to sign this gives us all the various theta functions with integer characteristics.

Recall also the following formula:

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\zeta, \Pi) = exp 2\pi i [\frac{\epsilon\epsilon'}{2}] \theta(\zeta, \Pi)$$

¿From this formula we may conclude that the 2^{2g} Theta functions with ϵ_i, ϵ_i' 0 or 1 partition into two classes, even and odd functions of ζ. The odd ones all vanish at $\zeta = 0$. Furthermore, if an even function vanishes at $\zeta = 0$ it vanishes to second order, so that if $\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$ is an even function and vanishes at $\zeta = 0$, then also $\frac{\partial \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}}{\partial \zeta_i}(0, \Pi) = 0$, $i = 1, \cdots, g$.

We shall also have a need in our discussions of the following formula:

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z + I\frac{\mu'}{2} + \Pi\frac{\mu}{2}, \Pi) =$$

$$exp\{2\pi i[-\frac{1}{8}{}^t\mu\Pi\mu - \frac{1}{4}{}^t\mu(\epsilon' + \mu') - \frac{1}{2}{}^t\mu z]\}\theta \begin{bmatrix} \epsilon + \mu \\ \epsilon' + \mu' \end{bmatrix} (z, \Pi) \tag{1}$$

The main result one has regarding these objects is due to Riemann. A discussion can be found in {FK}. Consider the local analytic function $\theta(\phi(P) - e)$ with $e \in \mathbb{C}^g$ on the Riemann surface. This either vanishes identically on the surface or vanishes at g points $P_1 \ldots P_g$ on the surface, s.t. $i(P_1 \ldots P_g) = 0$. Furthermore in the latter case we have $e = \phi_{P_0}(P_1, \ldots, P_g) + K_{p_0}$. The vector K_{P_0} is called the vector of Riemann constants with base point P_0.

In particular, for an integer characteristic $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$ we have: $\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\phi_{P_0}(P), \Pi)$ on the Riemann surface either vanishes identically on the surface or vanishes at g points $P_1 \ldots P_g$ on the surface, s.t. $i(P_1 \ldots P_g) = 0$. (This follows from equation (1) and the preceding result). In the latter case we have

$$\begin{pmatrix} \epsilon \\ \epsilon' \end{pmatrix} = \phi_{P_0}(P_1, \ldots, P_g) + K_{p_0}.$$

Note that if $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$ is even then $P_i \neq P_0$ for $i = 1, \ldots, g$, otherwise the function would vanish identically on the surface. This would follow from the Riemann vanishing theorem.

2 Points of Order 2 on the Jacobi variety of hyperelliptic Riemann surfaces

We return to hyperelliptic Riemann surfaces given by

$$w^2 = z(z - 1)\Pi_{i=1}^{2g-1}(z - \lambda_i).$$

It is well known that if the base point P_0 of the aforementioned mapping ϕ_{P_0} is chosen as any of the $2g + 2$ branch points $0, 1, \infty, \lambda_1, \ldots, \lambda_{2g-1}$ then the image of any other branch point is always of order 2.

In fact, it is easy to see that all points of order 2 are given as sums of k ($k \leq g$) distinct points of the $2g + 1$ points of order 2, $\phi_0(1), \phi_0(\infty), \phi_0(\lambda_1), \ldots, \phi_0(\lambda_{2g-1})$.

For the sake of completeness we shall prove this fact:

Take all sums of k such points $\sum_{i=1}^{k} \phi(P_i)$ for $k = 1, \cdots, g$. we thus get $2^{2g} - 1$ sums. Adding 0 we get exactly 2^{2g} sums, which are all points of order 2. Clearly there are exactly 2^{2g} points of order 2 ,

We now need to show that these are all distinct. This follows from the fact that for any divisor $P_1 \ldots P_k$ where $k \leq g$ and the P_i are distinct branch points of S we have $i(P_1 \ldots P_k) = g - k$.

This in turn follows for example from the fact that on a hyperelliptic surface, if a holomorphic differential vanishes at a branch point then it vanishes there to even order. Therefore $i(P_1 \ldots P_g) = 0$, and hence $i(P_1 \ldots P_k) = g - k$.

Suppose now that $\phi_{P_0}(P_1 \dots P_k) = \phi_{P_0}(Q_1 \dots Q_l)$ where $k \leq l$.

So $\phi_{P_0}(P_1 \dots P_k P_0^{l-k}) = \phi_{P_0}(Q_1 \dots Q_l)$. We then have by Abel's theorem that there exists a meromorphic function on S whose poles are $Q_1, \cdots, Q_l$.

But the Riemann-Roch theorem implies that $r[\frac{1}{Q_1 \dots Q_l}] = l - g + 1 + g - l = 1$ so this is impossible, unless $k = l$ and $\{P_1, \dots, P_k\} = \{Q_1, \dots, Q_k\}$.

We now fix the base point of ϕ_{P_0} to be 0, (that is we look at ϕ_0). One may compute explicitly the images of the various branch points under ϕ_0. A point of order 2 is called odd if in the representation

$$\begin{pmatrix} \epsilon \\ \epsilon' \end{pmatrix}$$

we have $\epsilon.\epsilon' = 1$. (In this case $\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$ is odd). Otherwise call it even. It turns out that $\phi_0(\lambda_{2k-1})$ is an odd point for all $k = 1, \dots, g$. We then have that $\theta[0](\phi_0(p), \Pi)$ vanishes at these g points (just use equation (1)) . As $\theta[0](\phi_0(p), \Pi)$ is non vanishing, these are precisely its zeroes. For more details details see {FK} (pp. 329-331). From the discussion in the end of section 1 we may now conclude that $\phi_0(\lambda_1 \lambda_3 \dots \lambda_{2g-1}) = -K_0$. We also conclude from the same discussion that the even non-vanishing characteristics are precisely the $\begin{pmatrix} 2g+1 \\ g \end{pmatrix}$ points of the form $\phi_0(P_1 \dots P_g) + K_0$ where $P_i \in (1, \infty, \lambda_1, \dots, \lambda_{2g-1})$, and $P_i \neq P_j$ for $i \neq j$.

Note that we may easily switch from this last representation of a non-vanishing even characteristic to a representation as a sum of k points $k \leq g$ using $\phi_0(\lambda_1, \lambda_3, \dots, \lambda_{2g-1}) = -K_0 = K_0$. In the following we shall use this relationship in order to characterize the non-vanishing even characteristic in terms of their representation as a sum of $k \leq g$ points. This will eventually allow us to prove Thomae's formulae using an inductive argument.

We begin by recaling a simple lemma:

Lemma 2.1 $\phi_0(1) + \phi_0(\infty) + \sum_{i=1}^{2g-1} \phi_0(\lambda_i) = 0$

Proof: This follows from the fact that the divisor of the function w on S is $\frac{0,1,\lambda_1,\dots,\lambda_{2g-1}}{\infty^{2g+1}}$, so $\phi_0(0) + \phi_0(1) + \sum_{i=1}^{2g-1} \phi_0(\lambda_i) - (2g+1)\phi_0(\infty) = 0$. This however can be rewritten as $\phi_0(1) + \phi_0(\infty) + \sum_{i=1}^{2g-1} \phi_0(\lambda_i) = 0$ because -$(2g+1)\phi_0(\infty) = -\phi_0(\infty) = \phi_0(\infty)$.

Suppose now we have $P_1, \dots, P_g$ where $P_i \in \{1, \infty, \lambda_1, \dots, \lambda_{2g-1}\}$ and $P_i \neq P_j$ for $i \neq j$.

Consider

$$\begin{pmatrix} \epsilon \\ \epsilon' \end{pmatrix} = \phi_0(P_1 \dots P_g) + K_0 = \phi_0(P_1 \dots P_g) + \phi_0(\lambda_1 \lambda_3 \dots \lambda_{2g-1}).$$

Suppose exactly k points of the P_i are odd points, that is, of the λ_i with i odd. Then we have

$$\phi_0(P_1 \dots P_g) + \phi_0(\lambda_1 \lambda_3 \dots \lambda_{2g-1}) =$$
$$\phi_0(P_{i_1} \dots P_{i_{g-k}}) + \phi_0(\lambda_{i_1} \dots \lambda_{i_{g-k}})$$

Where $P_{i_1} \dots P_{i_{g-k}}$ are all even points. If $2(g-k) \leq g$ then we get a representation of this even characteristic as a sum of $2(g-k) = r \leq g$ points where $\frac{r}{2}$ points are odd and $\frac{r}{2}$ are even points. If on the other hand $2(g-k) > g$ then using the lemma above, we may take the sum of the complement of the set $P_{i_1}, \dots, P_{i_{g-k}}, \lambda_{i_1}, \dots, \lambda_{i_{g-k}}$ in the set $1, \infty, \lambda_1, \dots, \lambda_{2g-1}$

to represent $\left(\begin{array}{c} \epsilon \\ \epsilon' \end{array} \right)$. This complement consists of $r = 2k+1$ points, of which $k+1$ are even and k are odd.

On the other hand we may go the other way round to show that each sum of one of these sorts, is a non-vanishing even characteristic. If we have a sum of k odd points and k even points s.t. $2k \leq g$ then add the "missing" odd points to the sum of the odd points and to the sum of the even points to get a sum of the form $\phi_0(P_1, \dots, P_g) + K_0$. If the sum is of k odd points and $2k+1$ even points, then turn over again to the sum of the complement set to get a sum of an equal number of odd and even points, and proceed as in the first case.

Thus we have proven the following

Lemma 2.2 $\left[\begin{array}{c} \epsilon \\ \epsilon' \end{array} \right]$ *corresponds to an even non-vanishing theta constant iff it is either of the form* $\sum_{i=1}^{k} \phi_0(P_i) + \sum_{i=1}^{k} \phi_0(Q_i)$*, where the* P_i *correspond to even points of order 2, and the* Q_i *correspond to odd points of order 2, and* $2k \leq g$

or of the form $\sum_{i=1}^{k+1} \phi_0(P_i) + \sum_{i=1}^{k} \phi_0(Q_i)$ *where the* P_i *and the* Q_i *are as before, and* $2k+1 \leq g$.

3 The proof of Thomae's Formulae

Lemma 3.1 *Let* $\theta \left[\begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] (\phi_0(P), \Pi)$ *and* $\theta \left[\begin{array}{c} \tilde{\epsilon} \\ \tilde{\epsilon'} \end{array} \right] (\phi_0(P), \Pi)$ *be two nonidentically vanishing theta functions with even characteristics with respective zeros* $P_1 ... P_g$ *and* $Q_1 ... Q_g$*. Suppose that*

$$\left(\begin{array}{c} \tilde{\epsilon} \\ \tilde{\epsilon'} \end{array} \right) = \left(\begin{array}{c} \epsilon \\ \epsilon' \end{array} \right) + \phi_0(R)$$

where R *is a branch point of the surface. Then* $R \neq P_i, Q_i$ *for any i=1,...,g and also* $P_i \neq Q_j$ *for all i,j=1,...,g . In other words* $\theta \left[\begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] (\phi_0(p), \Pi)$ *and* $\theta \left[\begin{array}{c} \tilde{\epsilon} \\ \tilde{\epsilon'} \end{array} \right] (\phi_0(p), \Pi)$ *have no common zeroes.*

Proof: The hypothesis implies that

$$\phi_0(P_1 ... P_g) + \phi_0(R) = \phi_0(Q_1 ... Q_g).$$

If $R = P_1$ (wlog) we would have

$$\phi_0(P_1 ... P_g P_1) = \phi_0(P_2 ... P_g) = \phi_0(0 P_2 ... P_g) = \phi_0(Q_1 ... Q_g)$$

This would imply that $i(Q_1 ... Q_g) \geq 1$ which is a contradiction. Hence $R \neq P_i$ for any i. In a similar manner one can show $R \neq Q_i$ for any i. (Just take $\phi_0(R)$ to the other side of the equation an proceed as before).

Suppose there indeed was a common zero (without loss of generality) $P_1 = Q_1$. We can then write

$$\phi_0(P_1 ... P_g R) = \phi_0(0 Q_1 ... Q_g)$$

which becomes

$$\phi_0(P_2 ... P_g R) = \phi_0(0 Q_2 ... Q_g).$$

We have already seen in section 2 that this is impossible.

Equipped with this lemma we turn now to our problem.

We start with a definition:

Definition 3.1 *Let* $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$ *be an even non-vanishing integer characteristic. Let* $P_1, \ldots, P_g$ *be the zeroes of* $\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\phi_0(P), \Pi)$*. Then we denote by* $Z \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$ *the set of points* $\{P_1, \ldots, P_g\} \cup \{0\}$

Now we prove:

Proposition 3.1 *Let* $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$ *and* $\begin{bmatrix} \tilde{\epsilon} \\ \tilde{\epsilon}' \end{bmatrix}$ *be as in lemma (3.1), then we have*

$$\frac{\theta^4 \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0, \Pi)}{\prod_{\lambda_i, \lambda_j \in Z \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}, \lambda_i < \lambda_j} (\lambda_i - \lambda_j) \cdot \prod_{\lambda_i, \lambda_j \notin Z \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}, \lambda_i < \lambda_j} (\lambda_i - \lambda_j)}$$
$$= \frac{\theta^4 \begin{bmatrix} \tilde{\epsilon} \\ \tilde{\epsilon}' \end{bmatrix} (0, \Pi)}{\prod_{\lambda_i, \lambda_j \in Z \begin{bmatrix} \tilde{\epsilon} \\ \tilde{\epsilon}' \end{bmatrix}, \lambda_i < \lambda_j} (\lambda_i - \lambda_j) \cdot \prod_{\lambda_i, \lambda_j \notin Z \begin{bmatrix} \tilde{\epsilon} \\ \tilde{\epsilon}' \end{bmatrix}, \lambda_i < \lambda_j} (\lambda_i - \lambda_j)}$$

Proof: In the following we shall use formula (1) of section 1.

We have: $\frac{\theta^2 \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\phi_0(P), \Pi)}{\Theta^2 \begin{bmatrix} \tilde{\epsilon} \\ \tilde{\epsilon}' \end{bmatrix} (\phi_0(P), \Pi)} = c \cdot \frac{\Pi_{i=1}^g (z - z(P_i))}{\Pi_{i=1}^g (z - z(Q_i))}$

where c is some constant, and $P_1, \ldots, P_g$ and $Q_1, \ldots, Q_g$ are again the zeroes of $\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\phi_0(P), \Pi)$ and $\theta \begin{bmatrix} \tilde{\epsilon} \\ \tilde{\epsilon}' \end{bmatrix} (\phi_0(P), \Pi)$ respectively - one can see this is a well defined function on S using elementary theta function identities (see e.g. {FK} p. 302), and it has the same divisor as the rational function of z on the right.

Note that according to lemma (3.1) all the P_i and Q_i are distinct and distinct from one another.

Moreover, none of them is 0 because an even characteristic s.t. $\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\phi_0(0), \Pi) = 0$ vanishes identically. (See the end of section 1).

We have now accounted for $2g + 1$ of the branch points and exactly one remains. This has to be R as defined in lemma (3.1).

Thus, to find c we may set $P = 0$ and we get:

$$c = \frac{\theta^2 \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0)}{\theta^2 \begin{bmatrix} \tilde{\epsilon} \\ \tilde{\epsilon}' \end{bmatrix} (0)} \cdot \frac{\Pi_{i=1}^g z(Q_i)}{\Pi_{i=1}^g z(P_i)}$$

In the next computation we warn the reader that we have not actually computed exactly but only up to sign. Set now $P = R$, then we have, using formula (1) (here is where we are neglecting the sign) in order to calculate the left hand side of the equation,

$$\frac{\theta^2\begin{bmatrix}\tilde{\epsilon}\\ \tilde{\epsilon}'\end{bmatrix}(\phi_0(0),\Pi)}{\theta^2\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\phi_0(0),\Pi)} = \frac{\theta^2\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\phi_0(R),\Pi)}{\theta^2\begin{bmatrix}\tilde{\epsilon}\\ \tilde{\epsilon}'\end{bmatrix}(\phi_0(R),\Pi)} = \frac{\theta^2\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0,\Pi)}{\theta^2\begin{bmatrix}\tilde{\epsilon}\\ \tilde{\epsilon}'\end{bmatrix}(0,\Pi)} \cdot \frac{\Pi_{i=1}^g z(Q_i)}{\Pi_{i=1}^g z(P_i)} \cdot \frac{\Pi_{i=1}^g (z(R)-z(P_i))}{\Pi_{i=1}^g (z(R)-z(Q_i))}$$

And so we have:

$$\frac{\theta^4\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0,\Pi)}{\theta^4\begin{bmatrix}\tilde{\epsilon}\\ \tilde{\epsilon}'\end{bmatrix}(0,\Pi)} = \frac{\Pi_{i=1}^g z(P_i)}{\Pi_{i=1}^g z(Q_i)} \cdot \frac{\Pi_{i=1}^g (z(R)-z(Q_i))}{\Pi_{i=1}^g (z(R)-z(P_i))} \tag{2}$$

Note that $R, Q_1, \ldots, Q_g \notin Z\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}$ whereas $0, P_1, \ldots, P_g \in Z\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}$, and again $R, P_1, \ldots, P_g \notin Z\begin{bmatrix}\tilde{\epsilon}\\ \tilde{\epsilon}'\end{bmatrix}$ and $0, Q_1, \ldots, Q_g \in Z\begin{bmatrix}\tilde{\epsilon}\\ \tilde{\epsilon}'\end{bmatrix}$.

To get the equality that the proposition states, we just have to multiply the denominator and the numerator of (2) by all the differences $z(P_i) - z(P_j)$ and $z(Q_i) - z(Q_j)$ (where $i \neq j$).

It is now clear that the polynomials we are looking for are

$$P_{\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}}(\lambda_1, \ldots, \lambda_{2g-1}) = \Pi_{\lambda_i \in Z\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}, \lambda_i < \lambda_j}(\lambda_i - \lambda_j) \cdot \Pi_{\lambda_i \notin Z\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}, \lambda_i < \lambda_j}(\lambda_i - \lambda_j)$$

What is left to do in order to prove the main theorem is to use an inductive argument so as to get the equality we look for for any two non-vanishing even characteristics. For this we use lemma (2.2):

Use induction on the minimal number r such that $\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}$ can be written as the sum of r terms $\phi(P_i)$ as in the lemma.

For $r = 1$ the theorem is a result of proposition (3.1).

Suppose we have proved the theorem for $r - 1$, and $\begin{bmatrix}\tilde{\epsilon}\\ \tilde{\epsilon}'\end{bmatrix}$ can be written as the some of r terms. Then, if r is even, we may subtract one 'odd' term, and the result will be a non vanishing even characteristic which can be written as a sum of $r-1$ terms. Using proposition (3.1), and the induction hypothesis, we get the result.

If r is odd, then we may subtract an 'even' term, and conclude in a similar manner . We have thus proved

Theorem 3.1 *Let* $\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0,\Pi)$ *be an even non vanishing theta constant on the hyperelliptic surface. Then*

$$\frac{\theta^4\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0,\Pi)}{\prod_{\lambda_i < \lambda_j \in Z\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}}(\lambda_i - \lambda_j) \cdot \prod_{\lambda_i < \lambda_j \notin Z\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}}(\lambda_i - \lambda_j)} = k$$

where k is independent of the characteristic.

4 Nonspecial Divisors on Z_3 Curves

Let $w^3 = \prod_{i=0}^{3r-2}(z-\lambda_i)$ be the algebraic equation of a three sheeted cover of the sphere branched over the 3r points $\lambda_0\lambda_1, ..., \lambda_{3r-1}, \infty$ where $\lambda_0 = 0$ and $\lambda_1 = 1$. This is a compact Riemann surface of genus g=3r-2. We would like to get Thomae type formulae for these surfaces. Recall that in the hyperelliptic case we were able to obtain the formulae due to the fact that a non vanishing even characteristic is of the form $\phi_{P_0}(P_1 \dots P_g) + K_{P_0}$ with $P_1, \dots, P_g$ branch points and $i(P_1 \dots P_g) = 0$. Thus, in order to find the analogue formulae for Z_3 curves, we would like to construct on this surface a set of integral divisors of degree g whose support is in the branch locus which are non special, that is, no holomorphic differential (non constant) vanishes at the g points which constitute the divisor, or there is no canonical divisor which is a multiple of this divisor. We shall show that the only such divisors are those constructed by choosing r-1 distinct points from the branch set and then an additional r distinct points from the remaining elements of the branch set and forming the divisor $P_1^2 ... P_{r-1}^2 Q_1 ... Q_r$. In other words we are constructing

$$\begin{pmatrix} 3r \\ r-1 \end{pmatrix}\begin{pmatrix} 2r+1 \\ r \end{pmatrix}$$

divisors.

We begin with some elementary and trivial observations: We think of the function z as the projection map of the compact surface onto the sphere and as such it has divisor $\frac{P_0^3}{P_\infty^3}$ where P_i is the point on the surface over the branch point λ_i. The divisor of the meromorphic differential dz is just $\frac{P_0^2 P_1^2 ... P_{3r-2}^2}{P_\infty^4}$ and the divisor of the meromorphic function w is just $\frac{P_0 P_1 ... P_{3r-2}}{P_\infty^{3r-1}}$.

It thus follows that the divisor of the differential

$$(\prod_{i=1}^{j}(z-\lambda_{n_i})\frac{dz}{w}) = P^3_{\lambda_{n_1}} ... P^3_{\lambda n_j} \prod_{i=1}^{3r-2} P_i P_\infty^{3r-3j-5}.$$

It thus follows that this differential is holomorphic whenever $j \le r-2$. Moreover it is also clear that the r-1 holomorphic differentials $\prod_{i=1}^{j}(z-\lambda_{n_i})\frac{dz}{w}, \quad j = 0, ..., r-2$ are r-1 linearly independent holomorphic differentials on the surface.

We can now do the same thing with $\frac{dz}{w^2}$ and conclude that the the 2r-1 differentials $\prod_{i=1}^{j}(z-\lambda_{n_i})\frac{dz}{w^2}, \quad j = 0, ...2r-2$ are 2r-1 linearly independent holomorphic differentials with divisors $P^3_{n_1} ... P^3_{n_j} P_\infty^{6r-6-3j}$ and that these 3r-2 differentials are a basis for the space of holomorphic differentials. As a special case we can choose the 3r-2 holomorphic differentials

$$\frac{dz}{w}, z\frac{dz}{w}, ..., z^{r-2}\frac{dz}{w}, \frac{dz}{w^2}, z\frac{dz}{w^2}, ...z^{2r-2}\frac{dz}{w^2}$$

and this as is easily seen is a basis adopted to either the point P_0 or P_∞. In fact, if we look at the orders of the zero at P_0 we see that for $r \ge 2$ the orders are precisely

$$0, 1, 3, 4, ..., 3r-6, 3r-5, 3r-3, 3r, 3r+3, ..., 3(2r-2).$$

The conclusion from this is that the Weierstass gaps at the Weierstrass point P_0 are precisely

$$1, 2, 4, 5, ...3r - 5, 3r - 4, 3r - 2, 3r + 1, 3r + 4, ..., 3(2r - 2) + 1$$

and in particular if a positive integer is $\leq 3r - 4$ and congruent to two mod three, we see that it is a gap. The argument just given for zero holds for all other branch points as well so we have thus seen that

Lemma 4.1 *If $2 \leq k \leq 3r - 4$ and k is congruent to two mod three then k is a gap at P_i for any point P_i in the branch set. Furthermore there are no orders of zero congruent to two mod three at any point in the branch set.*

Proof: The first statement has been proved before the statement of the lemma. The final statement follows from the fact that a zero of order congruent to two mod three would imply a gap congruent to zero mod three of which there are none.

We are now able to determine the integral divisors of degree g with support in the branch set which are nonspecial.

Theorem 4.1 *Let $P_1, ...P_{r-1}$ be r-1 distinct points of the branch set and let $Q_1, ..., Q_r$ be an additional set of r distinct points from the branch set which are also disjoint from the P_i already chosen. Then the integral divisor of degree g=3r-2 , $P_1^2...P_{r-1}^2 Q_1...Q_r$ satisfies the condition that there is no canonical divisor which is a multiple of it. Furthermore these are the only integral divisors of degree g=3r-2 with support in the branch set which have this property.*

Proof: Consider the divisor $\Delta = P_1^2...P_{r-1}^2 Q_1...Q_r$ and assume that $i(\Delta) \geq 1$. This means that some multiple of Δ is canonical and therefore we have

$$(\Delta\Omega) = (P_1^2...P_{r-1}^2 Q_1...Q_r\Omega) = (P_1^3...P_{r-1}^3 Q_1...Q_r\Omega')$$

is a canonical divisor with deg (Ω)=3r-4, and deg (Ω')= 2r-3. (The second equation is a consequence of the lemma).

We now use the fact that given any point in the branch set there is a holomorphic differential with support at that point. In other words $i(P_i^{6r-6}) = 1$. As a consequence of this if we take the Abel Jacobi map ϕ_{P_i} it is clear that if Z is a canonical divisor $\phi_{P_i}(Z) = 0$. Furthermore since for any two points in the branch set P_i, P_j we have the divisor $\frac{P_i^3}{P_j^3}$ is principal we also have $\phi_{P_i}(P_j^3) = 0$ for any pair of points P_i, P_j.

As a consequence of the preceding remarks we have $\phi_Q(Q_1...Q_r\Omega') = 0$ and the degree of $Q_1...Q_r\Omega'$ is 3r-3 where Q is any element of the branch set. By Abel's theorem this implies the existence of a meromorphic function with zeros at $Q_1...Q_r\Omega'$ and a pole of order 3r-3 at Q. Choosing Q however as one of the points in the branch set say one of the Q_i now gives a meromorphic function with a pole of order precisely 3r-4 at this point Q. (Note that we can always find a point Q_i that does not appear in Ω'). We have seen however that 3r-4 is a gap at any element of the branch set which is a contradiction.

We now need to prove that these are the only divisors of degree g with this property. From the fact that the holomorphic differential $\frac{dz}{w}$ has divisor $P_0P_1P_2...P_{3r-2}P_\infty^{3r-5}$ it is clear that there is no integral divisor of degree g with all points distinct and in the branch set which satisfies the condition of nonspeciality. Moreover if even one point of the divisor Δ has order 3 or more then once again the divisor is special since we will have $r(\frac{1}{\Delta}) \geq 2$. It thus follows that you can have only points with order at most two and must have at least one such point. In fact we see that you must have precisely r-1 such points as follows.

Consider the holomorphic differential $\prod_{i=1}^{k}(z-\lambda_i)\frac{dz}{w}$ with $k \leq r-2$. Its divisor is $P_1^4...P_k^4 P_{k+1}...P_{3r-2}P_\infty^{3r-3k-5}$. Therefore if $k \leq r-2$ we have $i(P_1^4...P_k^4 P_{k+1}...P_{3r-2}P_\infty^{3r-3k-5}) = 1$ and thus $i(P_1^2...P_k^2 P_{k+1}...P_{3r-2}) \geq 1$ as well. Hence we require at least r-1 points of order 2. Suppose now we had r points of order 2 or more. We then would have a divisor $P_1^2...P_k^2 Q_1...Q_{g-2k}$ with $k \geq r$. Note however that when $k=r$, $P_1^3...P_r^3 Q_1^3...Q_{r-2}^3$ is canonical and a multiple of the original divisor. If k is greater than r we get a similar result showing the divisors are all special. This concludes the proof of the theorem.

5 Thomae Formulae for Z_3 Curves

In this section we apply the main result of the previous section to get Thomae type formulae on the Riemann surface

$$w^3 = \prod_{i=0}^{3r-2}(z-\lambda_i).$$

We recall the following fact concerning the vector of Riemann constants K_P with respect to a base point P and canonical homology basis (details can be found in {FK}); as a point in the Jacobi variety of the surface it satisfies the condition that for any canonical divisor Z, $\phi_P(Z) = -2K_P$. $-2K_P$ is sometimes referred to as the canonical point.

It follows from what we have already done in the proof of the preceding theorem that if Q is any point in the branch set then K_Q is a point of order two in the Jacobi variety. Furthermore we saw that $\phi_Q(P_i)$ is a point of order three in the Jacobi variety. We therefore can conclude that for any Q in the branch set the point $\phi_Q(\Delta) + K_Q$ where Δ is any one of the nonspecial divisors of degree g with support in the branch set constructed in the previous section,

$$\phi_Q(\Delta) + K_Q$$

is a point of order 6 in the Jacobi variety of the surface, and it may be represented uniquely ,making some canonical choice once and for all, as $\Pi\frac{\frac{2\epsilon}{3}+\delta}{2} + I\frac{\frac{2\epsilon'}{3}+\delta'}{2}$, where δ, δ' are independent of Δ. We will represent such a point by the symbol $\begin{pmatrix} \frac{2\epsilon}{3}+\delta \\ \frac{2\epsilon'}{3}+\delta' \end{pmatrix}$. Hence we shall have a correspondence between the divisors constructed above and the theta characteristics $\begin{bmatrix} \frac{2\epsilon}{3}+\delta \\ \frac{2\epsilon'}{3}+\delta' \end{bmatrix}$.

Recalling the Riemann vanishing theorem cited in the introduction we conclude that if we choose the characteristics as above, taking $Q = P_0$,and taking $P_1, \ldots, P_{r-1}, P_r, \ldots, P_{2r-1} \neq P_0$, then we have that the theta functions with these characteristics do not vanish at the origin. We shall write such a theta function as $\theta[P_1..P_{r-1}; P_r..P_{2r-1}]$.

Note that, as in the hyperelliptic case we have that $\sum_{i=1}^{3r-2}\phi_{P_0}(P_i) + \phi_{P_0}(\infty) = 0$, Where we take the sum over all the branch points. As a consequence of this fact and the discussion above we have the following useful

Lemma 5.1 *Let $P_1, \ldots, P_{r-1}, Q_1, \ldots, Q_r$ be distinct branch points (not equal to P_0), and $\tilde{Q}_1, \ldots, \tilde{Q}_r$ denote the complement of these points in the set of branch points (minus P_0). Then*

$$-\phi_{P_0}(P_1^2 \ldots P_{r-1}^2 \cdot Q_1 \ldots Q_r) = \phi_{P_0}(P_1^2 \ldots P_{r-1}^2 \cdot \tilde{Q}_1 \ldots \tilde{Q}_r)$$

We now prove the following

Proposition 5.1 *Suppose $P_1, \ldots, P_{r-1}, Q_1, \ldots, Q_{r-1}, R$ are distinct branch points, corresponding to $\{\lambda_1, \ldots, \lambda_{r-1}, \mu_1, \ldots, \mu_{r-1}, \rho\}$ respectively then:*

$$\frac{\theta^6[P_1..P_{r-1}; Q_1..Q_{r-1}R](0,\Pi)}{\prod_{i=1}^{r-1}(\lambda_0-\lambda_i)^2(\rho-\mu_i)^2} = \frac{\theta^6[Q_1..Q_{r-1}; P_1..P_{r-1}R](0,\Pi)}{\prod_{i=1}^{r-1}(\lambda_0-\mu_i)^2(\rho-\lambda_i)^2}.$$

This equality is up to multiplication by a 6th root of unity.

Proof: Consider the following quotient of cubes of theta functions:

$$\frac{\theta^3[P_1..P_{r-1}; Q_1..Q_{r-1}R](\phi_{P_0}(P))}{\theta^3[Q_1..Q_{r-1}; P_1..P_{r-1}R](\phi_{P_0}(P))}$$

This is a well defined function. As in the hyperelliptic case this follows from elementary theta function identities. (We make here use of the fact that the characteristics in the denominator and numerator are of the form $\begin{bmatrix} \frac{2\epsilon}{3}+\delta \\ \frac{2\epsilon'}{3}+\delta' \end{bmatrix}$ and $\begin{bmatrix} \frac{2\tilde{\epsilon}}{3}+\delta \\ \frac{2\tilde{\epsilon}'}{3}+\delta' \end{bmatrix}$ for the same δ, δ')
There is though some ambiguity here, as the theta function itself is only fixed up to sign by an element in the Jacobi variety (see e.g. {FK} p. 303). (This was no problem in the hyperelliptic case as we took there the squares of theta functions)

It follows from the Riemann vanishing theorem that the zeros of this function on the Riemann surface are just sixth order zeros at $P_1, ...P_{r-1}$ and that the poles are sixth order poles at the points $Q_1, ..Q_{r-1}$. This is so because the zeros $R_1, \ldots, R_g$ of the function $\theta[P_1..P_{r-1}; Q_1..Q_{r-1}R](\phi_{P_0}(P))$ are the same as the zeros of the theta function $\theta(\phi_{P_0}(P)+e)$, where $e = \phi_{P_0}(P_1^2 \ldots, P_{r-1}^2 Q_1 \ldots Q_{r-1}R)$, and for the latter we have by Riemann: $-e = \phi_{P_0}(R_1 \ldots R_g) + K_{P_0}$. Denote by $\tilde{Q}_1, \ldots, \tilde{Q}_{r-1}$ the remaining branch points, as in the preceding lemma, then $-e = \phi_{P_0}(P_1^2 \ldots, P_{r-1}^2 \tilde{Q}_1 \ldots \tilde{Q}_{r-1}) + K_{P_0}$. It thus follows, that the zeros of the numerator are at the points $P_1..P_{r-1}$ all of second order and also at the points $\tilde{Q}_1...\tilde{Q}_{r-1}$ here simple vanishing. In the same manner the zeros of the denominator are at the points $Q_1...Q_{r-1}$ each one of second order, all by hypothesis distinct from the $P_1...P_{r-1}$ and also at the points $\tilde{Q}_1...\tilde{Q}_{r-1}$. The latter however cancel with those of the numerator.

We can therefore conclude that

$$\frac{\theta^3[P_1..P_{r-1}; Q_1..Q_{r-1}R](\phi_{P_0}(P))}{\theta^3[Q_1..Q_{r-1}; P_1..P_{r-1}R](\phi_{P_0}(P))} = c\frac{\prod_{i=1}^{r-1}(z-\lambda_i)^2}{\prod_{i=1}^{r-1}(z-\mu_i)^2}$$

If we now set $P = P_0$ we find that

$$c = \frac{\prod_{i=1}^{r-1}(\lambda_0-\mu_i)^2\theta^3[P_1 \ldots P_{r-1}; Q_1 \ldots Q_{r-1}R](0,\Pi)}{\prod_{i=1}^{r-1}(\lambda_0-\lambda_i)^2\theta^3[Q_1 \ldots Q_{r-1}; P_1 \ldots P_{k-1}R](0,\Pi)}$$

(Where $\lambda_0 = z(P_0)$). Now, letting $P = R$ we get

$$\frac{\theta^3[P_1 \ldots P_{r-1}; Q_1 \ldots Q_{r-1}R](\phi_{P_0}(R), \Pi)}{\theta^3[Q_1 \ldots Q_{r-1}; P_1 \ldots P_{k-1}R](\phi_{P_0}(R), \Pi)}$$
$$= \frac{\prod_{i=1}^{r-1}(\lambda_0 - \mu_i)^2 \theta^3[P_1 \ldots P_{r-1}; Q_1 \ldots Q_{r-1}R](0, \Pi)}{\prod_{i=1}^{r-1}(\lambda_0 - \lambda_i)^2 \theta^3[Q_1 \ldots Q_{r-1}; P_1 \ldots P_{k-1}R](0, \Pi)} \cdot \frac{\prod_{i=1}^{r-1}(\rho - \lambda_i)^2}{\prod_{i=1}^{r-1}(\rho - \mu_i)^2}$$

The left hand side is equal to $\frac{\theta^3[\phi_{P_0}((P_1 \ldots P_{k-1})^2 Q_1 \ldots Q_{k-1}R^2) + K_{P_0}](0,\Pi)}{\theta^3[\phi_{P_0}((Q_1 \ldots Q_{k-1})^2 P_1 \ldots P_{k-1}R^2) + K_{P_0}](0,\Pi)}$ multiplied by some 6th root of unity, which depends on the canonical choice for the representatives of the characteristics and the point $\phi(R)$. This equality is derived from formula (1) .Since it is always true that $\theta \begin{bmatrix} -\epsilon \\ -\epsilon' \end{bmatrix} (0, \Pi) = \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0, \Pi)$ and since
$-\phi_{P_0}((P_1 \ldots P_{k-1})^2 Q_1 \ldots Q_{k-1}R^2) - K_{P_0} = \phi_{P_0}((Q_1 \ldots Q_{k-1})^2 P_1 \ldots P_{k-1}R) + K_{P_0}$
$-\phi_{P_0}((Q_1 \ldots Q_{k-1})^2 P_1 \ldots P_{k-1}R^2) - K_{P_0} = \phi_{P_0}((P_1 \ldots P_{k-1})^2 Q_1 \ldots Q_{k-1}R) + K_{P_0}$
we obtain the conclusion.
Note though, that after making the last change (using the equality $\theta \begin{bmatrix} -\epsilon \\ -\epsilon' \end{bmatrix} (0, \Pi) = \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0, \Pi)$) we (may) get non-canonical representatives for the characteristics involved, and so we have to multiply again by some 6th root of unity in order to get the value of the theta functions corresponding to the canonical representatives.
If we would like to avoid the ambiguity caused by the 6th roots of unity, we may take the 6th power of each side of the equation in order to get:

$$\frac{\theta^{36}[P_1..P_{r-1}; Q_1..Q_{r-1}R](0, \Pi)}{\prod_{i=1}^{r-1}(\lambda_0 - \lambda_i)^{12}(\rho - \mu_i)^{12}} = \frac{\theta^{36}[Q_1..Q_{r-1}; P_1..P_{r-1}R](0, \Pi)}{\prod_{i=1}^{r-1}(\lambda_0 - \mu_i)^{12}(\rho - \lambda_i)^{12}}.$$

Turning back to the equation in the proposition we may now multiply both the denominators by
$\prod_{1 \le i < j \le r-1}(\lambda_i - \lambda_j)^2 \cdot \prod_{1 \le i < j \le r-1}(\mu_i - \mu_j)^2$ and get:

$$\frac{\theta^6[P_1..P_{r-1}; Q_1..Q_{r-1}R](0, \pi)}{[\lambda_0, \lambda_1, ..\lambda_{r-1}]^2 [\rho, \mu_1, ...\mu_{r-1}]^2} = \frac{\theta^6[Q_1..Q_{r-1}; P_1..P_{r-1}R](0, \pi)}{[\lambda_0, \mu_1, ..\mu_{r-1}]^2 [\rho, \lambda_1, ...\lambda_{r-1}]^2}$$

where we understand $[a_1, .., a_r]^2$ to be $\prod_{i,j=1 i<j}^{r}(a_i - a_j)^2$.

If $\tilde{Q}_1, \ldots, \tilde{Q}_r$ is the complement of $P_1, \ldots, P_{r-1}, Q_1, \ldots, Q_{r-1}, R$ in the branch set minus P_0, and $\tilde{Q}_i$ corresponds to ν_i then we may multiply both sides of the identity by $\frac{1}{[\nu_1, \ldots, \nu_r]^2}$. This is motivated by the following observation;

If $Q_1, \ldots, Q_{r-1}, Q_r, \ldots, Q_{2r-1}$ are distinct branch points (not equal to P_0), and $Q_{2r}, \ldots, Q_{3r-1}$ denote the complement of these points in the set of branch points (minus P_0), and suppose Q_i corresponds to μ_i then, by the lemma above we can write

$$\frac{\theta^6[Q_1..Q_{r-1}; Q_r..Q_{2r-1}](0, \pi)}{[\lambda_0, \mu_1, ..\mu_{r-1}]^2 [\mu_r, ..\mu_{2r-1}]^2 [\mu_{2r}, ..\mu_{3r-1}]^2} = \frac{\theta^6[Q_1..Q_{r-1}; Q_{2r}..Q_{3r-1}](0, \pi)}{[\lambda_0, \mu_1, ..\mu_{r-1}]^2 [\mu_{2r}, ..\mu_{3r-1}]^2 [\mu_r, ..\mu_{2r-1}]^2}$$

We may now conclude using the proposition and the last observation:

Theorem 5.1 *Let $Q_1, ..Q_{r-1}, Q_r, ..Q_{2r-1}$ be an arbitrary choice of 2r-1 of the points $P_1, ...P_{3r-1}$. Thus there remain r points which were not chosen. Denote these last r points by $Q_{2r}, ...Q_{3r-1}$. Suppose Q_i corresponds to μ_i Then we have*

$$\frac{\theta^6[Q_1..Q_{r-1}; Q_r..Q_{2r-1}](0, \pi)}{[\lambda_0, \mu_1, ..\mu_{r-1}]^2[\mu_r, ..\mu_{2r-1}]^2[\mu_{2r}, ..\mu_{3r-1}]^2} = k \cdot e(\epsilon, \epsilon')$$

where k is independent of the choice, and $e(\epsilon, \epsilon')$ is a 6th root of unity that does depend on the choice.

Note that we have equality of the quotients for two different choices of points (up to 6th roots of unity), if the choices differ by one of two moves; either one may replace $Q_1, \ldots, Q_{r-1}$ by $r-1$ of the points $Q_r, \ldots, Q_{2r-1}$ or one may replace the points $Q_r, \ldots, Q_{2r-1}$ by the points $Q_{2r}, \ldots, Q_{3r-1}$. The general formula follows readily from this by a simple combinatoric argument. Let there be given 3r-1 elements. Partition these elements into three distinct sets A,B,C with the cardinality of A being r-1 and the cardinalities of B and C being r. We will allow two types of moves. The first move is interchanging the elements of A with r-1 elements of B. There are clearly r such moves possible. A second move is interchanging the elements of B and C. It is not hard to see that this action is transitive on all the possible partitions and thus we conclude the proof of the theorem.
We mention again that in order to have equations that do not include possible multiplication by 6th roots of unity, one may raise the expressions in the equations to the 6th power. We then have the following:

Theorem 5.2 *Let $Q_1, ..Q_{r-1}, Q_r, ..Q_{2r-1}$ be an arbitrary choice of 2r-1 of the points $P_1, ...P_{3r-1}$, and let $Q_{2r}, ...Q_{3r-1}$ be the remaining r points. Suppose Q_i corresponds to μ_i Then we have*

$$\frac{\theta^{36}[Q_1..Q_{r-1}; Q_r..Q_{2r-1}](0, \pi)}{[\lambda_0, \mu_1, ..\mu_{r-1}]^{12}[\mu_r, ..\mu_{2r-1}]^{12}[\mu_{2r}, ..\mu_{3r-1}]^{12}} = k$$

where k is independent of the choice

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