Longest alternating subsequences in pattern-restricted k-ary words

Toufik Mansour

Department of Mathematics, University of Haifa, Haifa 31905, Israel

toufik@math.haifa.ac.il

Submitted: May 22, 2007; Accepted: January 20, 2008; Published: January 29, 2008

Abstract

We study the generating functions for pattern-restricted k-ary words of length n corresponding to the longest alternating subsequence statistic in which the pattern is any one of the six permutations of length three.

Keywords: alternating sequence, functional equation, kernel method, k-ary word
Subject Class: 05A05, 05A15, 05A16

1 Introduction

Permutations. Let S_n be the symmetric group of permutations of $1, 2, \ldots, n$ and let $\pi = \pi_1\pi_2\ldots\pi_n$ be any permutation of length n, that is, $\pi \in S_n$. An increasing subsequence in π of length ℓ is a subsequence $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_\ell}$ satisfying $\pi_{i_1} < \pi_{i_2} < \cdots < \pi_{i_\ell}$. Several authors have studied properties of the length of the longest increasing subsequence $is_n(\pi)$ of a permutation π. Logan-Shepp [4] and Vershik-Kerov [9] showed the asymptotic expectation $E(n)$ of $is_n(\pi)$ satisfies $E(n) = \frac{1}{n!}\sum_{\pi\in S_n} is_n(\pi) \sim 2\sqrt{n}$ when $n \to \infty$. Deutsch, Hildebrand and Wilf [3] generalized the above results to study the limiting distribution of the longest increasing subsequences in pattern-restricted permutations for which the pattern is any one of the six patterns of length three.

Recently, Stanley [8] developed an analogous theory for alternating subsequences, i.e., subsequences $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_\ell}$ of π satisfying $\pi_{i_1} > \pi_{i_2} < \pi_{i_3} > \pi_{i_4} < \cdots \pi_{i_\ell}$. He proved that the mean of the longest alternating subsequence $al_n(\pi)$ in a permutation $\pi \in S_n$ is $\frac{4n+1}{6}$ for $n \geq 2$, and the variance of $al_n(\pi)$ is $\frac{8}{45}n - \frac{13}{180}$ for $n \geq 4$. Also, Widom [10] showed the limiting distribution

$$\lim_{n\to\infty} \frac{1}{n!}|\{\pi \in S_n \mid al_n(\pi) \leq 2n/3 + t\sqrt{n}\}|$$

to be Gaussian with variance $\frac{8}{45}$. Firro and Mansour [7] generalized the above results to study the limiting distribution of the longest alternating subsequences in pattern-restricted permutations for which the pattern is any one of the six patterns of length three.

Words. Let $[k] = \{1, 2, \ldots, k\}$ be a (totally ordered) alphabet on k letters. We call the elements of $[k]^n$ words of length n (more generally, we call the elements of $\mathcal{A}^n$ words of length n on the alphabet $\mathcal{A}$). Consider two words, $\pi \in [k]^n$ and $\tau \in [\ell]^m$. That is, π is a k-ary word of length n and τ is an ℓ-ary word of length m. Assume additionally that τ contains all letters 1 through ℓ.

We say π *contains an occurrence* of τ, or simply that π *contains* τ, if π has a subsequence order-isomorphic to τ, i.e., if there exist $1 \le i_1 < \cdots < i_m \le n$ such that, for any relation $\phi \in \{<,=,>\}$ and indices $1 \le a, b \le m$, $\pi_{i_a}\phi\pi_{i_b}$ if and only if $\tau_a\phi\tau_b$. In this situation, the word τ is called a *pattern*. If π contains no occurrences of τ, we say that π *avoids* τ (or *is* τ*-avoiding*).

In this paper we develop an analogous theory for *alternating* subsequences in τ-avoiding k-ary words (see [6]), i.e., subsequences $\pi_{i_1}\pi_{i_2}\dots\pi_{i_\ell}$ of π satisfying $\pi_{i_1} > \pi_{i_2} < \pi_{i_3} > \pi_{i_4} < \cdots \pi_{i_\ell}$. Denote the longest alternating subsequence of π by $al_{n,k}(\pi)$. To achieve our main results we define $al'_{n,k}(\pi)$ to be the longest subsequence $\pi_{i_1}\pi_{i_2}\dots\pi_{i_\ell}$ of π satisfying $\pi_{i_1} < \pi_{i_2} > \pi_{i_3} < \pi_{i_4} > \cdots \pi_{i_\ell}$. Let $a_\tau^m(n,k)$ (resp. $a'^m_\tau(n,k)$) be the number of τ-avoiding k-ary words π of length n having $al_{n,k}(\pi) = m$ (resp. $al'_{n,k}(\pi) = m$), and let $a_\tau(n,k;q)$ and $a'_\tau(n,k;q)$ be the polynomials $\sum_{j=0}^n a_\tau^j(n,k)q^j$ and $\sum_{j=0}^n a'^j_\tau(n,k)q^j$, respectively. The corresponding generating functions are given by

$$\mathcal{A}_\tau(x,y,q) = \sum_{k\ge0} A_k^\tau(x,q)y^k = \sum_{k\ge0} y^k \left(\sum_{n\ge0} a_\tau(n,k;q)x^n\right),$$
$$\mathcal{A}'_\tau(x,y,q) = \sum_{k\ge0} A'^\tau_k(x,q)y^k = \sum_{k\ge0} y^k \left(\sum_{n\ge0} a'_\tau(n,k;q)x^n\right),$$

respectively.

The main result of this paper (in the next section) is an explicit formula for the generating function $\mathcal{A}_\tau(x,y,q)$ for the number of k-ary words π of length n that avoid τ satisfying $al_{n,k}(\pi) = m$, where τ is any pattern in S_3.

2 Pattern-restricted k-ary words

Our aim is to find an explicit formula for the generating functions $A_k^\tau(x,q)$ and $A'^\tau_k(x,q)$. The main technique used here is obtaining a functional equation for these generating functions and then applying the kernel method. In order to do this we must introduce the following definitions and notations. Let $a_\tau^m(n,k;j_1,\dots,j_s)$ (resp. $a'^m_\tau(n,k;j_1,\dots,j_s)$) be the number of τ-avoiding k-ary words π of length n such that $al_{n,k}(\pi) = m$ (resp. $al'_{n,k}(\pi) = m$) and $\pi_1\dots\pi_s = j_1\dots j_s$. For $n \ge s$ define

$$a_\tau(n,k;j_1,\dots,j_s;q) = \sum_{i=0}^n a_\tau^i(n,k;j_1,\dots,j_s)q^i,$$
$$a'_\tau(n,k;j_1,\dots,j_s;q) = \sum_{i=0}^n a'^i_\tau(n,k;j_1,\dots,j_s)q^i,$$

and for $k \ge 1$,

$$A_k^\tau(x,q;j_1,\dots,j_s) = \sum_{n\ge0} a_\tau(n,k;j_1,\dots,j_s;q)x^n,$$
$$A'^\tau_k(x,q;j_1,\dots,j_s) = \sum_{n\ge0} a'_\tau(n,k;j_1,\dots,j_s;q)x^n.$$

Also we define

$$\mathcal{B}_k^\tau(x,q,v) = 1 + \textstyle\sum_{j=1}^k A_k^\tau(x,q;j)v^{j-1}, \quad \mathcal{B}'^\tau_k(x,q,v) = 1 + \sum_{j=1}^k A'^\tau_k(x,q;j)v^{j-1}.$$

Our plan is to study the generating functions $\mathcal{A}_\tau(x,y,q)$ and $\mathcal{A}'_\tau(x,y,q)$ for each pattern $\tau \in S_3$. We first show how to reduce the number of cases by using some standard involutions.

Lemma 2.1 *Let $\tau = \tau_1\tau_2\dots\tau_\ell \in S_\ell$ be any pattern. Define*

$$\tau' = (\ell+1-\tau_1)(\ell+1-\tau_2)\dots(\ell+1-\tau_\ell).$$

Then $\mathcal{B}_k^\tau(x,q,v) = 1 + v^{k-1}(\mathcal{B}_k'^{\tau'}(x,q,1/v) - 1)$ and $\mathcal{A}_\tau(x,y,q) = \mathcal{A}'_{\tau'}(x,y,q)$.

Proof. A straightforward application of the *complement map*

$$\pi_1\pi_2\dots\pi_n \mapsto (k+1-\pi_1)(k+1-\pi_2)\dots(k+1-\pi_n).$$

□

Proposition 2.2 *We have*

(i) The number of 231-avoiding k-ary words π of length n having $al_{n,k}(\pi) = m$ is the same as the number of 132-avoiding k-ary words π of length n having $al_{n,k}(\pi) = m$. In other words,

$$\mathcal{A}_{231}(x,y,q) = \mathcal{A}_{132}(x,y,q).$$

(ii) The number of 213-avoiding k-ary words π of length n having $al_{n,k}(\pi) = m$ is the same as the number of 312-avoiding k-ary words π of length n having $al_{n,k}(\pi) = m$. In other words,

$$\mathcal{A}_{213}(x,y,q) = \mathcal{A}_{312}(x,y,q).$$

Proof. In each case we exhibit a bijection from $[k]^n(\tau)$ to $[k]^n(\tau')$ which does not change the value of $al_{n,k}$. We use the following notation. For an integer r and word $\sigma \in [k]^n$, we let $\sigma + r$ be the word obtained by replacing each letter σ_i by $\sigma_i + r$, $i = 1,2,\dots,n$.

(i) We first use a standard block decomposition. Let π be any k-ary word that avoids 231 and contains the letter k exactly $s \geq 1$ times. Then, π can be written as $\pi = \pi^0\ k\ \pi^1\ k\ \dots\ \pi^s\ k\ \pi^{s+1}$ such that

- $\pi_i^p \leq \pi_j^q$ for all i,j, $0 \leq p < q \leq s+1$,
- π^p avoids 231 for all $p = 0,1,\dots,s+1$,
- π^p, $p = 0,1,\dots,s+1$, is a word on the alphabet $\{a_p, a_p+1, \dots, a_{p+1}\}$ such that π^p contains at least one letter a_{p+1}, where $1 \leq a_0 < a_1 < \dots < a_{s+2} \leq k-1$.

We now define the map α recursively as follows. Define $\alpha(\pi)$ be the k ary word

$$\alpha(\sigma^0 + r_0)\ k\ \alpha(\sigma^1 + r_1)\ \dots k\ \alpha(\sigma^s + r_s)\ k\ \alpha(\sigma^{s+1} + r_{s+1}),$$

where $r_j = k + \sum_{i=1}^{j}(a_{i-1} - a_i) - a_{j+1} = k + a_0 - a_j - a_{j+1}$. In other words, we recursively ensure that the pattern 231 is avoided by the blocks, and then increase all elements in σ^j and decrease all elements in σ^{j+1} so that the pattern 132 does not straddle the divider k between blocks. Of course, we interpret each word σ^j as a k-ary word on the letters occurring in it. For example, $\alpha(1153355455) = 4452255155$. Since π avoids 231, every alternating subsequence that leaves σ^j, $0 \leq j \leq s$, must fall at that step, so by replacing the next element in the subsequence by k, we obtain an alternating subsequence of the same length that includes k. Every such sequence maps to an alternating subsequence via α. Thus $al_{n,k}(\alpha(\pi)) \leq al_{n,k}(\pi)$. Since the inverse has a similar property, α preserves the $al_{n,k}$.

(ii) This is similar to (i). Let π be any k-ary word that avoids 312 and contains the letter 1 exactly $s \geq 1$ times. Then, π can be written as $\pi = \pi^0\ 1\ \pi^1\ 1\ \dots\ \pi^s\ 1\ \pi^{s+1}$ such that

- $\pi_i^p \le \pi_j^q$ for all i,j, $0 \le p < q \le s+1$,
- π^p avoids 312 for all $p = 0,1,\dots,s+1$,
- π^p, $p = 0,1,\dots,s+1$, is a word on the alphabet $\{a_p, a_p+1, \dots, a_{p+1}\}$ such that π^p contains at least one letter a_{p+1}, where $2 \le a_0 < a_1 < \cdots < a_{s+2} \le k$.

We now define the map β recursively as follows. Define $\beta(\pi)$ be the k ary word

$$\beta(\sigma^0 + r_0)\ 1\ \beta(\sigma^1 + r_1)\ \dots 1\ \beta(\sigma^s + r_s)\ 1\ \beta(\sigma^{s+1} + r_{s+1}),$$

where $r_j = k + \sum_{i=1}^{j}(a_{i-1} - a_i) - a_{j+1} = k + a_0 - a_j - a_{j+1}$. Since π avoids 312, every alternating subsequence that leaves σ^j, $0 \le j \le s$, must rise at that step, so by replacing the next element in the subsequence by 1, we obtain an alternating subsequence of the same length that includes 1. Every such sequence maps to an alternating subsequence via α. Thus $al_{n,k}(\alpha(\pi)) \le al_{n,k}(\pi)$. Since the inverse has a similar property, α preserves $al_{n,k}$. □

Thus we need to consider in detail only the cases $\tau = 123$, $\tau = 321$, $\tau = 132$, and $\tau = 312$, which we now proceed to do.

2.1 The cases 123 and 321

First we study the generating functions $\mathcal{B}_k^\tau(x,q,v)$ and $\mathcal{B}_k'^\tau(x,q,v)$ for the pattern $\tau = 123$. From the definitions we have for all $j = 1,2,\dots,k$,

$$\begin{aligned} A_k^{123}(x,q;j) &= xq + \sum_{i=1}^{j-1} A_k^{123}(x,q;j,i) + A_k^{123}(x,q;j,j) + \sum_{i=j+1}^{k} A_k^{123}(x,q;j,i) \\ &= xq + xq\sum_{i=1}^{j-1} A_k'^{123}(x,q;i) + xA_k^{123}(x,q;j) + x\sum_{i=j+1}^{k} A_i^{123}(x,q;i) \end{aligned} \tag{1}$$

and

$$\begin{aligned} A_k'^{123}(x,q;j) &= xq + \sum_{i=1}^{j-1} A_k'^{123}(x,q;j,i) + A_k'^{123}(x,q;j,j) + \sum_{i=j+1}^{k} A_k'^{123}(x,q;j,i) \\ &= xq + x\sum_{i=1}^{j-1} A_k'^{123}(x,q;i) + xA_k'^{123}(x,q;j) + xq\sum_{i=j+1}^{k} A_i^{123}(x,q;i). \end{aligned} \tag{2}$$

Hence, for all $j = 1,2,\dots,k-1$,

$$\begin{cases} A_k^{123}(x,q;j) - A_{k-1}^{123}(x,q;j) \\ \qquad = \frac{x}{1-x}A_k^{123}(x,q;k) + \frac{xq}{1-x}\sum_{i=1}^{j-1}(A_k'^{123}(x,q;i) - A_{k-1}'^{123}(x,q;i)), \\ A_k'^{123}(x,q;j) - A_{k-1}'^{123}(x,q;j) \\ \qquad = \frac{xq}{1-x}A_k^{123}(x,q;k) + \frac{x}{1-x}\sum_{i=1}^{j-1}(A_k'^{123}(x,q;i) - A_{k-1}'^{123}(x,q;i)). \end{cases} \tag{3}$$

Also, Equations (1)-(2) for $j = k$ and the facts $\mathcal{B}_k^\tau(x,q,1) = 1 + \sum_{j=1}^{k} A_k^\tau(x,q;j)$ and $\mathcal{B}_k'^\tau(x,q,1) = 1 + \sum_{j=1}^{k} A_k'^\tau(x,q;j)$ give

$$A_k^{123}(x,q;k) = xq\mathcal{B}_k'^{123}(x,q,1) - \frac{x^2q(q-1)}{1-x} \text{ and } A_k'^{123}(x,q;k) = x\mathcal{B}_k'^{123}(x,q,1) + x(q-1).$$

Hence, multiplying (3) by v^{j-1} and summing over $j = 1, 2, \ldots, k-1$ we find that

$$\begin{aligned}
&\mathcal{B}_k^{123}(x,q,v) - \mathcal{B}_{k-1}^{123}(x,q,v) \\
&\quad = \tfrac{xqv}{(1-v)(1-x)}\left(\mathcal{B}_k'^{123}(x,q,v) - \mathcal{B}_{k-1}'^{123}(x,q,v)\right) - \tfrac{xqv^{k-1}}{(1-v)(1-x)}\left(\mathcal{B}_k'^{123}(x,q,1) - \mathcal{B}_{k-1}'^{123}(x,q,1)\right) \\
&\quad + \tfrac{xq}{1-x}\left(v^{k-1} + x\tfrac{1-v^{k-1}}{1-v}\right)\mathcal{B}_k'^{123}(x,q,1) - \tfrac{x^3q(q-1)(1-v^{k-1})}{(1-v)(1-x)^2}
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
&\mathcal{B}_k'^{123}(x,q,v) - \mathcal{B}_{k-1}'^{123}(x,q,v) \\
&\quad = \tfrac{xv}{(1-v)(1-x)}\left(\mathcal{B}_k'^{123}(x,q,v) - \mathcal{B}_{k-1}'^{123}(x,q,v)\right) - \tfrac{xv^{k-1}}{(1-v)(1-x)}\left(\mathcal{B}_k'^{123}(x,q,1) - \mathcal{B}_{k-1}'^{123}(x,q,1)\right) \\
&\quad + \left(\tfrac{xv^{k-1}}{1-x} + \tfrac{x^3q^2(1-v^{k-1})}{(1-v)(1-x)}\right)\mathcal{B}_k'^{123}(x,q,1) - \tfrac{x^3q^2(q-1)(1-v^{k-1})}{(1-v)(1-x)^2} + \tfrac{x(q-1)v^{k-1}}{1-x}.
\end{aligned} \tag{5}$$

The second functional equation in the above system may be solved systematically using the *kernel method*, as described in [1]. Assume that $v = 1-x$, then

$$\mathcal{B}_k'^{123}(x,q,1) = \tfrac{x(q-1)}{1-x} + \tfrac{1}{1-x(1-q^2)-xq^2(1-x)^{-k+1}}\mathcal{B}_{k-1}'^{123}(x,q,1).$$

Using this recurrence relation k times with the initial condition $\mathcal{B}_0'^{123}(x,q,1) = 1$ we arrive at

$$\mathcal{B}_k'^{123}(x,q,1) = \frac{x(q-1)}{1-x} + \frac{1 + \frac{x(q-1)}{1-x}\sum_{i=0}^{k-2}\prod_{j=1}^{k-1-i}(1-x(1-q^2)-xq^2(1-x)^{-j+1})}{\prod_{j=1}^{k}(1-x(1-q^2)-xq^2(1-x)^{-j+1})}. \tag{6}$$

Equations (4)-(5) give

$$\begin{aligned}
&(\mathcal{B}_k^{123}(x,q,v) - \mathcal{B}_{k-1}^{123}(x,q,v)) - q(\mathcal{B}_k'^{123}(x,q,v) - \mathcal{B}_{k-1}'^{123}(x,q,v)) \\
&\qquad = \tfrac{x^2q(1-q^2)(1-v^{k-1})}{(1-x)(1-v)}\mathcal{B}_k'^{123}(x,q,1) + \tfrac{x^3q(1-q)^2(1+q)(1-v^{k-1})}{(1-x)^2(1-v)} - \tfrac{xq(q-1)v^{k-1}}{1-x}
\end{aligned}$$

Therefore, for all $j = 1, 2, \ldots, k$,

$$\begin{aligned}
(\mathcal{B}_j^{123}(x,q,1) - \mathcal{B}_{j-1}^{123}(x,q,1)) &= q(\mathcal{B}_j'^{123}(x,q,1) - \mathcal{B}_{j-1}'^{123}(x,q,1)) \\
&\quad + \tfrac{(j-1)x^2q(1-q^2)}{1-x}\mathcal{B}_j'^{123}(x,q,1) + \tfrac{(j-1)x^3q(1-q)^2(1+q)}{(1-x)^2} - \tfrac{xq(q-1)}{1-x}.
\end{aligned}$$

Adding these relations for all $j = 1, 2, \ldots, k$ and using the initial conditions

$$\mathcal{B}_0^{123}(x,q,1) = \mathcal{B}_0'^{123}(x,q,1)$$

we obtain that

$$\begin{aligned}
\mathcal{B}_k^{123}(x,q,1) &= (1-q)(1 + \tfrac{kxq}{1-x}) \\
&\quad + \tfrac{\binom{k}{2}x^3q(q-1)^2(q+1)}{(1-x)^2} + q\mathcal{B}_k'^{123}(x,q,1) + \tfrac{x^2q(1-q^2)}{1-x}\sum_{j=1}^{k}(j-1)\mathcal{B}_j'^{123}(x,q,1).
\end{aligned} \tag{7}$$

Thus, Lemma 2.1, (6) and (7) give the following result.

Theorem 2.3 *Let $k \geq 0$. The generating function for the number of 321-avoiding k-ary words π of length n with $al_{n,k}(\pi) = m$ is given by*

$$\mathcal{B}_k^{321}(x,q,1) = \tfrac{x(q-1)}{1-x} + \frac{1+\frac{x(q-1)}{1-x}\sum\limits_{i=0}^{k-2}\prod\limits_{j=1}^{k-1-i}(1-x(1-q^2)-xq^2(1-x)^{-j+1})}{\prod\limits_{j=1}^{k}(1-x(1-q^2)-xq^2(1-x)^{-j+1})},$$

and the generating function for the number of 123*-avoiding* k*-ary words* π *of length* n *with* $al_{n,k}(\pi) = m$ *is given by*

$$\begin{aligned}\mathcal{B}_k^{123}(x,q,1) &= (1-q)(1+\tfrac{kxq}{1-x}) \\ &+\tfrac{\binom{k}{2}x^3q(q-1)^2(q+1)}{(1-x)^2} + q\mathcal{B}_k'^{123}(x,q,1) + \tfrac{x^2q(1-q^2)}{1-x}\sum_{j=1}^{k}(j-1)\mathcal{B}_j'^{123}(x,q,1).\end{aligned}$$

A number of results follow from Theorem 2.3. The first is an explicit expression for the generating function of the number of 321-avoiding (123-avoiding) k-ary words π of length n. Theorem 2.3 for $q=1$, gives $\mathcal{B}_k^{123}(x,1,1) = \mathcal{B}_k^{321}(x,1,1) = \frac{1}{\prod_{j=1}^{k}(1-\frac{x}{(1-x)^{j-1}})}$ (see [2]).

Another result is explicit expressions for the generating function $\mathcal{C}_{k,m}^{\tau}(x)$ for the number of τ-avoiding k-ary words π of length n having $al_{n,k}(\pi) = m$, where $\tau = 123, 321$. So by Theorem 2.3, when m is fixed $\mathcal{C}_{k,m}^{\tau}(x)$ is the coefficient of q^m in $\mathcal{B}_k^{\tau}(x,q,1)$. For instance, if $\tau = 321$ we can state the following result.

Corollary 2.4 *For all* $k \geq 1$,
$\mathcal{C}_{k,0}^{321} = 1$, $\mathcal{C}_{k,1}^{321} = \frac{1}{(1-x)^k} - 1$, $\mathcal{C}_{k,2}^{321} = \frac{kx-1}{(1-x)^{k+1}} + \frac{1}{1-x}$, $\mathcal{C}_{k,3}^{321} = \frac{1}{(1-x)^{2k}} - \frac{(2k-1)x}{(1-x)^{k+1}} - \frac{1}{1-x}$, *and*
$\mathcal{C}_{k,4}^{321} = \frac{-2+x(k+3)-x^2(k+1)}{(1-x)^{2k+2}} + \frac{2+4x(k-1)+x^2(2-3k-k^2)}{2(1-x)^{k+2}} + \frac{1}{1-x}$.

2.2 The cases 132 and 312

Our aim is to find an explicit formula for the generating functions $\mathcal{A}_{132}(x,y,q)$ and $\mathcal{A}_{312}(x,y,q)$. In order to do that we need the following lemma, which follows immediately from the definitions (see [5, Proposition 2.1]).

Lemma 2.5 *Let* σ *be any* 132*-avoiding* k*-ary word of length* n*. Then one of the following assertions holds:*

(a) σ *is a* 132*-avoiding* $(k-1)$*-ary word of length* n*,*

(b) $\sigma = kk\dots k$*,*

(c) $\sigma = \sigma'\underbrace{kk\dots k}_{j}$ *with* $1 \leq j \leq n-1$ *such that* σ' *is a nonempty* 132*-avoiding* $(k-1)$*-ary word of length* $n-j$*,*

(d) $\sigma = \underbrace{kk\dots k}_{j}\sigma''$ *with* $1 \leq j \leq n-1$ *such that* σ'' *is a nonempty* 132*-avoiding* k*-ary word of length* $n-j$ *and* $\sigma''_1 \neq k$*,*

(e) $\sigma = \sigma'\underbrace{kk\dots k}_{j}\sigma''$ *such that* σ' *is a nonempty* 132*-avoiding word with letters from the set* $\{k-p, k-p+1, \dots, k-1\}$ *and* σ'' *is a nonempty* 132*-avoiding word with letters from the set* $\{1, 2, \dots, k-p, k\}$*, where* σ' *contains the letter* $k-p$ *and* $\sigma''_1 \neq k$*.*

Rewriting the rules in the statement of Lemma 2.5 in terms of generating functions requires some new notation.

Notation 2.6 *For all $k \geq 0$ define the* even *and* odd *parts of the generating function $\mathcal{A}_\tau(x,y,q)$ to be*

$$\mathcal{E}_\tau(x,y,q) = \sum_{k\geq 0} E_k^\tau(x,q)y^k = \sum_{k\geq 0} y^k \left(\frac{A_k^\tau(x,q)+A_k^\tau(x,-q)}{2}\right),$$
$$\mathcal{O}_\tau(x,y,q) = \sum_{k\geq 0} O_k^\tau(x,q)y^k = \sum_{k\geq 0} y^k \left(\frac{A_k^\tau(x,q)-A_k^\tau(x,-q)}{2}\right),$$

respectively. Similarly, for all $k \geq 0$ we define the even *and* odd *parts of the generating function $\mathcal{A}'_\tau(x,y,q)$ to be*

$$\mathcal{E}'_\tau(x,y,q) = \sum_{k\geq 0} E_k'^\tau(x,q)y^k = \sum_{k\geq 0} y^k \left(\frac{A_k'^\tau(x,q)+A_k'^\tau(x,-q)}{2}\right),$$
$$\mathcal{O}'_\tau(x,y,q) = \sum_{k\geq 0} O_k'^\tau(x,q)y^k = \sum_{k\geq 0} y^k \left(\frac{A_k'^\tau(x,q)-A_k'^\tau(x,-q)}{2}\right),$$

respectively.

By Lemma 2.5, we have exactly five possibilities for the block decomposition of an arbitrary 132-avoiding k-ary word. Now let us write an equation for the generating function $A_k^{132}(x,q)$ (resp. $A_k'^{132}(x,q)$).

- The contribution of Lemma 2.5(a) equals $A_{k-1}^{132}(x,q)$ (resp. $A_{k-1}'^{132}(x,q)$),
- the contribution of Lemma 2.5(b) equals $\frac{xq}{1-x}$,
- the contribution of Lemma 2.5(c) equals $\frac{xq}{1-x}(E_{k-1}^{132}-1)+\frac{x}{1-x}O_{k-1}^{132}$ (resp. $\frac{x}{1-x}(E_{k-1}'^{132}-1)+\frac{xq}{1-x}O_{k-1}'^{132}$),
- the contribution of Lemma 2.5(d) equals $\frac{xq}{1-x}C_k(x,q)$ (resp. $\frac{x}{1-x}C_k(x,q)$),
- the contribution of Lemma 2.5(e) equals
$$\frac{xq}{1-x}\sum_{p=1}^{k-1}(E_p^{132}(x,q)-E_{p-1}^{132}(x,q))F_{k+1-p}(x,q)+\frac{x}{1-x}\sum_{p=1}^{k-1}(O_p^{132}(x,q)-O_{p-1}^{132}(x,q))F_{k+1-p}(x,q)$$
$$\left(\text{resp. } \frac{x}{1-x}\sum_{p=1}^{k-1}(E_p'^{132}(x,q)-E_{p-1}'^{132}(x,q))F_{k+1-p}(x,q)+\frac{xq}{1-x}\sum_{p=1}^{k-1}(O_p'^{132}(x,q)-O_{p-1}'^{132}(x,q))F_{k+1-p}(x,q)\right),$$

where $F_k(x,q)=(1-x)A_k'^{132}(x,q)+x(1-q)-1$ is the generating function for the number of nonempty k-ary words π of length n such that π avoids 132, $al'_{n,k}(\pi)=m$, and $\pi_1 \neq k$. Therefore, the generating functions $A_k^{132}(x,q)$ and $A_k'^{132}(x,q)$ satisfy the following recurrence relations

$$\begin{aligned}A_k^{132}(x,q) = {} & A_{k-1}^{132}(x,q)+\tfrac{xq}{1-x}E_{k-1}^{132}(x,q)+\tfrac{x}{1-x}O_{k-1}^{132}(x,q)+xq(A_k'^{132}(x,q)-1-\tfrac{xq}{1-x})\\ &+xq\textstyle\sum_{p=1}^{k-1}(E_p^{132}(x,q)-E_{p-1}^{132}(x,q))(A_{k+1-p}'^{132}(x,q)-1-\tfrac{xq}{1-x})\\ &+x\textstyle\sum_{p=1}^{k-1}(O_p^{132}(x,q)-O_{p-1}^{132}(x,q))(A_{k+1-p}'^{132}(x,q)-1-\tfrac{xq}{1-x}).\end{aligned}$$

and

$$\begin{aligned}A_k'^{132}(x,q) = {} & A_{k-1}'^{132}(x,q)+\tfrac{x(q-1)}{1-x}+\tfrac{x}{1-x}E_{k-1}'^{132}(x,q)+\tfrac{xq}{1-x}O_{k-1}'^{132}(x,q)+x(A_k'^{132}(x,q)-1-\tfrac{xq}{1-x})\\ &+x\textstyle\sum_{p=1}^{k-1}(E_p'^{132}(x,q)-E_{p-1}'^{132}(x,q))(A_{k+1-p}'^{132}(x,q)-1-\tfrac{xq}{1-x})\\ &+xq\textstyle\sum_{p=1}^{k-1}(O_p'^{132}(x,q)-O_{p-1}'^{132}(x,q))(A_{k+1-p}'^{132}(x,q)-1-\tfrac{xq}{1-x}).\end{aligned}$$

Taking the even and the odd parts of each of the above recurrence relations we arrive at the following result.

Lemma 2.7 *For all $k \geq 1$,*

$$E_k^{132}(x,q) = E_{k-1}^{132}(x,q) + xqO_k'^{132}(x,q) - \tfrac{x^2q}{1-x}M_{k-1} + x\sum_{p=1}^{k-1}(M_p - M_{p-1})O_{k+1-p}'^{132}(x,q),$$
$$O_k^{132}(x,q) = O_{k-1}^{132}(x,q) + xqE_k'^{132}(x,q) + \tfrac{x^2}{1-x}M_{k-1} + x\sum_{p=1}^{k-1}(M_p - M_{p-1})E_{k+1-p}'^{132}(x,q),$$

and

$$(1-x)E_k'^{132}(x,q) = -\tfrac{x}{1-x} + E_{k-1}'^{132}(x,q) + \tfrac{x^2}{1-x}N_{k-1} + x\sum_{p=1}^{k-1}(N_p - N_{p-1})E_{k+1-p}'^{132},$$
$$(1-x)O_k'^{132}(x,q) = \tfrac{xq}{1-x} + O_{k-1}'^{132}(x,q) - \tfrac{x^2q}{1-x}N_{k-1} + x\sum_{p=1}^{k-1}(N_p - N_{p-1})O_{k+1-p}'^{132},$$

where $M_k = M_k(x,q) = qE_k^{132}(x,q) + O_k^{132}(x,q)$ and $N_k = N_k(x,q) = E_k'^{132}(x,q) + qO_k'^{132}(x,q)$.

Lemma 2.7 gives

$$\begin{array}{l} M_k(x,q) = M_{k-1}(x,q) + xqN_k(x,q) + \tfrac{x^2(1-q^2)}{1-x}M_{k-1}(x,q) \\ \qquad\qquad\qquad +x\sum_{p=1}^{k-1}(M_p(x,q) - M_{p-1}(x,q))N_{k+1-p}(x,q), \\ (1-x)N_k(x,q) = \tfrac{x(q^2-1)}{1-x} + N_{k-1}(x,q) + \tfrac{x^2(1-q^2)}{1-x}N_{k-1} + x\sum_{p=1}^{k-1}(N_p - N_{p-1})N_{k+1-p}, \end{array} \tag{8}$$

for all $k \geq 1$. Multiplying by y^k and summing over all $k \geq 1$ while using the facts $N_0(x,q) = 1$ and $N_1(x,q) = 1 + \frac{xq^2}{1-x}$ we find that

$$M = q - \frac{qy^2}{y^2 - (x+y)(1-x) - x^2q^2y + x(1-x)(1-y)N}$$

and

$$\frac{x(1-y)}{y}(N-1)^2 - \frac{1-y+x^2(q^2-1)}{1-x}(N-1) + \frac{y(1-y+x(q^2-1))}{(1-x)(1-y)} = 0,$$

where $M = M(x,y,q) = \sum_{k\geq 0} M_k(x,q)y^k$ and $N = N(x,y,q) = \sum_{k\geq 0} N_k(x,q)y^k$. Solving the above two equations we obtain an explicit formula for the generating functions M and N.

Proposition 2.8 *The generating function $M(x,y,q)$ is given by*

$$q + \tfrac{qy}{2x(1-x)(1-y+x(q^2-1))}\left(1 - y + x^2(q^2-1) - \sqrt{((1-x)^2 + x^2q^2 - y)^2 - 4x^2(1-x)^2q^2}\right),$$

and the generating function $N(x,y,q)$ is given by

$$1 + \tfrac{y}{2x(1-x)(1-y)}\left(1 - y + x^2(q^2-1) - \sqrt{((1-x)^2 + x^2q^2 - y)^2 - 4x^2(1-x)^2q^2}\right).$$

Rewriting the fourth recurrence relation in the statement of Lemma 2.7 in terms of generating functions we have

$$\mathcal{O}_{132}'(x,y,q) = \frac{xyq}{(1-x)(1-y)} \cdot \frac{\frac{y}{1-y} + x - xN(x,y,q)}{y + x - xN(x,y,q)}.$$

Therefore, by Proposition 2.8 we find the generating function $\mathcal{O}_{132}'(x,y,q)$ to be

$$\tfrac{yq}{2(1-x)(1-y)(1-y+x(q^2-1))}\left(1 - y + x^2(q^2-1) - \sqrt{((1-x)^2 + x^2q^2 - y)^2 - 4x^2(1-x)^2q^2}\right). \tag{9}$$

From the definitions we have

$$\mathcal{A}_{132}'(x,y,q) = \mathcal{E}_{132}'(x,y,q) + \mathcal{O}_{132}'(x,y,q) = N(x,y,q) + (1-q)\mathcal{O}_{132}'(x,y,q).$$

Hence, applying Lemma 2.1 for $\tau = 312$ and using (9) we obtain the following result.

Theorem 2.9 *The generating function* $\mathcal{A}_{312}(x,y,q)=\mathcal{A}'_{132}(x,y,q)$ *is given by*

$$1+\tfrac{y(1-y+x(q-1))}{2x(1-x)(1-y)(1-y+x(q^2-1))}\left(1-y+x^2(q^2-1)-\sqrt{((1-x)^2+x^2q^2-y)^2-4x^2(1-x)^2q^2}\right).$$

Rewriting the second recurrence relation in the statement of Lemma 2.5 in terms of generating functions we find that the generating function $\mathcal{O}_{132}(x,y,q)$ satisfies

$$\begin{aligned}(1-y)\mathcal{O}_{132}(x,y,q)&=xq(\mathcal{E}'_{132}(x,y,q)-1)\\&+\tfrac{x^2y}{1-x}M(x,y,q)+\tfrac{x}{y}((1-y)M(x,y,q)-q)(\mathcal{E}'_{132}(x,y,z)-1-y).\end{aligned}$$

From the definitions we have that

$$\mathcal{E}'_{132}(x,y,q)+q\mathcal{O}'_{132}(x,y,q)=N(x,y,q)\text{ and }E'_{132}(x,y,q)+O'_{132}(x,y,q)=A'_{132}(x,y,q),$$

thus $\mathcal{E}'_{132}(x,y,q)=\frac{1}{1-q}(N(x,y,q)-q\mathcal{A}'_{132}(x,y,q))$. Therefore by combining Proposition 2.8 and Theorem 2.9, the generating function $\mathcal{O}_{132}(x,y,q)$ is given by

$$\begin{aligned}q&+\tfrac{q(x-1+y)(2x(1-x+xq^2)^2-(1+2x+3x^2(q^2-1))y+(2+x^2(q^2-1))y^2-y^3)}{2x(1-x)(1-y)(1-y+x(q^2-1))^2}\\&+\tfrac{(x-1+y)yq}{2x(1-x)(1-y+x(q^2-1))^2}\sqrt{((1-x)^2+x^2q^2-y)^2-4x^2(1-x)^2q^2}.\end{aligned}\tag{10}$$

Since

$$\mathcal{A}_{132}(x,y,q)=\mathcal{E}_{132}(x,y,q)+\mathcal{O}_{132}(x,y,q)=\frac{1}{q}(M(x,y,q)+(q-1)\mathcal{O}_{132}(x,y,q)),$$

we may now use Proposition 2.8 to give the following result.

Theorem 2.10 *The generating function* $\mathcal{A}_{132}(x,y,q)$ *is given by*

$$\begin{aligned}&\tfrac{1-x+xqy}{(1-x)(1-y)}\\&+\tfrac{yq(1-y+x(q-1))}{2x(1-x)(1-y+x(q^2-1))^2}\left(1-y-2x-\tfrac{1+y}{1-y}(q^2-1)x^2-\sqrt{((1-x)^2+x^2q^2-y)^2-4x^2(1-x)^2q^2}\right).\end{aligned}$$

A number of corollaries follow from Theorems 2.9 and 2.10. The first is an explicit expression for the generating function of the number of 132-avoiding (312-avoiding) k-ary words π of length n. Theorem 2.9 and Theorem 2.10 for $q=1$, give

$$\mathcal{A}_{132}(x,y,1)=\mathcal{A}_{312}(x,y,1)=1+\frac{y}{2x(1-x)}\left(1-\sqrt{\frac{(1-2x)^2-y}{1-y}}\right)$$

(see [2, 5]).

Another result is explicit expressions for the generating function $\mathcal{D}^{\tau}_m(x,y)$ for the number of τ-avoiding k-ary words π of length n having $al_{n,k}(\pi)=m$, where $\tau=132,312$. So by Theorems 2.9 and 2.10, when m is fixed, $\mathcal{D}^{\tau}_m(x,y)$ is the coefficient of q^m in $\mathcal{A}_\tau(x,y,1)$. Thus, we can state the following result.

Corollary 2.11 *We have*

$\mathcal{D}^{132}_0(x,y)=\mathcal{D}^{312}_0(x,y)=\frac{1}{1-y}$, $\mathcal{D}^{132}_1(x,y)=\mathcal{D}^{312}_1(x,y)=\frac{xy}{(1-y)(1-y-x)}$,

$\mathcal{D}^{132}_2(x,y)=\frac{y^2x^2}{(1-y)(1-x)(1-y-x)^2}$, $\mathcal{D}^{312}_2(x,y)=\frac{y^2x^2}{(1-y)(1-x-y)((1-x)^2-y)}$,

$\mathcal{D}^{132}_3(x,y)=\frac{y^2x^3(1+y-x)}{(1-y)((1-2x)^2-y)(1-x)(1-y-x)^2}$, $\mathcal{D}^{312}_3(x,y)=\frac{y^2x^3}{(1-y)(1-x-y)^2((1-x)^2-y)}$,

$\mathcal{D}^{132}_4(x,y)=\frac{y^2x^4(1+y-x)}{(1-y)((1-x)^2-y)(1-x)(1-y-x)^3}$, *and* $\mathcal{D}^{312}_4(x,y)=\frac{y^2x^4((1-x)^3-y^2)}{(1-y)(1-y-x)^2((1-x)^2-y)^3}$.

References

[1] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps, Generating functions for generating trees. Formal Power Series and Algebraic Combinatorics (Barcelona, 1999). *Discrete Math.* **246:1-3** (2002) 29–55.

[2] A. Burstein, Enumeration of words with forbidden patterns, Ph.D. thesis, University of Pennsylvania, 1998.

[3] E. Deutsch, A.J. Hildebrand, and H.S. Wilf, Longest increasing subsequences in pattern-restricted permutations, *Elect. J. Combin.* **9:2** (2002) #R12.

[4] B. F. Logan and L. A. Shepp, A variational problem for random Young tableaux, *Adv. Math.* **26** (1977) 206-222.

[5] T. Mansour, Restricted 132-avoiding k-ary words, Chebyshev polynomials, and Continued fractions, *Adv. Appl. Math.* **36:2** (2006) 175–193.

[6] T. Mansour, Longest alternating subsequences of k-ary words, *Discrete Appl. Math.* **156:1** (2008) 119–124.

[7] G. Firro, T. Mansour and M. Wilson, Longest alternating subsequences in pattern-restricted permutations, *Elect. J. Combin.* **14** (2007) #R34.

[8] R. Stanley, Longest alternating subsequences of permutations, preprint, arXiv/0511419v1.

[9] A. M. Vershik and K. V. Kerov, Asymptotic behavior of the Plancherel measure of the symmetric group and the limit form of Young tableaux (Russian), *Dokl. Akad. Nauk SSSR* **223** (1977) 1024-1027. English translation: *Soviet Math. Dokl.* **233** (1977) 527-531.

[10] H. Widom, On the limiting distribution for the longest alternating sequence in a randon permutation, *Elect. J. Combin.* **13:1** (2006) #R25.