ANALYTIC EXTENSION OF HYPERHARMONIC NUMBERS

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Abstract. We define the analytic extension of hyperharmonic numbers involving the Pochhammer symbol, gamma and digamma functions. In addition, some sum of hyperharmonic series have been calculated. Surprisingly, the Lerch transcendent appears in the closed form of the sums.

1. Introduction

Although the harmonic numbers and their connections with the transcendental functions mentioned in the abstract are well known, this is not true for hyperharmonic numbers because of their novelty. The author points out that there is a close relationship between hyperharmonic numbers and transcendental functions, too. In 1996, J. H. Conway and R. K. Guy in [3] have defined the notion of hyperharmonic numbers.

The n-th harmonic number is the n-th partial sum of the harmonic series:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

$H_n^{(1)} := H_n$, and for all $r > 1$ let

$$H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)}$$

be the n-th hyperharmonic number of order r. These numbers can be expressed by binomial coefficients and ordinary harmonic numbers:

$$H_n^{(r)} = \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}). \tag{1}$$

We use this form later to construct the analytic extension of these numbers.

The hyperharmonic numbers have combinatorial connections. To present this fact, we need to introduce the notion of r-Stirling numbers. Getting deeper insight, see [1] and the references given there.

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Definition 1. $\left[{n \atop k} \right]_r$ is the number of permutations of the set $\{1, \ldots, n\}$ having k disjoint, non-empty cycles, in which the elements 1 through r are restricted to appear in different cycles.

The following identity integrates the hyperharmonic- and the r-Stirling numbers.

$$\frac{\left[{n+r \atop r+1} \right]_r}{n!} = H_n^{(r)}.$$

Now we turn our attention to formulate the results of this paper. Let us introduce the necessary notions. The gamma function for $\Re(z) > 0$ is

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

It can be extended to the whole complex plane, except the non-positive integers. It is well known that

$$\Gamma(n) = (n-1)! \quad n = 1, 2, \ldots. \tag{2}$$

The logarithmic derivative of the gamma function is called digamma function:

$$\Psi(z) = \frac{d}{dz} \log(\Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

For integers,

$$\Psi(n) = H_{n-1} - \gamma. \tag{3}$$

(γ is the Euler-Mascheroni constant.) Ψ's derivatives are called polygamma functions.

The Pochhammer symbol is

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\cdots(x+n-1).$$

It's derivatives are

$$\frac{d}{dx}(x)_n = (x)_n(\Psi(x+n) - \Psi(x)), \tag{4}$$

$$\frac{d}{dn}(x)_n = (x)_n \Psi(x+n). \tag{5}$$

2. The hyperharmonic function

According to (1), (2) and (3) one can write

$$H_n^{(r)} = \frac{(n+1)(n+2)\cdots(n+r-1)}{(r-1)!}(\Psi(n+r) - \Psi(r)) =$$

$$= \frac{(n+1)_{r-1}}{\Gamma(r)}(\Psi(n+r) - \Psi(r)) = \frac{(n)_r}{n\Gamma(r)}(\Psi(n+r) - \Psi(r)).$$

Since these functions are analytic except Γ and Ψ have poles at $z \in \mathbb{Z}^- = \{0, -1, -2, \ldots\}$ the following definition is correct.

Definition 2. The function

$$H_z^{(w)} = \frac{(z)_w}{z\Gamma(w)}(\Psi(z+w) - \Psi(w))$$

is called hyperharmonic function ($w, z+w \in \mathbb{C} \setminus \mathbb{Z}^-$).

Some interesting values are computed:

$$H_{\frac{1}{2}}^{(\frac{1}{2})} = \frac{4\ln(2)}{\pi}, \quad H_{\frac{1}{3}}^{(\frac{1}{3})} = \frac{3\sqrt{3}\Gamma\left(\frac{2}{3}\right)^3}{4\pi}$$

$$H_{\frac{2}{3}}^{(\frac{2}{3})} = \frac{\pi\sqrt{3}\left(3 - \frac{1}{3}\pi\sqrt{3}\right)}{3\Gamma\left(\frac{2}{3}\right)^3},$$

$$H_{\frac{3}{4}}^{(\frac{3}{4})} = \frac{2\sqrt{\pi}\left(2 + \ln(2) - \frac{1}{2}\pi\right)}{3\Gamma\left(\frac{3}{4}\right)^2},$$

$$H_{\frac{1}{3}}^{(\frac{1}{2})} = \frac{3\Gamma\left(\frac{5}{6}\right)\sqrt{3}\Gamma\left(\frac{2}{3}\right)\left(-\frac{3}{2}\ln(3) + \frac{1}{2}\pi\sqrt{3}\right)}{2\sqrt{\pi^3}},$$

$$H_{\sqrt{2}}^{(2)} = (1+\sqrt{2})(\Psi(2+\sqrt{2}) - 1 + \gamma),$$

$$H_{1+i}^{(2+i)} \approx 0.355 + 0.965i.$$

Using a formula given for the digamma function, the complex hyperharmonic function can be written as follows.

Proposition 3. *If* $w, z+w \in \mathbb{C} \setminus \mathbb{Z}^-$, *then*

$$H_z^{(w)} = \frac{(z)_w}{\Gamma(w)} \sum_{n=0}^{\infty} \frac{1}{(z+w+n)(w+n)}.$$

Proof. According to [7], the digamma function satisfies the equality

$$\Psi(z) = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+n-1}\right) \quad (z \in \mathbb{C} \setminus \mathbb{Z}^-),$$

whence

$$\Psi(z+w) - \Psi(w) = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+w+n-1} - \frac{1}{n} + \frac{1}{w+n-1}\right) =$$

$$= \sum_{n=1}^{\infty}\left(-\frac{1}{z+w+n-1} + \frac{1}{w+n-1}\right) =$$

$$= \sum_{n=1}^{\infty}\left(\frac{-w-n+1+z+w+n-1}{(z+w+n-1)(w+n-1)}\right) =$$

$$= z\sum_{n=1}^{\infty}\frac{1}{(z+w+n-1)(w+n-1)},$$

changing the range of summation, the result follows. □

The derivatives of the hyperharmonic function can be easily computed:

Proposition 4.

$$\frac{d}{dz}H_z^{(w)} = H_z^{(w)}(\Psi(z+w)-\Psi(z+1)) + \frac{(z)_w}{z\Gamma(w)}\Psi'(z+w),$$

$$\frac{d}{dw}H_z^{(w)} = H_z^{(w)}(\Psi(z+w)-\Psi(w)) + \frac{(z)_w\Psi(w)}{z\Gamma(w)}(\Psi'(z+w)-\Psi'(z)).$$

Proof. The formulas (4), (5) and the definition of the digamma function gives

$$\frac{(z)_w}{z\Gamma(w)}(\Psi(z+w)-\Psi(w)) = \frac{(z+1)_{w-1}}{\Gamma(w)}(\Psi(z+w)-\Psi(w)) =$$

$$= \frac{(z+1)_{w-1}}{\Gamma(w)}(\Psi(z+w)-\Psi(z+1))(\Psi(z+w)-\Psi(w)) +$$

$$+\frac{(z+1)_{w-1}}{\Gamma(w)}\Psi'(z+w) =$$

$$= \frac{(z)_w}{z\Gamma(w)}(\Psi(z+w)-\Psi(w))(\Psi(z+w)-\Psi(z+1)) + \frac{(z)_w}{z\Gamma(w)}\Psi'(z+w) =$$

$$= H_z^{(w)}(\Psi(z+w)-\Psi(z+1)) + \frac{(z)_w}{z\Gamma(w)}\Psi'(z+w).$$

The second derivative can be calculated in the same way. □

3. Series involving hyperharmonic numbers

Some interesting sums with respect to harmonic numbers are known, see [2] and [4]. For example,

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2} = 2\zeta(3),$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n2^n} = \frac{1}{12}\pi^2.$$

Our goal is to derive similar equalities for hyperharmonic series.

The main tool is the generating function. Since (see [6])

$$-\frac{\ln(1-x)}{(1-x)^r} = \sum_{n=1}^{\infty}H_n^{(r)}x^n, \tag{6}$$

the following results come immediately:

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{2^n} = 4\ln(2), \quad \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{3^n} = -\frac{9}{4}\ln\left(\frac{2}{3}\right),$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{2^n} = 8\ln(2), \quad \sum_{n=1}^{\infty}\frac{H_n^{(3)}}{3^n} = -\frac{27}{8}\ln\left(\frac{2}{3}\right),$$

and so on. The following proposition helps us to calculate more diffcult sums.

Proposition 5. *We have*

$$\int \frac{\ln(t)}{(1-t)t}dt = \mathrm{Li}_2(t) + \frac{1}{2}\ln^2(t),$$

and for all $2 \le r \in \mathbb{N}$

$$\int \frac{\ln(t)}{(1-t)t^r}dt = \int \frac{\ln(t)}{(1-t)t^{r-1}}dt - \frac{\ln(t)}{(r-1)t^{r-1}} - \frac{1}{(r-1)^2 t^{r-1}},$$

or, equivalently,

$$\int \frac{\ln(t)}{(1-t)t^r}dt = \mathrm{Li}_2(t) + \frac{1}{2}\ln^2(t) - \sum_{k=1}^{r-1}\left(\frac{\ln(t)}{kt^k} + \frac{1}{k^2t^k}\right).$$

up to additive constants, where

$$\mathrm{Li}_2(x) = \int_1^x \frac{\ln(t)}{1-t}dt$$

is the dilogarithm function.

Proof. The definition of Li_2 gives that

$$\mathrm{Li}_2'(t) = \frac{\ln(t)}{1-t}.$$

Moreover,

$$\left[\frac{1}{2}\ln^2(t)\right]' = \frac{\ln(t)}{t},$$

whence

$$\mathrm{Li}_2'(t) + \left[\frac{1}{2}\ln^2(t)\right]' = \frac{t\ln(t) + (1-t)\ln(t)}{(1-t)t} = \frac{\ln(t)}{(1-t)t}.$$

The first statement is proved. The second one also can be deduced by differentiation. The derivative of the right-hand side has the form

$$\frac{\ln(t)}{(1-t)t^{r-1}} - \frac{(r-1)t^{r-2} - (r-1)^2\ln(t)t^{r-2}}{(r-1)^2(t^{r-1})^2} - \frac{-(r-1)}{(r-1)^2t^r} =$$

$$= \frac{t\ln(t)}{(1-t)t^r} - \frac{t^r - (r-1)\ln(t)t^r}{(r-1)(t^r)^2} + \frac{1}{(r-1)t^r} =$$

$$= \frac{t^{r+1}\ln(t)(r-1) - t^r(1-t) + (1-t)(r-1)\ln(t)t^r + t^r(1-t)}{(r-1)t^{2r}(1-t)} =$$

$$= \frac{t\ln(t) + (1-t)\ln(t)}{t^r(1-t)} = \frac{\ln(t)}{t^r(1-t)},$$

as we want. Using this result, the verification of the summation formula is straightforward.

□

Example 6. Let us see some examples. For instance, let the order equal 2 and 3.

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{2^n n}=\frac{1}{12}\pi^2+2\ln(2)-1,$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{3^n n}=\mathrm{Li}_2\left(\frac{2}{3}\right)+\frac{1}{2}\left(\ln^2\left(\frac{2}{3}\right)-3\ln\left(\frac{2}{3}\right)-3\right)$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{2^n n}=\frac{1}{12}\pi^2+4\ln(2)-\frac{7}{4},$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{3^n n}=\frac{21}{8}\ln(3)-\frac{21}{8}\ln(2)+$$

$$+\mathrm{Li}_2\left(\frac{2}{3}\right)+\frac{1}{2}\ln^2(2)+\frac{1}{2}\ln^2(3)-\frac{13}{16}-\ln(3)\ln(2).$$

Proof. The formula (6) can be transformed into the form

$$-\frac{\ln(1-x)}{x(1-x)^r}=\sum_{n=1}^{\infty}H_n^{(r)}x^{n-1}.$$

Formal integration gives that

$$\int-\frac{\ln(1-x)}{x(1-x)^r}dx=\sum_{n=1}^{\infty}\frac{H_n^{(r)}}{n}x^n.$$

It is known that – for a fixed x – the sum of a series equals to the integral of the corresponding generating function from 0 to x. Therefore

$$\sum_{n=1}^{\infty}\frac{H_n^{(r)}}{n}x^n=\int_0^x-\frac{\ln(1-t)}{t(1-t)^r}dt.$$

The substitution $1-t=x$ gives that

$$\int_0^a-\frac{\ln(1-t)}{t(1-t)^r}dt=\int_1^{1-a}\frac{\ln(x)}{(1-x)x^r}dx,$$

therefore the last proposition is applicable.

Let us fix the order: $r=2$ and let $x=\frac{1}{2}$.

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{2^n n}=\left[\mathrm{Li}_2(t)+\frac{1}{2}\ln^2(t)-\frac{\ln(t)}{t}-\frac{1}{t}\right]_{t=1}^{t=\frac{1}{2}}=$$

$$=\mathrm{Li}_2\left(\frac{1}{2}\right)+\frac{1}{2}\ln^2\left(\frac{1}{2}\right)-2\ln\left(\frac{1}{2}\right)-1=\frac{1}{12}\pi^2+2\ln(2)-1,$$

since $\mathrm{Li}_2\left(\frac{1}{2}\right)=\frac{1}{12}\pi^2-\frac{1}{2}\ln^2(2)$. The next case is $x=\frac{1}{3}$. As above,

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{3^n n}=\mathrm{Li}_2\left(\frac{2}{3}\right)+\frac{1}{2}\left(\ln^2\left(\frac{2}{3}\right)-3\ln\left(\frac{2}{3}\right)-3\right).$$

Since there is no known closed form of $\mathrm{Li}_2\left(\frac{2}{3}\right) \approx 0.3662$, it is not possible to simplify this sum.

Let r be 3. In this case

$$\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{2^n n} = \left[\mathrm{Li}_2(t) + \frac{1}{2}\ln^2(t) - \frac{\ln(t)}{t} - \frac{1}{t} - \frac{1}{4t^2} - \frac{\ln(t)}{2t^2}\right]_{t=1}^{t=\frac{1}{2}} =$$

$$\mathrm{Li}_2\left(\frac{1}{2}\right) + \frac{1}{2}\ln^2\left(\frac{1}{2}\right) - 2\ln\left(\frac{1}{2}\right) - 2 - 1 - 2\ln\left(\frac{1}{2}\right) + 1 + \frac{1}{4} =$$

$$= \frac{1}{12}\pi^2 + 4\ln(2) - \frac{7}{4}.$$

The last identity can be proved as the former ones. The approach described above gives a method to calculate an arbitrary sum which has the form

$$\sum_{n=1}^{\infty} \frac{H_n^{(r)}}{c^n n} \quad (r = 1, 2, \dots).$$

□

In the next section we extend this result to arbitrary $r \in \mathbb{C} \setminus \mathbb{Z}$.

4. Hyperharmonic series and the Lerch transcendent

Now, we extend our result to real, even complex orders. As we shall prove, there is a general form to express the integral

$$\int -\frac{\ln(1-x)}{x(1-x)^r} dx$$

except the cases $r = 0, 1, 2, \dots$ described in the last section. We introduce the notion of the Lerch transcendent (see [5], for example).

$$\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(a+n)^s}$$

The next proposition helps us to calculate the sum of some type of hyperharmonic series with non-integer order.

Proposition 7. *If $r \in \mathbb{C} \setminus \mathbb{Z}$, then the integral of the generating function of $H_n^{(r)}$ divided by x can be written as follows:*

$$\int \frac{\ln(t)}{(1-t)t^r} dt = \frac{r\ln(t) + 1 + r^2\ln(t)\Phi(t, 1, -r) - r^2\Phi(t, 2, -r)}{t^r r^2}.$$

Proof. To prove our statement, let us differentiate the right hand side of the formula given in the proposition. It equals to

$$-\frac{r\ln(t) + 1 + r^2\ln(t)\Phi(t, 1, -r) - r^2\Phi(t, 2, -r)}{t^{r+1} r} +$$

$$+\frac{\frac{r}{t} + r^2\frac{\Phi(t,1,-r)}{t} + r^2\ln(t)\left(\frac{1}{t(t-1)} + \frac{r\Phi(t,1,-r)}{t}\right)}{t^r r^2} -$$

$$-\frac{r^2\left(\frac{\Phi(t,1,-r)}{t}+r\frac{\Phi(t,2,-r)}{t}\right)}{t^r r^2}=$$

$$=\frac{-r^2\ln(t)-r-r^3\ln(t)\Phi(t,1,-r)+r^3\Phi(t,2,-r)+r+r^2\Phi(t,1,-r)}{t^{r+1}r^2}+$$

$$+\frac{r^2\ln(t)\left(\frac{1}{t-1}+r\Phi(t,1,-r)\right)-r^2\left(\Phi(t,1,-r)+r\Phi(t,2,-r)\right)}{t^{r+1}r^2}=$$

$$=\frac{-r^2\ln(t)+r^2\ln(t)\frac{1}{1-t}}{t^{r+1}r^2}=\frac{-\ln(t)+\frac{\ln(t)}{1-t}}{t^{r+1}}=\frac{\ln(t)}{(1-t)t^r},$$

as we expect. We used the facts

$$\frac{d}{dz}\Phi(z,1,a)=\frac{1}{z(1-z)}-\frac{a\Phi(z,1,a)}{z},$$

$$\frac{d}{dz}\Phi(z,2,a)=\frac{\Phi(z,1,a)}{z}-\frac{a\Phi(z,2,a)}{z}.$$

Since the $\Phi(z,s,a)$ function have singularities in the points $a\in\mathbb{Z}^-$ if $\Re(s)>0$ and the hyperharmonic function has poles at negative integers, we have to restrict us the cases $r\in\mathbb{C}\setminus\mathbb{Z}$. $\square$

Example 8. Let us see a concrete series. For instance, let $r=\frac{3}{2}$.

First of all, we make the substitution $1-x=t$. It yields that

$$\int-\frac{\ln(1-x)}{x(1-x)^r}dx=\int\frac{\ln(t)}{(1-t)t^r}dt.$$

Using this substitution to find the interval of the integration, we get that

$$\sum_{n=1}^\infty\frac{H_n^{\left(\frac{3}{2}\right)}}{2^n n}=\int_0^{\frac{1}{2}}-\frac{\ln(1-x)}{x(1-x)^{\frac{3}{2}}}dt=\int_1^{\frac{1}{2}}\frac{\ln(t)}{(1-t)t^{\frac{3}{2}}}dt=$$

$$=\left[\frac{\frac{3}{2}\ln(t)+1+\left(\frac{3}{2}\right)^2\ln(t)\Phi\left(t,1,-\frac{3}{2}\right)-\left(\frac{3}{2}\right)^2\Phi\left(t,2,-\frac{3}{2}\right)}{t^{\frac{3}{2}}\left(\frac{3}{2}\right)^2}\right]_{t=1}^{t=\frac{1}{2}}=$$

$$\frac{8}{9}\sqrt{2}\left(-\frac{3}{2}\ln(2)+1-\frac{9}{4}\ln(2)\Phi\left(\frac{1}{2},1,-\frac{3}{2}\right)-\frac{9}{4}\Phi\left(\frac{1}{2},2,-\frac{3}{2}\right)\right)+$$

$$+4+\frac{1}{2}\pi^2\approx 0.99222242.$$

An other exotic sum is computed with the same method.

$$\sum_{n=1}^\infty\frac{H_n^{(\sqrt{2})}}{3^n n}=\int_0^{\frac{1}{3}}-\frac{\ln(1-x)}{x(1-x)^{\sqrt{2}}}=\int_1^{\frac{2}{3}}\frac{\ln(t)}{(1-t)t^{\sqrt{2}}}=$$

$$=\left[\frac{\sqrt{2}\ln(t)+1+2\ln(t)\Phi(t,1,-\sqrt{2})-2\Phi(t,2,-\sqrt{2})}{2t^{\sqrt{2}}}\right]_{t=1}^{t=\frac{2}{3}}=$$

$$= \frac{\sqrt{2}\ln\left(\frac{2}{3}\right) + 1 + 2\ln\left(\frac{2}{3}\right)\Phi(\frac{2}{3},1,-\sqrt{2}) - 2\Phi(\frac{2}{3},2,-\sqrt{2})}{2\left(\frac{2}{3}\right)^{\sqrt{2}}} -$$
$$-\frac{1+2\Phi(1,2,-\sqrt{2})}{2} \approx 0.48960626.$$

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