On sumsets of dissociated sets *

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Annotation.

In the paper we are studying some properties of subsets $Q \subseteq \Lambda_1 + \dots + \Lambda_k$, where Λ_i are dissociated sets. The exact upper bound for the number of solutions of the following equation

$$q_1 + \dots + q_p = q_{p+1} + \dots + q_{2p} \,, \quad q_i \in Q \tag{1}$$

in groups $\mathbf{F}_2^n$ is found. Using our approach, we easily prove a recent result of J. Bourgain on sets of large exponential sums and obtain a tiny improvement of his theorem. Besides an inverse problem is considered in the article. Let Q be a set belonging to a sumset of two dissociated sets such that equation (1) has many solutions. We prove that in the case the large proportion of Q is highly structured.

1. Introduction.

Let $G = (G,+)$ be a finite Abelian group with additive group operation $+$. Suppose that A is a subset of G. It is very convenient to write $A(x)$ for such a function. Thus $A(x) = 1$ if $x \in A$ and $A(x) = 0$ otherwise. By $\widehat{G}$ denote the Pontryagin dual of G, in other words the space of homomorphisms ξ from G to $\mathbf{R}/\mathbf{Z}$, $\xi : x \to \xi \cdot x$. It is well known that $\widehat{G}$ is an additive group which is isomorphic to G. Also denote by N the cardinality of G. Let f be a function from G to $\mathbf{C}$, $N = |G|$. By $\widehat{f}(\xi)$ denote the Fourier transformation of f

$$\widehat{f}(\xi) = \sum_{x \in G} f(x) e(-\xi \cdot x) \,, \tag{2}$$

where $e(x) = e^{2\pi i x}$ and $\xi \in \widehat{G}$.

Let δ, α be real numbers, $0 < \alpha \le \delta \le 1$ and let A be a subset of $\mathbb{Z}_N$ of cardinality δN. Consider the set $\mathcal{R}_\alpha$ of all r, where Fourier transform of A is large

$$\mathcal{R}_\alpha = \mathcal{R}_\alpha(A) = \{ \, r \in \widehat{G} \ : \ |\widehat{A}(r)| \ge \alpha N \, \} \,. \tag{3}$$

In many problems of combinatorial number theory is important to know the structure of the set $\mathcal{R}_\alpha$ (see [1]). In other words what kind of properties does $\mathcal{R}_\alpha$ have? Clearly, this question is an inverse problem of additive number theory (see [2, 24]).

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The first non–trivial result in the direction was obtained by M.– C. Chang [6] in 2002. Recall that a set $\mathcal{D} = \{d_1, \dots, d_{|\mathcal{D}|}\} \subseteq G$ is called *dissociated* if any equality of the form

$$\sum_{i=1}^{|\mathcal{D}|} \varepsilon_i d_i = 0\,, \tag{4}$$

where $\varepsilon_i \in \{-1, 0, 1\}$ implies that all ε_i are equal to zero.

Let log stand for the logarithm to base 2. Let p be a positive integer. By $[p]$ denote the segment of natural numbers $\{1, \dots, p\}$.

Theorem 1.1 (Chang) *Let δ, α be real numbers, $0 < \alpha \le \delta \le 1$, A be a subset of G, $|A| = \delta N$, and the set $\mathcal{R}_\alpha$ is defined by (3). Then any dissociated set Λ, $\Lambda \subseteq \mathcal{R}_\alpha$ has the cardinality at most* $2(\delta/\alpha)^2 \log(1/\delta)$.

A simple consequence of Parseval's identity gives $|\Lambda| \le \delta/\alpha^2$. Hence Chang's Theorem is nontrivial if δ is small.

Using the approach of the paper [5] (see also [4]) Chang applied her result to prove the famous Freiman's Theorem [3] on sets with small doubling. Other applications of Theorem 1.1 were obtained by B. Green in paper [7] by B. Green and I. Ruzsa in [9], T. Sanders (see e.g. [12, 13, 14]), and also T. Schoen in [23]. If the parameter α is close to δ then the structural properties of the set $\mathcal{R}_\alpha$ were studied in papers [17, 18, 19] (see also survey [20]).

By $A_1 \dot{+} A_2 \dot{+} \dots \dot{+} A_d$ denotes the set of sums of different elements of the sets $A_1, \dots, A_d$. If all A_i are equal to A then we shall write $d\dot{A}$.

In paper [26] J. Bourgain used sumsets of a dissociated set Λ and obtained an extension of Chang's theorem. He used the extension in proving of his beautiful result on density of subsets of $[N]$ without arithmetic progressions of length three. Further applications on the Theorem below were obtained in [15].

Theorem 1.2 (Bourgain) *Let d be a positive integer, δ, α be real numbers, $0 < \alpha \le \delta \le 1$, A be a subset of G, $|A| = \delta N$, and the set $\mathcal{R}_\alpha$ is defined by (3). Suppose that Λ is a dissociated set. Then for any $d \ge 1$, we have $|d\dot{\Lambda} \bigcap \mathcal{R}_\alpha| \le 8(\delta/\alpha)^2 \log^d(1/\delta)$.*

In articles [28, 29, 30] another results on sets of large exponential sums were obtained. In particular, the following theorem was proved in these papers.

Theorem 1.3 *Let δ, α be real numbers, $0 < \alpha \le \delta$, A be a subset of $\mathbb{Z}_N$, $|A| = \delta N$, and $k \ge 2$ be a positive integer. Let also $B \subseteq \mathcal{R}_\alpha \setminus \{0\}$ be an arbitrary set. Then the number*

$$T_k(B) := |\{\ (r_1, \dots, r_k, r'_1, \dots, r'_k) \in B^{2k}\ :\ r_1 + \dots + r_k = r'_1 + \dots + r'_k\ \}| \tag{5}$$

is at least

$$\frac{\delta \alpha^{2k}}{2^{4k} \delta^{2k}} |B|^{2k}\,. \tag{6}$$

In article [29] was showed that Theorem 1.3 and an inequality of W. Rudin [21, 22] on dissociated sets imply M.– C. Chang's theorem. Similarly in the paper we show that an appropriate analog of Rudin's result and Theorem 1.3 gives us Theorem 1.2 in $\mathbf{F}_2^n$ (see section 2). Our approach is elementary and does not require sufficiently difficult hypercontractivity technique from [26]. We show that for any $Q \subseteq d\dot{\Lambda}$, where Λ is a dissociated, the value $T_k(Q)$ does not exceed $C^{dk} k^{dk} |Q|^k$. Here $C > 0$ is an absolute constant. Applying this result to the set $d\dot{\Lambda} \bigcap \mathcal{R}_\alpha$ and using Theorem 1.3, we get Theorem 1.2. Actually a tiny improvement of the last result was obtained (see Theorem 2.9).

In section 4 an inverse problem is considered (inverse problems are discussed in [2] and [24], for example). Let Q be a subset of $2\dot{\Lambda}$, where Λ is an arbitrary dissociated set. Suppose that the value $T_k(Q)$ is large in the sense that $T_k(Q) \gg C^{dk} k^{dk} |Q|^k$. What can we say about the structure of Q? We show that in the case the set Q contains a sumset of two dissociated sets. In some sense we give a full description of large subsets of $2\dot{\Lambda}$ with large value of T_k. Here an approximate result (for exact formulation see Theorem 4.9).

Theorem 1.4 *Let K be a real number, k be a positive integer. Let also $\Lambda \subseteq \mathbf{F}_2^n$ be a dissociated set, $Q \subseteq \Lambda \dotplus \Lambda$, and*

$$T_k(Q) \ge \frac{k^{2k}|Q|^k}{K^k} \,. \tag{7}$$

Suppose that $k \le \log|\Lambda| / \log\log|\Lambda|$ and $|Q| \gg |\Lambda|^{1+\varepsilon}$, where $\varepsilon = \varepsilon(k, K) \in (0,1)$. Then there are sets $\mathcal{L}_1, \mathcal{L}'_1, \dots, \mathcal{L}_h, \mathcal{L}'_h \subseteq \Lambda$ such that $\mathcal{L}_i \bigcap \mathcal{L}'_j = \emptyset$, $\mathcal{L}_i + \mathcal{L}'_i \subseteq Q$, $i = 1, \dots, h$, $j = 1, \dots, h$, $|\mathcal{L}_i|, |\mathcal{L}'_i| \gg_k \log(|Q|/|\Lambda|)/\log K$, and

$$\Big| Q \bigcap \Big((\mathcal{L}_1 + \mathcal{L}'_1) \bigsqcup \dots \bigsqcup (\mathcal{L}_h + \mathcal{L}'_h) \Big) \Big| \gg_K |Q| \,. \tag{8}$$

Thus (7) implies that large proportion of Q is a union of sums of disjoint subsets $\mathcal{L}_i$, $\mathcal{L}'_i \subseteq \Lambda$. The lower bound $|\mathcal{L}_i|, |\mathcal{L}'_i| \gg_k \log(|Q|/|\Lambda|)/\log K$ from the theorem above is best possible (see Note 4.11).

The obtained results are formulated in groups $\mathbf{F}_2^n$ but they can be extended to any Abelian group (see discussion of using $\mathbf{F}_p^n$, p is a prime, in [11]). In our forthcoming papers we are going to obtain these extensions.

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2. An elementary proof of a result of Bourgain.

Denote by G the group $\mathbf{F}_2^n$. Let $A \subseteq G$ be a set, and $k \ge 2$ be a positive integer. By $T_k(A)$ denote the number

$$T_k(A) := |\{a_1 + \dots + a_k = a'_1 + \dots + a'_k \ : \ a_1, \dots, a_k, a'_1, \dots, a'_k \in A\}| \,.$$

If $A_1, \dots, A_{2k} \subseteq G$ are any sets, then denote by $T_k(A_1, \dots, A_{2k})$ the following number

$$T_k(A_1, \dots, A_{2k}) := |\{a_1 + \dots + a_k = a_{k+1} + \dots + a_{2k} \ : \ a_i \in A_i,\ i = 1, \dots, 2k\}| \,.$$

We shall write $\sum_x$ instead of $\sum_{x \in G}$ for simplicity.

Using the notion of convolution, we can calculate $T_k(A)$.

Definition 2.1 Let $f, g : G \to \mathbb{C}$ be any functions. Denote by $(f * g)(x)$ the function

$$(f * g)(x) = \sum_s f(s) g(x - s) \,. \tag{9}$$

Clearly, $(f * g)(x) = (g * f)(x)$, $x \in G$. Further, using induction, we get the operation $*_k$, where k is a positive integer. So $*_k := *(*_{k-1})$.

If $A, B \subseteq G$ are arbitrary sets, then $(A * B)(x) \neq 0$ iff $x \in A + B$. Hence $T_2(A) = \sum_x (A * A)^2(x)$. Let $f : G \to \mathbb{C}$ be a function. By $T_k(f)$ denote $T_k(f) = \sum_x |(f *_{k-1} f)(x)|^2$.

Lemma 2.2 *Let $s, t \geq 2$ be positive integers, and let $f_1, \dots, f_s, g_1, \dots, g_t : G \to \mathbb{R}$ be functions. Then*

$$\left| \sum_x (f_1 * \dots * f_s)(x) \cdot (g_1 * \dots * g_t)(x) \right| \leq$$

$$\leq (T_s(f_1))^{1/2s} \dots (T_s(f_s))^{1/2s} (T_t(g_1))^{1/2t} \dots (T_t(g_t))^{1/2t} . \qquad (10)$$

Proof. Since $\widehat{(f * g)}(r) = \widehat{f}(r)\widehat{g}(r)$, it follows that

$$\sigma := \sum_x (f_1 * \dots * f_s)(x) \cdot (g_1 * \dots * g_t)(x) = \frac{1}{N} \sum_r \widehat{f_1}(r) \dots \widehat{f_s}(r) \overline{\widehat{g_1}(r)} \dots \overline{\widehat{g_t}(r)} .$$

Using Hölder's inequality several times, we obtain

$$\sigma \leq \left(\frac{1}{N} \sum_r |\widehat{f_1}(r)|^{2s} \right)^{\frac{1}{2s}} \dots \left(\frac{1}{N} \sum_r |\widehat{f_s}(r)|^{2s} \right)^{\frac{1}{2s}} \cdot$$

$$\cdot \left(\frac{1}{N} \sum_r |\widehat{g_1}(r)|^{2t} \right)^{\frac{1}{2t}} \dots \left(\frac{1}{N} \sum_r |\widehat{g_t}(r)|^{2t} \right)^{\frac{1}{2t}} =$$

$$= (T_s(f_1))^{1/2s} \dots (T_s(f_s))^{1/2s} (T_t(g_1))^{1/2t} \dots (T_t(g_t))^{1/2t} .$$

This completes the proof.

Corollary 2.3 *Let A, B be finite subsets of G. Then*

$$T_k^{1/2k}(A \cup B) \leq T_k^{1/2k}(A) + T_k^{1/2k}(B) . \qquad (11)$$

We need in the notion of dissociativity in $\mathbf{F}_2^n$.

Definition 2.4 Let $R \subseteq \mathbf{F}_2^n$ be a set, $R = -R$ and $\{0\} \in R$. We say that a set $\Lambda = \{\lambda_1, \dots, \lambda_{|\Lambda|}\} \subseteq \mathbf{F}_2^n$ belongs to the family $\mathbf{\Lambda}_R(k)$ if the equality

$$\sum_{i=1}^{|\Lambda|} \varepsilon_i \lambda_i \in R , \qquad (12)$$

where $\varepsilon_i \in \{0, 1\}$ and $\sum_{i=1}^{|\Lambda|} |\varepsilon_i| \leq k$ implies that all ε_i are equal to zero. If $R = \{0\}$ then Λ belongs to the family $\mathbf{\Lambda}(k)$.

Proposition 2.5 *Let $k \geq 2$ be a positive integer, and $\Lambda \subseteq \mathbf{F}_2^n$ be an arbitrary set, belonging to the family $\mathbf{\Lambda}(2k)$. Then for any integer p, $2 \leq p \leq k$, we have*

$$T_p(\Lambda) \leq p^p |\Lambda|^p . \qquad (13)$$

Proof. Let $m = |\Lambda|$. Consider the equation

$$\lambda_1 + \dots + \lambda_{2p} = 0, \quad \lambda_i \in \Lambda, \quad i = 1, \dots, 2p . \qquad (14)$$

Let us consider any partitions $\mathcal{M} = \{M_1, \dots, M_p\}$ of the segment $[2p]$ onto sets M_j, $|M_j| = 2$, $j = 1, \dots, p$. It is easy to see that the number of such partitions equals $\frac{(2p)!}{2^p p!} \leq \frac{(2p)^p}{2^p} = p^p$. Further, let us mark any set M_j by an element $\lambda^{(j)}$ of the set Λ. Then the number of these *labelled* partitions does not exceed $p^p m^p$. By assumption the set Λ belongs to the family $\mathbf{\Lambda}(2k)$. Hence if $(\lambda_1, \dots, \lambda_{2p})$ is an arbitrary solution of (14) then any λ_i, $i \in [2p]$ appears an even number of times in this solution. So a solution $(\lambda_1, \dots, \lambda_{2p})$ of (14) corresponds to a labelled partition $\mathcal{M}' = \{(M_1, \lambda^{(1)}), \dots, (M_p, \lambda^{(p)})\}$. To see this let us construct a labelled partition $\mathcal{M}' = \{(M_1, \lambda^{(1)}), \dots, (M_p, \lambda^{(p)})\}$ such that for any $M_j = \{\alpha, \beta\}$, $j = 1, \dots, p$, we have $\lambda_\alpha = \lambda_\beta = \lambda^{(j)}$. Clearly, if we have two different solutions of (14) then we get different labelled partitions. Hence the total number of solutions of (14) does not exceed $p^p m^p$. This completes the proof.

Note 2.6 Rudin's Theorem (see [21, 22]) asserts that for any function $f : G \to \mathbb{C}$, $\operatorname{supp} \widehat{f} \subseteq \Lambda$, Λ is a dissociated set, we have $\|f\|_k \le C\sqrt{k}\|f\|_2$, where $C > 0$ is an absolute constant and $k \ge 2$. In other words, for an arbitrary a_λ the following holds

$$\frac{1}{N} \sum_x \left| \sum_{\lambda \in \Lambda} a_\lambda e(-\lambda \cdot x) \right|^k \le C^k k^{k/2} \left(\sum_{\lambda \in \Lambda} |a_\lambda|^2 \right)^{k/2} . \tag{15}$$

Certainly, inequality (15) implies Proposition 2.5 (up to constants) : to see this one can put $k = 2p$, $a_\lambda = 1$ and note that $T_p(\Lambda) = \frac{1}{N} \sum_x \left| \sum_{\lambda \in \Lambda} e(-\lambda \cdot x) \right|^{2p}$. On the other hand, we can use a slightly modified arguments from Proposition 2.5 to prove (15). Indeed, to obtain inequality (15), we need to calculate the number of solutions of (14) such that any solutions has weight $a_{\lambda_1} \dots a_{\lambda_{2p}}$ By assumption the set Λ belongs to the family $\mathbf{\Lambda}(2k)$. Hence if $(\lambda_1, \dots, \lambda_{2p})$ is an arbitrary solution of (14) then any λ_i, $i \in [2p]$ appears even number of times in this solution. It follows that if a partition $\mathcal{M} = \{M_1, \dots, M_p\}$ of the segment $[2p]$ onto the sets M_j, $|M_j| = 2$, $j = 1, \dots, p$ is fixed then we get weight $\left(\sum_{\lambda \in \Lambda} |a_\lambda|^2 \right)^p$. We know that the number of such partitions $\mathcal{M}$ does not exceed p^p. Thus, we have proved (15) in the case $k = 2p$. Using standard methods (see e.g. [10], Lemma 19), we obtain inequality (15) for all $k \ge 2$.

Now we can prove an analog of Proposition 2.5 for subsets of sums of dissociated sets and obtain Theorem 2.9.

Proposition 2.7 *Let k, d be positive integers, $k \ge 2$, and $\Lambda \subseteq \mathbf{F}_2^n$ be an arbitrary set, $\Lambda \in \mathbf{\Lambda}(2dk)$ such that $|\Lambda| \ge 4d^2$. Let also Q be a subset of $d\dot{\Lambda}$. Then for all integers p, $2 \le p \le k$, we have*

$$T_p(Q) \le 2^{8dp} p^{dp} |Q|^p . \tag{16}$$

Proof. We use induction. If $d = 1$, then the bound for $T_p(Q)$ was obtained in Proposition 2.5. Let $d \ge 2$, and let $m = |Q|$. Put $c_d := 8d$, $d \ge 1$.

Let $a = [|\Lambda|/2d]$. By assumption $|\Lambda| \ge 4d^2$. Hence $|\Lambda|/a \le 4d$. Also

$$\binom{|\Lambda| - d}{a - 1}^{-1} \binom{|\Lambda|}{a} = \frac{|\Lambda|(|\Lambda| - 1) \dots (|\Lambda| - d + 1)}{a(|\Lambda| - a)(|\Lambda| - a - 1) \dots (|\Lambda| - a - d + 2)} \le$$

$$\le 4d \cdot e^{\frac{1}{|\Lambda|} (\sum_{i=1}^{d-1} i + 2\sum_{i=0}^{d-2} (a+i))} \le 2^4 d . \tag{17}$$

Let E be any set. By E^c denote $\Lambda \setminus E$. Using dissociativity of Λ and the definition of the operation $\dot{+}$, we get

$$Q(x) = d^{-1} \binom{|\Lambda| - d}{a - 1}^{-1} \sum_{\Lambda_0 \subseteq \Lambda, \, |\Lambda_0| = a} \left(Q \bigcap (\Lambda_0 + (d-1)\dot{\Lambda}_0^c \right) (x) .$$

Using Hölder's inequality, we obtain

$$T_p(Q) \le d^{-2p} \binom{|\Lambda| - d}{a-1}^{-2p} \binom{|\Lambda|}{a}^{2p-1} \sum_{\Lambda_0 \subseteq \Lambda,\, |\Lambda_0| = a} T_p(Q \bigcap (\Lambda_0 + (d-1)\dot{\Lambda}_0^c)\,. \tag{18}$$

If we prove for any $\Lambda_0 \subseteq \Lambda$ the following inequality

$$T_p(Q \bigcap (\Lambda_0 + (d-1)\dot{\Lambda}_0^c) \le 2^{c_{d-1}p} p^{dp} |Q \bigcap (\Lambda_0 + (d-1)\dot{\Lambda}_0^c|^p\,,$$

then using (18) and (17), we obtain

$$T_p(Q) \le d^{-2p} \binom{|\Lambda| - d}{a-1}^{-2p} \binom{|\Lambda|}{a}^{2p} 2^{c_{d-1}p} p^{pd} m^p \le 2^{(c_{d-1}+8)p} p^{pd} m^p = 2^{c_d p} p^{pd} m^p$$

and Proposition 2.7 is proved.

Let $\Lambda_1 = \tilde{\Lambda}$, $\Lambda_2 = \Lambda \setminus \tilde{\Lambda}$, and $Q' \subseteq \Lambda_1 + (d-1)\dot{\Lambda}_2$. We have to prove that $T_p(Q') \le 2^{c_{d-1}p} p^{pd} |Q'|^p$. Let λ be an element from Λ_1. Consider the sets

$$D_\lambda = D(\lambda) = \{\ \lambda' \ : \ \lambda \dotplus \lambda' \in Q',\ \lambda' \in (d-1)\dot{\Lambda}_2\ \}\,,$$

$$Q_\lambda = Q(\lambda) = \{\ q \in Q' \ : \ q = \lambda \dotplus \lambda'_2 \dotplus \dots \dotplus \lambda'_d, \quad \lambda'_i \in \Lambda,\ i = 2, \dots, d\ \}\,,$$

Clearly, $Q(\lambda) = D(\lambda) + \lambda$. Let s_1 be a number of nonempty sets D_λ. Let these sets be $D_{\lambda_1}, \dots, D_{\lambda_{s_1}}$. We shall write D_j instead of D_{λ_j}. Let also $s_2 = |\Lambda_2|$. Obviously, $Q \subseteq \{\lambda_1, \dots, \lambda_s\} + (d-1)\dot{\Lambda}_2$

Consider the equation

$$q_1 + \dots + q_p = q_{p+1} + \dots + q_{2p}\,, \tag{19}$$

where $q_i \in Q'$, $i = 1, \dots, 2p$. Denote by σ the number of solutions of (19). Since $Q' \subseteq \Lambda_1 + (d-1)\dot{\Lambda}_2$, it follows that for all $q \in Q'$, we have $q = \lambda + \mu$, where $\lambda \in \Lambda_1$, $\mu \in (d-1)\dot{\Lambda}_2$.

Let $i_1, \dots, i_{2p} \in [s_1]$ be arbitrary numbers. Denote by $\sigma_{\vec{i}}$, $\vec{i} = (i_1, i_2, \dots, i_{2p})$ the set of solutions of equation (19) such that for all $j \in [2p]$, we have the restriction $q_j \in D(\lambda_{i_j})$, $\lambda_{i_j} \in \Lambda_1$. By assumption the sets Λ_1 and Λ_2 belong to the family $\mathbf{\Lambda}(2dk)$. Also L_1, L_2 have empty intersection. It follows that if $q_1, \dots, q_{2p}$ is a solution of equation (19) such that this solution belongs to the set $\sigma_{\vec{i}}$, then any component of vector $\vec{i}$ appears an even number of times in the vector. We have

$$\sigma \le \sum_{\mathcal{M},\, \mathcal{M} = \{M_1, \dots, M_r\},\ [2p] = M_1 \bigsqcup \dots \bigsqcup M_p} \ \sum_{\vec{i} \in \mathcal{M}} |\sigma_{\vec{i}}|\,. \tag{20}$$

Summation in (20) is taken over families of sets $\mathcal{M}$, $\mathcal{M} = \{M_1, \dots, M_p\}$, $[2p] = M_1 \bigsqcup \dots \bigsqcup M_p$ such that for all $j = 1, \dots, p$, we have $|M_j| = 2$. Let $M_j = \{\alpha_1^{(j)}, \alpha_2^{(j)}\}$, $j = 1, \dots, p$. By definition $\vec{i} \in \mathcal{M}$ if for all $j \in [p]$, we have $i_{\alpha_1^{(j)}} = i_{\alpha_2^{(j)}}$.

Using Lemma 2.2 and induction, we get

$$|\sigma_{\vec{i}}| \le 2^{c_{d-1}} p^{d(p-1)} \prod_{j=1}^{2p} |D(\lambda_{i_j})|^{1/2}\,.$$

Hence

$$\sigma \le 2^{c_{d-1}} p^{d(p-1)} \sum_{\mathcal{M}} \sum_{\vec{i} \in \mathcal{M}} \prod_{j=1}^{2p} |D(\lambda_{i_j})|^{1/2}\,. \tag{21}$$

Let $m' = |Q'|$, and let q be an arbitrary element of the set Q'. By assumption $\Lambda_1 \bigcap \Lambda_2 = \emptyset$ and Λ is a dissociated set, so it is easy to see that the sets $Q(\lambda)$ are disjoint. Hence

$$\sum_{\lambda \in \Lambda_1} |D(\lambda)| = \sum_{\lambda \in \Lambda_1} |Q(\lambda)| = m' \,. \tag{22}$$

For any $\lambda \in \Lambda_1$, we have $|D_\lambda| \le m'$. Let $x \ge 1$ be an arbitrary number. Using formula (22), we get

$$\sum_{\lambda \in \Lambda_1} |D(\lambda)|^x = \sum_{\lambda \in \Lambda_1} |Q(\lambda)|^x \le (m')_2^{x-1} \sum_{\lambda \in \Lambda_1} |Q(\lambda)| = (m')^x \,. \tag{23}$$

The number of partitions $\mathcal{M}$ in inequality (21) does not exceed p^p. Any component of vector $\vec{i}$ appears even number of times in the vector. Combining inequality (21) and bound (23), we obtain $\sigma \le 2^{c_{d-1}} p^{dp} (m')^p$. This completes the proof.

In some sense the last proposition is best possible.

Proposition 2.8 *Let k, d be positive integers, and let $\Lambda \subseteq \mathbf{F}_2^n$ belong to the family $\mathbf{\Lambda}(2d)$. Let also Λ_1 be an arbitrary subset of Λ, and $Q = d\dot{\Lambda}_1 \subseteq d\dot{\Lambda}$. Then for all $k \le |\Lambda_1|/(2d)$ and for any $2 \le p \le k$, we have $T_p(Q) \ge 2^{-3pd} p^{pd} |Q|^p$.*

Proof. Consider the equation

$$q_1 + \dots + q_p = q_{p+1} + \dots + q_{2p} \,, \tag{24}$$

where $q_i \in Q$, $i = 1, \dots, 2p$. Let us prove that equation (24) has at least $2^{-3pd} p^{pd} |Q|^p$ solutions. Since $q_i \in Q$, it follows that $q_i = \sum_{j=1}^d \lambda_j^{(i)}$, $i = 1, \dots, 2p$. Consider tuples $(q_1, \dots, q_p)$ such that *all* $\lambda_j^{(i)}$ for all q_i are different. Clearly, there are exactly $\binom{|\Lambda_1|}{pd} \frac{(pd)!}{(d!)^p}$ such tuples. For any tuple $(q_1, \dots, q_p)$ there are at least $\frac{(pd)!}{(d!)^p}$ solutions of equation (24). Indeed we have $\frac{(pd)!}{(d!)^p}$ number of ways to partition the set $\{\lambda_1^{(1)}, \dots, \lambda_d^{(1)}, \dots, \lambda_1^{(p)}, \dots, \lambda_d^{(p)}\} = \{\lambda_1, \dots, \lambda_{pd}\}$ onto p sets $M_1, \dots, M_p$ of the same cardinality. Put $q_i = \sum_{j \in M_i} \lambda_j$, $i = p+1, \dots, 2p$, we get a tuple $(q_{p+1}, \dots, q_{2p}) \in Q^p$ such that $q_1 + \dots + q_p = q_{p+1} + \dots + q_{2p}$. By assumption Λ is a dissociated set, thus any collection of sets $M_1, \dots, M_p$ corresponds to a tuple $(q_{p+1}, \dots, q_{2p})$. Hence $T_p(Q) \ge \binom{|\Lambda_1|}{pd} \frac{(pd)!}{(d!)^p} \cdot \frac{(pd)!}{(d!)^p}$. Since $\Lambda \in \mathbf{\Lambda}(2d)$, it follows that $|Q| = \binom{|\Lambda_1|}{d}$. Using the inequality $2kd \le |\Lambda_1|$, we get

$$T_p(Q) \ge \binom{|\Lambda_1|}{pd} \frac{(pd)!}{(d!)^p} \cdot \frac{(pd)!}{(d!)^p} \ge 2^{-pd} \frac{(pd)!}{(d!)^p} |Q|^p \ge 2^{-3pd} p^{pd} |Q|^p \,.$$

This completes the proof.

At the end of the section we show that Theorem 1.3 (actually Theorem 5.1, see Appendix) and Proposition 2.7 imply Theorem 1.2 in the case $G = \mathbf{F}_2^n$.

Theorem 2.9 *Let δ, α be real numbers, $0 < \alpha \le \delta \le 1/4$, d be a positive integer, $d \le \log(1/\delta)/4$, A be an arbitrary subset of $\mathbf{F}_2^n$ of the cardinality δN, and let $\mathcal{R}_\alpha$ be as in (3). Suppose that a set $\Lambda \subseteq \mathbf{F}_2^n$ belongs to the family $\mathbf{\Lambda}(2 \log(1/\delta))$. Then for all $1 \le d \le \log(1/\delta)/4$, we have*

$$|d\dot{\Lambda} \bigcap \mathcal{R}_\alpha| \le \left(\frac{\delta}{\alpha} \right)^2 \left(\frac{2^{12} \log(1/\delta)}{d} \right)^d \,. \tag{25}$$

Proof. Let $k = [\ln(1/\delta)/d] \ge 2$, $Q = d\dot{\Lambda} \bigcap \mathcal{R}_\alpha$ and $m = |Q|$. We need to prove that $m \le (\delta/\alpha)^2 (\frac{2^{12} \log(1/\delta)}{d})^d$. Using Theorem 5.1, we get

$$T_k(Q) \ge \frac{\delta \alpha^{2k}}{\delta^{2k}} m^{2k} \,. \tag{26}$$

On the other hand, by Proposition 2.7, we obtain $T_k(Q) \le 2^{8kd} k^{kd} m^k$. Combining the last inequality and (26), we get $m \le (\delta/\alpha)^2 (\frac{2^{12}\log(1/\delta)}{d})^d$. This concludes the proof.

So an upper bound for $|d\dot{\Lambda} \bigcap \mathcal{R}_\alpha|$ was obtained in Theorem 2.9. The next simple proposition gives us a lower estimate for the quantity. It is turn out this lower bound is close to the upper one.

Proposition 2.10 *Let δ be a real number, $1/N \le \delta \le 1/16$, and $\alpha = 2^{-12}\delta/\sqrt{n}$, $n \ge 32$. Then there exist a set $A \subseteq \mathbf{F}_2^n$, $\delta N \le |A| \le 8\delta N$ and a dissociated set $\Lambda \subseteq \mathcal{R}_\alpha(A)$ such that for all integers $d \ge 1$, we have*

$$|d\dot{\Lambda} \bigcap \mathcal{R}_\alpha| \ge 2^{-30} \left(\frac{\delta}{\alpha}\right)^2 \left(\frac{\log(1/\delta)}{16d}\right)^{d-1} . \tag{27}$$

Proof. Let $\vec{e}_1 = (1,0,\dots,0), \dots, \vec{e}_n = (0,\dots,0,1)$ be the standard basis for $\mathbf{F}_2^n$. Let also $k = [\log 1/(4\delta)]$, and $H, H^\perp$ be subspaces spanned by vectors $\vec{e}_1, \dots, \vec{e}_{n-k}$ and $\vec{e}_{n-k+1}, \dots, \vec{e}_n$, correspondingly. Let $A \subseteq H$ be a set of $\vec{x} = (x_1, \dots, x_n) \in H$ such that the number $x_i = 1$, $i = 1, \dots, n-k$ is at least $(n-k)/2$. Clearly, $|A| \ge 2^{n-k-2} \ge \delta N$ and $|A| \le |H| \le 2^{n-k} \le 8\delta N$. Let H' be a space spanned by vectors of the length $n-k$, namely $(1,0,\dots,0), \dots, (0,\dots,0,1)$. Let also $n' = n-k$, and let $A' \subseteq H'$ be the restriction of A on H'. Let us find Fourier coefficients of A'. We have

$$\widehat{A}'(r) = \sum_x A'(x)(-1)^{<r,x>} = |A' \bigcap H_r^{(0)}| - |A' \bigcap H_r^{(1)}| = 2|A' \bigcap H_r^{(0)}| - |A'| , \tag{28}$$

where $H_r^{(0)} = \{x \in H' \ : < r, x >= 0\}$ and $H_r^{(1)} = \{x \in H' \ : < r, x >= 1\}$. Let $l \ge 0$ be a positive integer. Consider the sets

$$\mathcal{H}_l = \{x = (x_1, \dots, x_{n'}) \ : \ \# x_i = 1 \text{ equals } l\} .$$

Let $r \in \mathcal{H}_1$. Put $\binom{x}{y} = 0$ for $y > x$. Using Stirling's formula and (28), we get

$$|\widehat{A}'(r)| = \left| \sum_{s=\lceil n'/2 \rceil}^{n'} \left(2\binom{n'-1}{s} - \binom{n'}{s} \right) \right| = \sum_{s=\lceil n'/2 \rceil}^{n'} \left(\frac{2s-n'}{n'} \right) \binom{n'}{s} \ge$$

$$\ge \sum_{s=\lceil n'/2 + \sqrt{n'}/2 \rceil}^{[n'/2+\sqrt{n'}]} \left(\frac{2s-n'}{n'} \right) \binom{n'}{s} \ge e^{-8} \frac{1}{2\sqrt{\pi}} \frac{2^{n'}}{\sqrt{n'}} \ge 2^{-14} \frac{2^{n'}}{\sqrt{n'}} \ge 2^{-12} \frac{\delta N}{\sqrt{n}} . \tag{29}$$

It is easy to see that for any $r \in H'$ and for all $h^\perp \in H^\perp$, we have $\widehat{A}(r + h^\perp) = \widehat{A}'(r)$. Hence $\mathcal{H}_1 + H^\perp \subseteq \mathcal{R}_\alpha(A)$, $\alpha = 2^{-12}\delta/\sqrt{n}$ and $|\mathcal{R}_\alpha(A)| \ge n' 2^k \ge n/(16\delta) \ge 2^{-28} \cdot \delta/\alpha^2$. Thus we have a lower bound for the cardinality of $\mathcal{R}_\alpha(A)$, which is close to an upper bound — δ/α^2. Clearly, the set A' is invariant under all permutations. Using this fact one can prove (assuming some restrictions on parameters) that the following holds $\mathcal{R}_\alpha(A) = (\{0\} \sqcup \mathcal{H}_1) + H^\perp$. We do not need in the fact.

Let $\Lambda = \{\vec{e}_1, \dots, \vec{e}_n\} \subseteq \mathcal{R}_\alpha(A)$, and $\Lambda^* = \{\vec{e}_{n-k+1}, \dots, \vec{e}_n\}$. Clearly, $\bigsqcup_{h_1 \in \mathcal{H}_1} (h_1 + (d-1)\dot{\Lambda}^*) \subseteq \mathcal{R}_\alpha(A) \bigcap d\dot{\Lambda}$. Hence

$$|\mathcal{R}_\alpha(A) \bigcap d\dot{\Lambda}| \ge n' \binom{k}{d-1} \ge \frac{n}{4} \cdot \frac{k^{d-1}}{d^{d-1} e^{d-1}} \ge 2^{-30} \left(\frac{\delta}{\alpha}\right)^2 \left(\frac{\log(1/\delta)}{16d}\right)^{d-1} .$$

This completes the proof.

Note 2.11 Certainly, we can change the value of the parameter α in Proposition 2.10. For example one can consider sets $\mathcal{H}_2$ instead of $\mathcal{H}_1$ and choose the parameter α smaller than $2^{-12}\delta/\sqrt{n}$.

3. On connected subsets of $d\dot{\Lambda}$.

Let G be an Abelian group, and $A \subseteq G$ be an arbitrary finite set. In paper [31], so–called "connected" sets A were studied (see also article [8]). Let us give a definition from [31].

Definition 3.1 Let $k \geq 2$ be a positive integer, and $\beta_1, \beta_2 \in [0,1]$ be real numbers, $\beta_1 \leq \beta_2$. Nonempty set $A \subseteq G$ is called (β_1, β_2)*–connected of degree* k if there exists an absolute constant $C \in (0,1]$ such that for any $B \subseteq A$, $\beta_1|A| \leq |B| \leq \beta_2|A|$ we have

$$T_k(B) \geq C^{2k} \left(\frac{|B|}{|A|} \right)^{2k} T_k(A) \,. \tag{30}$$

By $\zeta_k(A)$ denote the quantity $\zeta_k(A) := \frac{\log T_k(A)}{\log |A|}$. In paper [31] (see also [16]) the following result was obtained.

Theorem 3.2 *Let $\beta_1, \beta_2 \in (0,1)$ be real numbers, $\beta_1 \leq \beta_2$. Then there exists a set $A' \subseteq A$ such that*
1) *A' is (β_1, β_2)–connected of degree k set such that (30) holds for any $C \leq 1/32$.*
2) *$|A'| \geq m \cdot 2^{\frac{\log((2k-1)/\zeta)}{\log(1+\kappa)} \log(1-\beta_2)}$, where $\kappa = \frac{\log((1-\beta_1)^{-1})}{\log m}(1-16C)$.*
3) *$\zeta_k(A') \geq \zeta_k(A)$.*

In the section we prove an analog of Theorem 3.2 for subsets of dissociated sets.

Let $\Lambda \subseteq \mathbf{F}_2^n$ be an arbitrary set from the family $\mathbf{\Lambda}(2dk)$, and $A \subseteq d\dot{\Lambda}$. Denote by $D_k(A)$ the quantity

$$D_k(A) = \log \left(\frac{T_k(A)}{k^k |A|^k} \right) \,. \tag{31}$$

In other words $T_k(A) = 2^{D_k(A)} k^k |A|^k$. Since for all sets A with sufficiently large cardinality, we have $T_k(A) \geq \binom{|A|}{k} (k!)^2 \geq e^{-2k} k^k |A|^k$, it follows that the quantity $D_k(A)$ is at least $-2k \log e$. On the other hand, by Proposition 2.7, we get $D_k(A) \leq 8d \log d + k(d-1) \log k$.

Theorem 3.3 *Let $K > 0$ be a real number, $k, d \geq 2$ be positive integers. Let $\Lambda \subseteq \mathbf{F}_2^n$ be an arbitrary set, $\Lambda \in \mathbf{\Lambda}(2dk)$, and Q be a subset of $d\dot{\Lambda}$ such that $T_k(Q) \geq \frac{k^{dk}|Q|^k}{K^k}$. Let also $\beta_1, \beta_2 \in (0,1)$ be real numbers, $\beta_1 \leq \beta_2$. Then there is a set $Q' \subseteq Q$ such that*
1) *Q' is a (β_1, β_2)–connected of degree k such that (30) holds for any $C \leq 1/8$.*
2) *$|Q'| \geq |Q| \cdot 2^{\frac{8d \log d + k(d-1) \log k - D_k(Q)}{k \log(1+\beta_1(1-4C))} \log(1-\beta_2)}$.*
3) *$T_k(Q') \geq \frac{k^{dk}|Q'|^k}{K^k}$.*

Proof. Let $m = |Q|$, and $C \leq 1/8$ be a real number. The proof of Theorem 3.3 is a sort of algorithm. If Q is (β_1, β_2)–connected of degree k and (30) is true with the constant C then there is nothing to prove. Suppose that Q is not (β_1, β_2)–connected of degree k set (with the constant C). Then there exists a set $B \subseteq Q$, $\beta_1|Q| \leq |B| \leq \beta_2|Q|$ such that (30) does not hold. Note that $|Q| > 2$. Let $\overline{B} = Q \setminus B$ and $c_B = |B|/|Q|$. We have $\beta_1 \leq c_B \leq \beta_2$. Using Corollary 2.3, we get

$$T_k(\overline{B}) > T_k(Q)(1 - Cc_B)^{2k} \,. \tag{32}$$

Let $b = |B|$ and $\overline{b} = |\overline{B}| = m - b$, $D = D_k(Q)$, $\overline{D} = D_k(\overline{B})$. By inequality (32), we obtain

$$\overline{D} > D + k \log m - k \log \overline{b} + 2k \log(1 - Cc_B) = D + k \left(\log(\frac{m}{m-b}(1 - Cc_B)^2) \right) \geq$$

$$\geq D + k\log((1+c_B)(1-2Cc_B)) \geq D + k\log(1+\beta_1(1-4C))\,. \qquad (33)$$

Besides, by the definition of (β_1,β_2)–connectedness of degree k, we have

$$|\overline{B}| \geq (1-\beta_2)m = (1-\beta_2)|Q|\,. \qquad (34)$$

Thus if the set Q is not (β_1,β_2)–connected of degree k then there is a set $\overline{B} \subseteq Q$ such that (33), (34) hold. Put $Q_1 = \overline{B}$ and apply the arguments above to Q_1. And so on. We get the sets $Q_0 = Q, Q_1, Q_2, \dots, Q_s$. Clearly, for any Q_i, we have $D_k(Q_i) \leq 8d\log d + k(d-1)\log k$. Using this and (33), we obtain that the total number of steps of our algorithm does not exceed $\frac{8d\log d + k(d-1)\log k - D_k(Q)}{k\log(1+\beta_1(1-4C))}$. At the last step of the algorithm, we find the set $Q' = Q_s \subseteq Q$ such that Q' is (β_1,β_2)–connected of degree k and such that $D_k(Q') \geq D_k(Q)$. Thus Q' has the properties 1) and 3) of the Theorem. Let us prove 2). Using (34), we obtain

$$|Q'| \geq (1-\beta_2)^s m \geq m \cdot 2^{\frac{8d\log d + k(d-1)\log k - D_k(Q)}{k\log(1+\beta_1(1-4C))}\log(1-\beta_2)}\,.$$

This concludes the proof.

We shall use Theorem 3.3 in the next section.

4. On large subsets of sum of two dissociated sets.

To prove Theorem 4.9 we need in some lemmas. Lemma 4.3 is the most important one, other of them are technical. Actually, we just replace crude Proposition 2.5 by the Lemma 4.3 in our proof. The advantage of the last lemma consist in the following. At some step of our proof we need to count the number of solutions of the equation $\lambda_1 + \dots + \lambda_{2p} = 0$, where $\lambda_i \in E_i$, $E_i \subseteq \Lambda$, $i = 1, \dots, 2p$ and Λ is a dissociated set. Proposition 2.5 gives an upper bound for the number of solutions of the last equation in terms of the cardinalities of E_i whereas Lemma 4.3 uses information about the cardinalities of *intersections* $E_i \cap E_j$ of these sets. It turns out to be more economical than dealing with $|E_i|$ (see also discussion in Note 4.4). Further, in Proposition 4.6 we express our quantity $T_p(Q)$ in terms of such intersections and obtain Theorem 4.9.

Let $H = (h_{ij})$ be a matrix of the size $x \times y$, $x \leq y$. By $\operatorname{per} H$ denote the permanent of matrix H. Recall that

$$\operatorname{per} H = \sum_{\sigma} h_{1\sigma(1)} \dots h_{x\sigma(x)}\,, \qquad (35)$$

where the summation in (35) is taken over all injective maps $\sigma : [x] \to [y]$. We need in a well–known Frobenius–König's Theorem on nonnegative matrices (see [25]).

Theorem 4.1 *Let p and r be positive integers, $r \leq p$, and let H be a nonnegative matrix of size $p \times r$. Then the permanent of matrix H equals zero iff H contains a zero matrix of size $p-s+1 \times s$.*

Using Theorem 4.1, we prove a simple lemma.

Lemma 4.2 *Let p and r be positive integers, and let $H = (h_{ij})$ be a nonnegative matrix of size $p \times r$. Let also*
1) *For all $i \in [p]$, we have $\sum_{j=1}^{r} h_{ij} \geq 2$.*
2) *For all $j \in [r]$, we have $\sum_{i=1}^{p} h_{ij} \geq 1$, and, finally,*
3) $\sum_{i=1}^{p}\sum_{j=1}^{r} h_{ij} = 2p$.
Deleting from H all columns such that $\sum_{i=1}^{p} h_{ij} = 1$, we get matrix H_0. Then the permanent of H_0 does not equal zero.

Proof. Let the number of j such that $\sum_{i=1}^{p} h_{ij} = 1$ equals e. Without loss of generality we can suppose that the matrix H_0 was obtain from H by deletion of the last e columns. Let $H_0 = (h^0_{ij})$, $i = 1, \ldots, p$, $j = 1, \ldots, r - e = r_0$. Applying condition 3) of the Lemma, we get $\sum_{i=1}^{p} \sum_{j=1}^{r_0} h^0_{ij} = 2p - e$. Using condition 2), we obtain $r_0 \le p$. Suppose that our Lemma is false. If the permanent of H_0 equals zero then by Theorem 4.1 the matrix contains a submatrix of the size $s \times t$, $s + t = p + 1$. Using permutations of rows and columns, we can suppose that H_0 is

$$H_0 = \begin{pmatrix} \mathbf{X} & \mathbf{Z} \\ \mathbf{0} & \mathbf{Y} \end{pmatrix},$$

where zero matrix $\mathbf{0}$ has the size $s \times t$, $s + t = p + 1$. Denote by s_1 the number of $i \in \{p - s + 1, \ldots, p\}$ such that $\sum_{j=1}^{r_0} h^0_{ij} = 1$, and by s_2 the number of $i \in \{p - s + 1, \ldots, p\}$ such that $\sum_{j=1}^{r_0} h^0_{ij} \ge 2$. Clearly, $s_1 \le e$. By 2), we obtain $s = s_1 + s_2$. Using condition 1) of the lemma, we get

$$2p - e = \sum_{i=1}^{p} \sum_{j=1}^{r_0} h^0_{ij} \ge \sum_{j=1}^{t} \sum_{i=1}^{p} h^0_{ij} + \sum_{i=p-s+1}^{p} \sum_{j=1}^{r_0} h^0_{ij} \ge 2t + s_1 + 2s_2 = 2t + 2s - s_1 = 2p + 2 - s_1 \,.$$

The last inequality implies $s_1 \ge e + 2$ with contradiction. This completes the proof.

Let p be a positive integer, $\Lambda \subseteq \mathbf{F}_2^n$ be an arbitrary set, $\Lambda \in \mathbf{\Lambda}(2p)$, and $\mathcal{E} = \{E_1, \ldots, E_{2p}\}$ be a tuple of subsets of Λ. In the proof of Proposition 2.7 we estimated the number of solutions of the equation

$$\lambda_1 + \cdots + \lambda_{2p} = 0, \qquad \text{where} \qquad \lambda_i \in E_i, \quad i = 1, \ldots, 2p \,. \tag{36}$$

To calculate the number of such solutions, we used Lemma 2.2 — a simple corollary of Hölder's inequality. In the proof of the main result of this section — Theorem 4.10, we need in a more delicate result on the number of solutions of equation (36).

Lemma 4.3 *Let p a positive integer, $\Lambda \subseteq \mathbf{F}_2^n$ be an arbitrary set, $\Lambda \in \mathbf{\Lambda}(2p)$, and $\mathcal{E} = \{E_1, \ldots, E_{2p}\}$ be a tuple of subsets of Λ. Suppose that we have a partition of the segment $[2p]$ onto r classes $\mathcal{C}_1, \ldots, \mathcal{C}_r$. Let $S^* \subseteq [2p]$ be an arbitrary set, and $\bar{S}^* = [2p] \setminus S$. Let also $M(S^*) = (m_{ij})$ be a matrix of size $p \times p$, $m_{ij} = |E_i \bigcap E_j|$, $i \in S$, $j \in \bar{S}^*$. Then number of solutions of the equation*

$$\lambda_1 + \cdots + \lambda_{2p} = 0, \qquad \textit{where} \qquad \lambda_i \in E_i, \quad i = 1, \ldots, 2p \tag{37}$$

does not exceed

$$\sum_{S^* \subseteq [2p],\, |S^*| = p} \operatorname{per} M(S^*) \,, \tag{38}$$

and the summation in formula (38) is taken over all sets S^* such that S^* contains an element from any class $\mathcal{C}_i$ such that $|\mathcal{C}_i| \ge 2$.

Proof. Denote by Z the number of solutions of equation (37). By assumption the set Λ belongs to the family $\mathbf{\Lambda}(2p)$. Hence if $(\lambda_1, \ldots, \lambda_{2p})$ is an arbitrary solution of (37) then any λ_i, $i \in [2p]$ appears even number of times in this solution. Thus (see proof of Proposition 2.5), we get

$$Z \le \sum_{\mathcal{K},\, \mathcal{K} = \{K_1, \ldots, K_p\},\ [2p] = K_1 \bigsqcup \cdots \bigsqcup K_p} \ \prod_{j=1}^{p} \left| \bigcap_{\alpha \in K_j} E_\alpha \right| = Z_1 \,. \tag{39}$$

The summation in (39) is taken over families of sets $\mathcal{K}, \mathcal{K} = \{K_1, \dots, K_p\}$, $[2p] = K_1 \bigsqcup \dots \bigsqcup K_p$ such that for any $j \in [p]$, we have $|K_j| = 2$. Let us prove that

$$Z_1 \leq \sum_{S^* \subseteq [2p],\, |S^*|=p} \operatorname{per} M(S^*)\,, \tag{40}$$

and the summation in formula (40) is taken over all sets S^* such that S^* contains an element from any class $\mathcal{C}_i$ such that $|\mathcal{C}_i| \geq 2$. Clearly,

$$\sum_{\mathcal{K}, \mathcal{K}=\{K_1,\dots,K_p\},\ [2p]=K_1 \bigsqcup \dots \bigsqcup K_p} \prod_{j=1}^{p} \left| \bigcap_{\alpha \in K_j} E_\alpha \right| \leq \sum_{S^* \subseteq [2p],\, |S^*|=p} \operatorname{per} M(S^*) \tag{41}$$

Indeed if x is a summand from the left hand side of (41) which corresponds some partition $\mathcal{K}$ then x is present in the right hand side too. To see this let S^* be the set of all first elements of K_j, $j = 1, \dots, p$. Let α is any of such numbers, $\alpha \in K_j$. Then there is quantity $|E_\alpha \bigcap E_\beta|$ with $\beta \in K_j$ in the right hand side of (41). Taking a product of such quantitaes, we get x. Further if y is an arbitrary summand from the right hand side of (41) then it is easy to form a partition $\mathcal{K}$ which corresponds to the y.

If x is a summand from the left hand side of (40) which corresponds some partition $\mathcal{K}$ and we shall find a set S^* such that for all $j \in [p]$, we have $|K_j \bigcap S^*| = 1$ and such that S^* contains an element from any class $\mathcal{C}_i$ with restriction $|\mathcal{C}_i| \geq 2$ then we shall prove (40). Let $H = (h_{\gamma\delta})$ be a nonnegative matrix $p \times r$ such that any element $h_{\gamma\delta}$ of H equals $|K_\gamma \bigcap \mathcal{C}_\delta|$. Clearly, for all $\gamma \in [p]$, we have $\sum_\delta h_{\gamma\delta} = |K_\gamma| = 2$ and $\sum_{\gamma,\delta} h_{\gamma\delta} = \sum_\gamma |K_\gamma| = 2p$. Since the sets $\mathcal{C}_1, \dots, \mathcal{C}_r$ form a partition of the segment $[2p]$, it follows that for all $\delta \in [r]$ the following holds $\sum_\gamma h_{\gamma\delta} = |\mathcal{C}_\delta| \geq 1$. Using Lemma 4.2, we obtain that the permanent of the matrix H_0 does not equal zero. Hence H_0 contains a diagonal of nonzero elements. Let the size of H_0 be $p \times r_0$. Without loss of generality we can assume that the matrix H_0 is formed by first r_0 columnes of H. Then nonzero diagonal H_0 is $(\gamma_1, 1), \dots, (\gamma_{r_0}, r_0)$ and for any $i \in [r]$ there is a number $\alpha_i \in K_{\gamma_i}$ such that $\alpha_i \in \mathcal{C}_i$ and $|\mathcal{C}_i| \geq 2$. Let us add elements $\alpha_1, \dots, \alpha_r$ into the set S^*. Besides let us add the first elements of all K_γ, $\gamma \neq \gamma_1, \dots, \gamma_r$ in S^*. It is easy to see that S^* contains an element from any class $\mathcal{C}_i$ such that $|\mathcal{C}_i| \geq 2$. This completes the proof.

Note 4.4 Lemma 4.3 gives us an upper bound for $T_p(E_1, \dots, E_{2p})$. This estimate implies (up to constants) the bound $p^p \prod_{\alpha=1}^{2p} |E_\alpha|^{1/2}$ for $T_p(E_1, \dots, E_{2p})$, which can be derived from Lemma 2.2. Indeed for any sets A and B, we get

$$|A \bigcap B| \leq \min\{|A|, |B|\} \leq |A|^{1/2} |B|^{1/2}\,. \tag{42}$$

So any summand in $\operatorname{per} M(S^*)$ does not exceed $\prod_{\alpha=1}^{2p} |E_\alpha|^{1/2}$. We have exactly $p!$ of such summands. Thus by Lemma 4.3, we obtain that $T_p(E_1, \dots, E_{2p}) \leq 2^{2p} p! \prod_{\alpha=1}^{2p} |E_\alpha|^{1/2}$.

The quantity $\pi(t_1, \dots, t_r)$ below will appear in the proof of Theorem 4.9. We need in an upper bound for the last expression.

Lemma 4.5 *Let $\delta_0 > 0$ be a real number, r, p be positive integers, $p \geq 2\delta_0 + 3$, $r \geq p - \delta_0$. Let $t_1, \dots, t_r$ be a sequence of natural numbers such that $t_j \geq 2$, $j = 1, \dots, r$ and $\sum_{j=1}^{r} t_j = 2p$. Let also $T = \max_{j \in [r]} t_j$, and $\alpha_j = |\{j \in [r] \ :\ t_j \geq T - i\}|$, $i = 0, 1, \dots, T-2$. Let z be a nonnegative number such that $\sum_{i=0}^{z-1} \alpha_i \leq p < \sum_{i=0}^{z} \alpha_i$, and let $q_z = p - \sum_{i=0}^{z-1} \alpha_i$. Then the quantity*

$$\pi(t_1, \dots, t_r) := T^{\alpha_0} (T-1)^{\alpha_1} \dots (T-(z-1))^{\alpha_{z-1}} (T-z)^{q_z}$$

does not exceed $2^{3p}\max\{\,\delta_0^{4\delta_0},1\,\}$.

Proof. Suppose that $\delta_0 \ge 1$. It is easy to see that the sequence $\alpha_0,\alpha_1,\dots,\alpha_{T-2}$ is non-decreasing and $\sum_{i=0}^{T-2}\alpha_i = \sum_{j=1}^{r} t_j = 2p$. Using the condition $\sum_{j=1}^{r} t_j = 2p$ one more and inequalities $r \ge p-\delta_0$, $t_j \ge 2$, $j\in[r]$, we get $T+2(r-1) \le 2p$ and $T \le 2\delta_0+2 \le 4\delta_0$. Suppose that $\alpha_0=\dots=\alpha_{z-1}=q_z=1$. Then $p=\sum_{i=0}^{z-1}\alpha_i + q = z+1$. On the other hand, there are exactly $T-1$ numbers α_i. Hence z does not exceed $T-1$ and we get inequality $p\le T\le 2\delta_0+2$ with contradiction. Thus either $\alpha_{z-1}\ge 2$ or $q_z\ge 2$.

Let π^* be the maxiamal value of the function $\pi(t_1,\dots,t_r)$ such that all t_i satisfy

$$t_1+\dots+t_r = 2p, \quad t_j \ge 2, \quad r \ge p-\delta_0\,. \tag{43}$$

If (43) holds for a tuple $t_1,\dots,t_r$ then we shall say that this tuple is *admissible.* Let $\pi^* = \pi(t_1^0,\dots,t_r^0)$. Without loss of generality we can assume that $t_1^0\ge t_2^0\ge\dots\ge t_r^0$. We have that either $\alpha_{z-1}\ge 2$ or $q_z \ge 2$. Suppose that $t_2^0\ge 3$. Then we can consider an admissible tuple $\tilde t_1 = t_1^0+1$, $\tilde t_2 = t_1^0-1$, $\tilde t_3 = t_3^0,\dots,\tilde t_r = t_r^0$. Clearly, $\pi^*=\pi(t_1^0,\dots,t_r^0) < \pi(\tilde t_1,\dots,\tilde t_r)$. Whence $t_2^0 = 2$ and $\pi^* \le T^T 2^p \le 2^{3p}\delta_0^{4\delta_0}$.

Now suppose that $\delta_0 < 1$. In the case we have $T\le 4$. Using a trivial estimate $\pi(t_1,\dots,t_r)\le T^p \le 2^{2p}$ we get the required result. This completes the proof.

Let $k\ge 2$ be positive integer, and $\Lambda_1,\Lambda_2\subseteq \mathbf{F}_2^n$ be arbitrary *disjoint* sets such that $\Lambda_1\bigsqcup\Lambda_2$ belongs to the family $\mathbf{\Lambda}(4k)$. Let also Q be a subset of $\Lambda_1\dot{+}\Lambda_2 = \Lambda_1+\Lambda_2$. Define the sets $D(\lambda)=D_\lambda$ and $Q(\lambda)=Q_\lambda$, $\lambda\in\Lambda_1$ (see proof of Proposition 2.7). Let $\lambda\in\Lambda_1$ and

$$D(\lambda) = \{\ \lambda' \ :\ \lambda+\lambda' \in Q,\ \lambda'\in\Lambda_2\ \}\,,$$

$$Q(\lambda) = \{\ q\in Q \ :\ q=\lambda+\lambda',\ \lambda'\in\Lambda_2\ \}\,.$$

Clearly, $Q(\lambda) = D(\lambda)+\lambda$. Let s_1 be a number of nonempty sets D_λ. Let these sets are $D_{\lambda_1},\dots,D_{\lambda_{s_1}}$. We shall write D_j instead of D_{λ_j}. Let also $s_2 = |\Lambda_2|$. Obviously, $Q\subseteq \{\lambda_1,\dots,\lambda_s\}+\Lambda_2$.

Now we express our quantity $T_p(Q)$ in terms of the cardinalities of intersections $|D_\alpha\bigcap D_\beta|$.

Proposition 4.6 *Let* $M>0$ *be a real number,* $p\ge 5$ *be a positive integer, and* $\Lambda_1,\Lambda_2\subseteq\mathbf{F}_2^n$ *be arbitrary disjoint sets from the family* $\mathbf{\Lambda}(4p)$. *Let also* Q *be a subset of* $\Lambda_1+\Lambda_2$, $|Q|\ge\max\{2s_2p, 2^8 s_2 p M^8\}$, $\delta_0 = \max\{(p\log(2eM))/\log(|Q|/(s_2p)),1\}$, *and* $X=\max\{\,\delta_0^{4\delta_0},1\,\}$. *Then*

$$T_p(Q) \le 2^{5p} X p^{3p} s_2^p \cdot \sum_{r=p-\lceil\delta_0\rceil}^{p}\left(\frac{1}{ps_2}\right)^r\cdot\left(\sum_{S\subseteq[s_1],\,|S|=r}\ \prod_{\alpha\in S}\left(\sum_{\beta\in S}|D_\alpha\bigcap D_\beta|\right)\right) + \frac{p^{2p}|Q|^p}{2M^p}\,. \tag{44}$$

Proof. Let $m=|Q|$. Consider the equation

$$q_1+\dots+q_{2p} = 0\,, \tag{45}$$

where $q_i\in Q$, $i=1,\dots,2p$. Denote by σ the number of solutions of equation (45). Since $Q\subseteq\Lambda_1+\Lambda_2$, it follows that for all $q\in Q$, we have $q=\lambda_1+\lambda_2$, where $\lambda_1\in\Lambda_1$, $\lambda_2\in\Lambda_2$.

Let $i_1,\dots,i_{2p}\in[s_1]$ be arbitrary numbers. By $\sigma_{\vec i}$, $\vec i = (i_1,i_2,\dots,i_{2p})$ denote the set of solutions of equation (45) such that for all $j\in[2p]$ we have the restriction $q_j\in D(\lambda_{i_j})$, $\lambda_{i_j}\in\Lambda_1$. By assumption the set $\Lambda_1\bigsqcup\Lambda_2$ belongs to the family $\mathbf{\Lambda}(4k)$ and $\Lambda_1\bigcap\Lambda_2=\emptyset$. Hence if $(q_1,\dots,q_{2p}) \in \sigma_{\vec{i}}$ is an arbitrary solution of (45) then any component of vector $\vec{i}$ appears even number of times in this vector. We have

$$\sigma \le \sum_{\mathcal{N},\, \mathcal{N}=\{N_1,\dots,N_r\},\ [2p]=N_1 \bigsqcup \dots \bigsqcup N_r} \ \sum_{\vec{i}\in\mathcal{N}} |\sigma_{\vec{i}}| \,. \tag{46}$$

The summation in the right hand side of (46) is taken over families of sets $\mathcal{N}$, $\mathcal{N} = \{N_1,\dots,N_r\}$, $[2p] = N_1 \bigsqcup \dots \bigsqcup N_r$ such that for all $j = 1,\dots,r$ the cardinality of N_j is an even number and $|N_j| \ge 2$. Let $N_j = \{\alpha_1^{(j)},\dots,\alpha_{|N_j|}^{(j)}\}$, $j = 1,\dots,r$. By definition $\vec{i} \in \mathcal{N}$ if for all $j \in [r]$ the following holds $i_{\alpha_1^{(j)}} = \dots = i_{\alpha_{|N_j|}^{(j)}}$ and for any different sets N_{j_1}, N_{j_2} from the partition $\mathcal{N}$, we have $i_\alpha \ne i_\beta$, where α is an arbitrary element from N_{j_1}, and β is an element from N_{j_2}.

By $r = r(\mathcal{N})$ denotes the number of the sets N_j in the partition $\mathcal{N}$. We have

$$\sigma = \sum_{r=0}^{p-\lceil\delta_0\rceil} \ \sum_{\mathcal{N},\, r(\mathcal{N})=r} \ \sum_{\vec{i}\in\mathcal{N}} |\sigma_{\vec{i}}| + \sum_{r=p-\lceil\delta_0\rceil+1}^{p} \ \sum_{\mathcal{N},\, r(\mathcal{N})=r} \ \sum_{\vec{i}\in\mathcal{N}} |\sigma_{\vec{i}}| = \sigma_1 + \sigma_2 \,. \tag{47}$$

Let us estimate the sum σ_1. Let q be an arbitrary element of the set Q. Using the condition $\Lambda_1 \bigcap \Lambda_2 = \emptyset$ and dissociativity of Λ, it is easy to see that the sets $Q(\lambda)$ are disjoint. Hence

$$\sum_{\lambda\in\Lambda_1} |D(\lambda)| = \sum_{\lambda\in\Lambda_1} |Q(\lambda)| = m \,. \tag{48}$$

For any $\lambda \in \Lambda_1$, we have $|D_\lambda| \le s_2$. Let $x \ge 1$ be an arbitrary number. Using formula (48), we get

$$\sum_{\lambda\in\Lambda_1} |D(\lambda)|^x = \sum_{\lambda\in\Lambda_1} |Q(\lambda)|^x \le s_2^{x-1} \sum_{\lambda\in\Lambda_1} |Q(\lambda)| = s_2^{x-1} m \,. \tag{49}$$

Let $S_{\vec{i}} = \{i_j\}_{j\in[2p]}$. Applying Lemma 2.2 and inequality (49), we get

$$\sigma_1 \le \sum_{r=0}^{p-\lceil\delta_0\rceil} \ \sum_{\mathcal{N},\, r(\mathcal{N})=r} \ \sum_{\vec{i}\in\mathcal{N}} \prod_{\alpha\in[2p]} |D_{i_\alpha}|^{1/2} \le p^p \sum_{r=0}^{p-\lceil\delta_0\rceil} \ \sum_{\mathcal{N},\, r(\mathcal{N})=r} s_2^{p-r} \sum_{\vec{i}\in\mathcal{N}} \prod_{\alpha\in S_{\vec{i}}} |D_\alpha| \,.$$

Note that if the lengths of the sets N_j are fixed then the set $S_{\vec{i}}$ does not change after any permutation of these sets. Let $t_j = |N_j|$, $j = 1,\dots,r$. Using the inequality $m \ge 2s_2p$, the definition of the quantity δ_0 and identity (48), we obtain

$$\sigma_1 \le p^p \sum_{r=0}^{p-\lceil\delta_0\rceil} \ \sum_{t_1+\dots+t_r=2p} \frac{(2p)!}{t_1!\dots t_r!} \frac{1}{r!} s_2^{p-r} m^r \le e^p p^p s_2^p \sum_{r=0}^{p-\lceil\delta_0\rceil} \left(\frac{m}{s_2}\right)^r r^{2p-r} \le$$

$$\le 2e^p p^{3p} s_2^p \left(\frac{m}{ps_2}\right)^{p-\delta_0} = 2e^p p^{2p} m^p \left(\frac{s_2 p}{m}\right)^{\delta_0} \le \frac{p^{2p} m^p}{2M^p} \,. \tag{50}$$

Thus partitions $\mathcal{N}$ with small number $r(\mathcal{N})$ make a small contribution in $T_p(Q)$. At the second part of the proof we consider partitions $\mathcal{N}$ with large number of the sets N_j.

Let us estimate the sum σ_2. Consider the sets $D_{i_1},\dots,D_{i_{2p}}$. Let $\mathcal{C}_j = N_j$. So we get a partition of $[2p]$ onto the sets $\mathcal{C}_j$. Using Lemma 4.3, we obtain

$$\sigma_2 \le \sum_{r=p-\lceil\delta_0\rceil}^{p} \ \sum_{\mathcal{N},\, r(\mathcal{N})=r} \ \sum_{\vec{i}\in\mathcal{N}} \left(\sum_{S^*\subseteq[2p],\, |S^*|=p} \operatorname{per} M_{\vec{i}}(S^*) \right) \,. \tag{51}$$

By Lemma 4.3 the summation in formula (51) is taken over all sets S^* such that S^* contains an element from any set N_j. Further let $M_{\vec{i}}(S^*) = (m_{\alpha\beta})$ be a matrix of size $p \times p$, $m_{\alpha\beta} = |D_{i_\alpha} \bigcap D_{i_\beta}|$, $\alpha \in S^*$, $\beta \in \bar{S}^*$. Let $M'_{\vec{i}}$ be a matrix of the size $r \times 2p$, $M'_{\vec{i}} = (|D_\alpha \bigcap D_{i_\beta}|)_{\alpha \in S_{\vec{i}}, \beta \in [2p]}$. Using formula (35), we get

$$\operatorname{per} M_{\vec{i}}(S^*) \le \prod_{\alpha \in S^*,\, i_\alpha \notin S_{\vec{i}}} \left(\sum_{\beta \in \bar{S}^*} |D_{i_\alpha} \bigcap D_{i_\beta}| \right) \cdot \operatorname{per} M'_{\vec{i}} \,. \tag{52}$$

Applying the last inequality and a trivial estimate $|D_\lambda| \le s_2$, $\lambda \in \Lambda_1$, we have

$$\operatorname{per} M_{\vec{i}}(S^*) \le \prod_{\alpha \in S^*,\, i_\alpha \notin S_{\vec{i}}} \left(\sum_{x \in \Lambda_2} D_{i_\alpha}(x) \sum_{\beta \in \bar{S}^*} D_{i_\beta}(x) \right) \cdot \operatorname{per} M'_{\vec{i}} \le p^{p-r} s_2^{p-r} \operatorname{per} M'_{\vec{i}} \,. \tag{53}$$

Using (35), it is easy to see that

$$\operatorname{per} M'_{\vec{i}} \le \pi(t_1, \dots, t_r) \prod_{\alpha \in S_{\vec{i}}} \left(\sum_{\beta \in S_{\vec{i}}} |D_\alpha \bigcap D_\beta| \right) ,$$

where the quantity $\pi(t_1, \dots, t_r)$ was defined in Lemma 4.5. Using the bound for $\pi(t_1, \dots, t_r)$ from the lemma and inequalities (51), (53), we get

$$\sigma_2 \le 2^{5p} \max\{\, \delta_0^{4\delta_0}, 1 \,\} \cdot \sum_{r = p - \lceil \delta_0 \rceil}^{p} p^{p-r} s_2^{p-r} \sum_{\mathcal{N},\, r(\mathcal{N}) = r} \sum_{\vec{i} \in \mathcal{N}} \prod_{\alpha \in S_{\vec{i}}} \left(\sum_{\beta \in S_{\vec{i}}} |D_\alpha \bigcap D_\beta| \right) .$$

If we make a permutation of components of the vector $\vec{i}$ by different parts N_j of our partition $\mathcal{N}$ then the set $S_{\vec{i}}$ does not change. Besides, if the lengths of sets N_j are fixed then the set $S_{\vec{i}}$ does not change after any permutation of these sets. Hence

$$\sigma_2 \le 2^{5p} \max\{\, \delta_0^{4\delta_0}, 1 \,\} \cdot \sum_{r = p - \lceil \delta_0 \rceil}^{p} p^{p-r} s_2^{p-r} \sum_{t_1 + \dots + t_r = 2p} \frac{(2p)!}{t_1! \dots t_r!} \frac{1}{r!} r! \cdot$$

$$\cdot \left(\sum_{S \subseteq [s_1],\, |S| = r} \prod_{\alpha \in S} \left(\sum_{\beta \in S} |D_\alpha \bigcap D_\beta| \right) \right) \le 2^{5p} \max\{\, \delta_0^{4\delta_0}, 1 \,\} p^{3p} s_2^{p} \cdot$$

$$\cdot \sum_{r = p - \lceil \delta_0 \rceil}^{p} \left(\frac{1}{p s_2} \right)^{r} \cdot \left(\sum_{S \subseteq [s_1],\, |S| = r} \prod_{\alpha \in S} \left(\sum_{\beta \in S} |D_\alpha \bigcap D_\beta| \right) \right) . \tag{54}$$

Combining inequalities (50), (54) and formula (47), we get inequality (44). This completes the proof.

To prove Theorem 4.10 we need in a combinatorial lemma and a well–known lemma of E. Bombieri (see e.g. [27]).

Let p be a positive integer, and $A_1, \dots, A_p$ be a sequence of sets such that any two of them A_i and A_j either disjoint or equals. By ρ denote the number of different sets among

$A_1, \dots, A_p$. Let the set A_1^* appears in the sequence $A_1, \dots, A_p$ exactly l_1 times, A_2^* — exactly l_2 times, …, A_ρ^* exactly l_ρ times.

Lemma 4.7 *Let w be a positive integer, $2 \le p \le a$, $\zeta \in (0,1]$ be a real number, and $S_1, \dots, S_q$ be some different sets, $|S_i| = p$, $S_i = \{s_i^{(1)}, \dots, s_i^{(p)}\}$, $i = 1, \dots, q$. Let also for all $i \in [q]$ and for all sets S_i, we have $s_i^{(j)} \in A_j$, $j = 1, \dots, p$. Suppose that*

$$q \ge 2 \sum_{\omega = \lceil \zeta p \rceil}^{p} \frac{(pw)^\omega}{\omega!} \sum_{n_1 + \dots + n_\rho = p - \omega,\, n_i \le l_i} \frac{|A_1^*|^{n_1} \dots |A_\rho^*|^{n_\rho}}{n_1! \dots n_\rho!} \,.$$

Then there are sets $S_{n_1}, \dots, S_{n_w}$ from the sequence $S_1, \dots, S_q$ such that for an arbitrary $l = 2, \dots, w$, we have $|(\bigcup_{i=1}^{l-1} S_{n_i}) \bigcap S_{n_l}| \le \zeta p$.

Proof. We use a greedy algorithm. Let $S_{n_1} = S_1$. Suppose that sets $S_{n_1}, \dots, S_{n_{l-1}}$ have been constructed and find S_{n_l}. Let $C_l = \bigcup_{i=1}^{l-1} S_{n_i}$. Clearly, $|C_l| \le wp$. Let $C_l = C_1^* \bigsqcup \dots \bigsqcup C_\rho^*$, where $C_i^* \subseteq A_i^*$, $i = 1, \dots, \rho$. Let also $a_i = |A_i^*|$, $c_i = |C_i^*|$, $i = 1, \dots, \rho$. The number of sets E belong to $A_1^* \bigsqcup \dots \bigsqcup A_\rho^*$, $|E| = p$ and such that $|E \bigcap C_l| \ge \zeta p$ does not exceed

$$\sigma := \sum_{\omega = \lceil \zeta p \rceil}^{p} \sum_{m_1 + \dots + m_\rho = \omega,\, m_i \le c_i} \sum_{n_1 + \dots + n_\rho = p - \omega,\, n_i \le \min\{a_i - c_i, l_i\}} \binom{c_1}{m_1} \dots \binom{c_\rho}{m_\rho} \times$$

$$\times \binom{a_1 - c_1}{n_1} \dots \binom{a_\rho - c_\rho}{n_\rho} \le \sum_{\omega = \lceil \zeta p \rceil}^{p} \sum_{m_1 + \dots + m_\rho = \omega} \sum_{n_1 + \dots + n_\rho = p - \omega,\, n_i \le l_i} \frac{c_1^{m_1} \dots c_\rho^{m_\rho}}{m_1! \dots m_\rho!} \cdot \frac{a_1^{n_1} \dots a_\rho^{n_\rho}}{n_1! \dots n_\rho!} \le$$

$$\le \sum_{\omega = \lceil \zeta p \rceil}^{p} \frac{(c_1 + \dots + c_\rho)^\omega}{\omega!} \cdot \sum_{n_1 + \dots + n_\rho = p - \omega,\, n_i \le l_i} \frac{a_1^{n_1} \dots a_\rho^{n_\rho}}{n_1! \dots n_\rho!} \le$$

$$\le \sum_{\omega = \lceil \zeta p \rceil}^{p} \frac{(pw)^\omega}{\omega!} \cdot \sum_{n_1 + \dots + n_\rho = p - \omega,\, n_i \le l_i} \frac{a_1^{n_1} \dots a_\rho^{n_\rho}}{n_1! \dots n_\rho!} = \sigma^* \,.$$

By assumption $q \ge 2\sigma^*$. Hence $q \ge w$ and, consequently, $q - (l-1) > q - w \ge \sigma^*$. Thus there is a set S_{n_l} from $S_1, \dots, S_q$ such that S_{n_l} does not equal $S_{n_1}, \dots, S_{n_{l-1}}$ and such that $|(\bigcup_{i=1}^{l-1} S_{n_i}) \bigcap S_{n_l}| \le \zeta p$. This completes the proof.

Lemma 4.8 (Bombieri) *Let q be a positive integer, $\lambda > 0$ be a real number, B be a finite set. Suppose that $B_1, \dots, B_q$ are subsets of B such that $|B_i| \ge \lambda |B|$. Then for all $t \le \lambda q$ there are different positive integers $j_1, \dots, j_t \in [q]$ such that*

$$|B_{j_1} \bigcap \dots \bigcap B_{j_t}| \ge \left(\lambda - \frac{t}{q} \right) \binom{q}{t}^{-1} |B| \,.$$

Now we can prove the main result of the section.

Theorem 4.9 *Let $K, \eta > 0$ be real numbers, $\eta \in (0, 1/2]$, p be a positive integer, and $\Lambda \subseteq \mathbf{F}_2^n$ be an arbitrary set from the family $\mathbf{\Lambda}(4p)$. Let also Q be a subset of $\Lambda \dotplus \Lambda$, $K^* := \max\{1, K\}$, $p \ge 2^{30} K^* / \eta$, and*

$$T_p(Q) \ge \frac{p^{2p} |Q|^p}{K^p} \,. \qquad (55)$$

Suppose that $p \le \log|\Lambda|/\log\log|\Lambda|$ and

$$|Q| \ge 2^{60+\frac{2}{\eta}}(K^*)^{17}p^3|\Lambda| \cdot \max\left\{ (2^{30}(K^*)^{11}p)^{\eta p}|\Lambda|^{\eta}\log|\Lambda|, \exp\left(\frac{\log(2^{30}(K^*)^{20})\log(\frac{p\log K^*}{\log p})}{\log(\frac{2^{-25}\eta p}{K^*})} \right) \right\}$$

Then there are sets $\mathcal{L}_1, \mathcal{L}'_1, \dots, \mathcal{L}_h, \mathcal{L}'_h$ from Λ such that $\mathcal{L}_i \bigcap \mathcal{L}'_j = \emptyset$, $\mathcal{L}_i + \mathcal{L}'_i \subseteq Q$, $i = 1, \dots, h$, $j = 1, \dots, h$,

$$|\mathcal{L}_i| \ge \frac{\log(\frac{|Q|}{16(K^*)^9|\Lambda|})}{2^{10}\log(2^{20}K^*)}, \quad |\mathcal{L}'_i| \ge \frac{1}{2^{10}p^2}\left(\frac{|Q|}{(K^*)^9|\Lambda|}\right)^{\eta}, \tag{56}$$

$(\mathcal{L}_i + \mathcal{L}'_i)\bigcap(\mathcal{L}_j + \mathcal{L}'_j) = \emptyset$, $i, j = 1, \dots, h$, $i \ne j$ and

$$\left| Q \bigcap \left((\mathcal{L}_1 + \mathcal{L}'_1) \bigsqcup \dots \bigsqcup (\mathcal{L}_h + \mathcal{L}'_h) \right) \right| \ge \frac{|Q|}{16(K^*)^9}. \tag{57}$$

If p is an arbitrary and

$$\log\left(\frac{|Q|}{16(K^*)^9 p|\Lambda|}\right) \ge 2^{20}\log(2^{10}K^*)\log p, \tag{58}$$

then there are sets $\mathcal{L}_1, \mathcal{L}'_1, \dots, \mathcal{L}_h, \mathcal{L}'_h$ from Λ satisfying (57) and such that

$$|\mathcal{L}_i| \ge \min\{2^{-18}\frac{p}{K^*}, 2^{-5}\log\left(\frac{|Q|}{16(K^*)^9 p}\right)\}, \quad |\mathcal{L}'_i| \ge \frac{1}{32p^2}\left(\frac{|Q|}{(K^*)^9|\Lambda|}\right)^{1/2}. \tag{59}$$

Note 4.10 If $K = O(1)$ e.g. $K \le 1$ then inequalities (57), (59) hold if we have more weaker bound than (58), namely $|Q| \ge 2^{60+\frac{2}{\eta}}(K^*)^{17}p^3|\Lambda|$.

Proof of the theorem. Let $m = |Q|$, $\beta_1 = 1/4$, $\beta_2 = 1/2$. Let also

$$\mathbf{M} = 2^{52+\frac{2}{\eta}}(K^*)^{17}p^3|\Lambda| \cdot \max\left\{ (2^{27}(K^*)^{11}p)^{\eta p}|\Lambda|^{\eta}\log|\Lambda|, \exp\left(\frac{\log(2^{24}(K^*)^{20})\log(\frac{p\log K^*}{\log p})}{\log(\frac{2^{-22}\eta p}{K^*})} \right) \right\}.$$

Since $T_p(Q) \ge p^{2p}|Q|^p/K^p$, it follows that $D_p(Q) \ge p\log(p/K)$. Using Theorem 3.3 with parameters $d = 2$ and $C = 1/8$, we get (β_1, β_2)–connected set $Q_1 \subseteq Q$ of degree p such that $m_1 := |Q_1| \ge m/(2K^9)$ and $T_p(Q_1) \ge p^{2p}m_1^p/K^p$. Let $a = \lceil |\Lambda|/2 \rceil$. We have

$$\sum_{\tilde{\Lambda} \subseteq \Lambda,\, |\tilde{\Lambda}| = a} \; \sum_{\lambda_1 \in \tilde{\Lambda},\, \lambda_2 \in \Lambda \setminus \tilde{\Lambda}} Q_1(\lambda_1 + \lambda_2) = 2\binom{|\Lambda| - 2}{a - 1}|Q_1|. \tag{60}$$

Using (60), it is easy to see that there is a set $\tilde{\Lambda} \subseteq \Lambda$, $|\tilde{\Lambda}| = a$ such that $|Q_1 \bigcap (\tilde{\Lambda} + (\Lambda \setminus \tilde{\Lambda}))| \ge 2m_1\binom{|\Lambda|-2}{a-1}\binom{|\Lambda|}{a}^{-1} = 2m_1\frac{a(|\Lambda|-a)}{|\Lambda|(|\Lambda|-1)} \ge m_1/2$. Put $\Lambda_1 = \tilde{\Lambda}$, $\Lambda_2 = \Lambda \setminus \tilde{\Lambda}$ and $Q_2 = Q_1 \bigcap (\Lambda_1 + \Lambda_2)$. Certainly, we can find a set $Q_3 \subseteq Q_2$ such that $Q_3 = \lceil m_1/2 \rceil$. Let $m_3 = |Q_3|$. Since the set Q_1 is (β_1, β_2)–connected of degree p and $C = 1/8$, it follows that

$$T_p(Q_3) \ge 2^{-6p}\left(\frac{m_3}{m_1}\right)^{2p} T_p(Q_1) \ge \frac{p^{2p}m_3^p}{(2^7K)^p}.$$

Using notation of Proposition 4.6, taking $M = 2^7K$ and applying this Proposition to the set $Q_3 \subseteq \Lambda_1 + \Lambda_2$, we get

$$2^{5p}Xp^{3p}s_2^p\cdot\sum_{r=p-\lceil\delta_0\rceil}^{p}\left(\frac{1}{ps_2}\right)^r\cdot\left(\sum_{S\subseteq[s_1],\,|S|=r}\prod_{\alpha\in S}\left(\sum_{\beta\in S}|D_\alpha\bigcap D_\beta|\right)\right)\ge\frac{p^{2p}m_3^p}{2(2^7K)^p}\,. \tag{61}$$

Recall that the quantity δ_0 equals $\delta_0 = \max\{(p\log(2eM))/\log(|Q_3|/(s_2p)),1\}$, and the number X is $\max\{\,\delta_0^{4\delta_0},1\,\}$. If $m \ge \mathbf{M}$ or m satisfy (58) then $\delta_0 \le \max\{(p\log(2^{10}K))/(2\log p),1\} \le p/2$ and $X^{1/p} \le 2^8K$. Let $K_1 = 2^{13}KX^{1/p} \le 2^{21}K^2$. Suppose that either $m \ge \mathbf{M}$ or m satisfy (58). Then $m_3 \ge 2K_1p|\Lambda|$. Using the last inequality and (61), we obtain that there is a positive integer $p_1 \in [p-\lceil\delta_0\rceil, p]$ such that

$$\sum_{S\subseteq[s_1],\,|S|=p_1}\prod_{\alpha\in S}\left(\sum_{\beta\in S}|D_\alpha\bigcap D_\beta|\right)\ge\frac{m_3^{p_1}}{K_1^{p_1}}\,. \tag{62}$$

Let $S \subseteq [s_1]$ be a set, $|S| = p_1$, and $\alpha \in S$ be an arbitrary element of the set S. Let also $\varepsilon = 1/(16K_1)$. If $M \le 1/2$ then $X = 1$, and by inequality $p \ge 2^{30}K^*/\eta$, we get $\varepsilon \ge 1/p_1$. Suppose that $M > 1/2$ and either $m \ge \mathbf{M}$ or m satisfy (58). In the case the inequality $\varepsilon \ge 1/p_1$ can be derived from the condition $p \ge 2^{30}K^*/\eta$. A slightly more accurate computations show that in the both cases, we have $\varepsilon \ge 16/(\eta p)$. Define the sets

$$G_{S,\alpha} = \{\ x \in D_\alpha \ :\ \sum_{\beta\in S} D_\beta(x) \ge \varepsilon p_1\ \}\,.$$

In other words, $G_{S,\alpha}$ is the set of x from D_α such that x belongs to at least εp_1 the sets D_β, $\beta \in S$. We have

$$\sum_{\beta\in S}|D_\alpha\bigcap D_\beta| = \sum_{x\in\Lambda_2} D_\alpha(x)\sum_{\beta\in S}D_\beta(x) =$$

$$= \sum_{x\in G_{S,\alpha}} D_\alpha(x)\sum_{\beta\in S}D_\beta(x) + \sum_{x\notin G_{S,\alpha}} D_\alpha(x)\sum_{\beta\in S}D_\beta(x) \le p_1|G_{S,\alpha}| + \varepsilon p_1|D_\alpha|\,. \tag{63}$$

Let $\mathcal{S}$ be the family of sets S, $S \subseteq [s_1]$, $|S| = p_1$ such that for any $S \in \mathcal{S}$ there is $\alpha \in S$ such that $|G_{S,\alpha}| \ge \varepsilon|D_\alpha|$ and $|D_\alpha| \ge \varepsilon m_3/s_2$. Let also $\bar{\mathcal{S}}$ be the family of sets from S, $S \subseteq [s_1]$, $|S| = p_1$ do not belong to the family $\mathcal{S}$. Let us prove that

$$\sigma_1 := \sum_{S\in\bar{\mathcal{S}}}\prod_{\alpha\in S}\left(\sum_{\beta\in S}|D_\alpha\bigcap D_\beta|\right) \le \frac{m_3^{p_1}}{2K_1^{p_1}}\,. \tag{64}$$

Let $Y(S) = \{\alpha \in S\ :\ |G_{S,\alpha}| < \varepsilon|D_\alpha|\}$, and $\bar{Y}(S) = S \setminus Y(S)$. Using (63), we get

$$\sigma_1 = \sum_{S\in\bar{\mathcal{S}}}\ \prod_{\alpha\in\bar{Y}(S),\,|D_\alpha|<\varepsilon m_3/s_2}\left(\sum_{\beta\in S}|D_\alpha\bigcap D_\beta|\right)\cdot\prod_{\alpha\in Y(S)}\left(\sum_{\beta\in S}|D_\alpha\bigcap D_\beta|\right) \le$$

$$\le \sum_{l=0}^{p_1}\ \sum_{S\subseteq[s_1],\,|S|=p_1,\,|Y(S)|=p_1-l}\left(\frac{\varepsilon p_1 m_3}{s_2}\right)^{|\bar{Y}(S)|}\cdot\prod_{\alpha\in Y(S)}(2\varepsilon p_1|D_\alpha|) \le$$

$$\leq \sum_{l=0}^{p_1} \left(\frac{\varepsilon p_1 m_3}{s_2}\right)^l (2\varepsilon p_1)^{p_1-l} \binom{s_1-(p_1-l)}{l} \sum_{S'\subseteq[s_1],\, |S'|=p_1-l} \prod_{\alpha\in S'} |D_\alpha| \leq$$

$$\leq 2^{p_1}\varepsilon^{p_1} \sum_{l=0}^{p_1} \left(\frac{\varepsilon p_1 m_3}{s_2}\right)^l p_1^{p_1-l} \frac{s_1^l}{l!} \varepsilon^{-l} \frac{1}{(p_1-l)!} m_3^{p_1-l} \leq 2(2e)^{p_1}\varepsilon^{p_1} m_3^{p_1} \sum_{l=0}^{p_1} \frac{p_1!}{l!(p_1-l)!} =$$

$$= 2(4e)^{p_1}\varepsilon^{p_1} m_3^{p_1} \leq 2^{4p_1-1}\varepsilon^{p_1} m_3^{p_1} = \frac{m_3^{p_1}}{2K_1^{p_1}}$$

and inequality (64) is proved. Hence

$$\sigma_2 = \sum_{S\in\mathcal{S}} \prod_{\alpha\in S} \left(\sum_{\beta\in S} |D_\alpha \bigcap D_\beta|\right) \geq \frac{m_3^{p_1}}{2K_1^{p_1}}\,. \tag{65}$$

Consider the case $p \leq \log|\Lambda|/\log\log|\Lambda|$. Let $u_0 = [\log s_2]$, and $\Lambda^{(j)} = \{\alpha \in [s_1] \;:\; 2^{j-1} \leq |D_\alpha| \leq 2^j\}$, $j = 1, \dots, u_0$. Applying inequality $\sum_{\alpha\in\Lambda_1} |D_\alpha| \leq m$, we derive that $|\Lambda^{(j)}| \leq 2m2^{-j}$. Let $(j_1, \dots, j_{p_1})$ be a tuple from $[u_0]^{p_1}$. By $\rho = \rho(j_1, \dots, j_{p_1})$ denote the number of different j_t in $(j_1, \dots, j_{p_1})$ and let an element j_1^* appears in the tuple exactly l_1 times, an element j_2^* appears exactly l_2 times, $\dots$, and an element j_ρ^* appears exactly l_ρ times, and all elements $j_1^*, j_2^*, \dots, j_\rho^*$ are different. We have

$$\sigma_2 \leq \sum_{S\in\mathcal{S}} p_1^{p_1} \prod_{\alpha\in S} |D_\alpha| = \frac{p_1^{p_1}}{p_1!} \sum_{S\in\mathcal{S}} \sum_{\alpha_1,\dots,\alpha_{p_1}\text{— different}} S(\alpha_1)\dots S(\alpha_{p_1})|D_{\alpha_1}|\dots|D_{\alpha_{p_1}}| =$$

$$= p_1^{p_1} \sum_{S\in\mathcal{S}} \sum_{j_1,\dots,j_{p_1}=1}^{u_0} \frac{1}{l_1!\dots l_\rho!}\times$$

$$\times \sum_{\alpha_1\in\Lambda^{(j_1)},\dots,\alpha_{p_1}\in\Lambda^{(j_{p_1})},\, \alpha_1,\dots,\alpha_{p_1}\text{— different}} S(\alpha_1)\dots S(\alpha_{p_1})|D_{\alpha_1}|\dots|D_{\alpha_{p_1}}|\,. \tag{66}$$

Using formula (66), we obtain that there is a tuple $(j_1, \dots, j_{p_1})$ such that

$$\sum_{S\in\mathcal{S},\, S=\{s^{(1)},\dots,s^{(p_1)}\},\, s^{(j)}\in\Lambda^{(j)}} \prod_{\alpha\in S} |D_\alpha| \geq \frac{l_1!\dots l_\rho!}{p_1^{p_1} u_0^{p_1}} \cdot \frac{m_3^{p_1}}{4K_1^{p_1}}\,.$$

By the definition of the sets $\Lambda^{(j)}$, we get

$$q_0 := |\{S\in\mathcal{S} \;:\; S=\{s^{(1)},\dots,s^{(p_1)}\},\, s^{(j)}\in\Lambda^{(j)}\}| \geq \frac{l_1!\dots l_\rho!}{p_1^{p_1} u_0^{p_1} 2^{j_1+\dots+j_{p_1}}} \cdot \frac{m_3^{p_1}}{4K_1^{p_1}}\,.$$

Using Dirichlet's principle, we obtain that there is $\alpha \in [s_1]$ such that

$$q := |\{S\in\mathcal{S} \;:\; S=\{s^{(1)},\dots,s^{(p_1)}\},\, s^{(j)}\in\Lambda^{(j)},\, \alpha\in S\}| \geq \frac{l_1!\dots l_\rho!}{p_1^{p_1} u_0^{p_1} 2^{j_1+\dots+j_{p_1}}} \cdot \frac{m_3^{p_1}}{4s_1K_1^{p_1}}\,.$$

Let $A_i = \Lambda^{(j_i)}$, $i = 1, \dots, p_1$, and $A_i^* = \Lambda^{(j_i^*)}$, $i = 1, \dots, \rho$. We want to apply Lemma 4.7 to the sets A_i, A_i^* with parameter $w = [\log(m_3/s_2)/(\varepsilon^2 p_1 \log(2^6/\varepsilon^2))]$. Since $m \geq \mathbf{M}$ and $m_3 \geq m/(8K^9)$, it follows that

$$q \geq \frac{l_1!\dots l_\rho!}{p_1^{p_1} u_0^{p_1} 2^{j_1^* l_1+\dots+j_\rho^* l_\rho}} \cdot \frac{m_3^{p_1}}{4s_1K_1^{p_1}} \geq 2 \sum_{\omega=\lceil\zeta p_1\rceil}^{p_1} \frac{(p_1 w)^\omega}{\omega!} \sum_{n_1+\dots+n_\rho=p_1-\omega,\, n_i\leq l_i} \frac{|A_1^*|^{n_1}\dots|A_\rho^*|^{n_\rho}}{n_1!\dots n_\rho!} = 2\sigma^*\,. \tag{67}$$

Indeed, by assumption $m \geq \mathbf{M} \geq p_1 w s_2$. Hence

$$2^{j_1^* l_1 + \cdots + j_\rho^* l_\rho} \sigma^* \leq 2^{p_1} \sum_{\omega = \lceil \zeta p_1 \rceil}^{p_1} \frac{(p_1 w)^\omega}{\omega!} \sum_{n_1 + \cdots + n_\rho = p_1 - \omega,\, n_i \leq l_i} \frac{s_2^\omega m^{p_1 - \omega}}{n_1! \dots n_\rho!} \leq$$

$$\leq 2^{p_1} m^{p_1} \sum_{\omega = \lceil \zeta p_1 \rceil}^{p_1} \frac{(p_1 w)^\omega}{\omega!} \left(\frac{s_2}{m} \right)^\omega \frac{\rho^{p_1 - \omega}}{(p_1 - \omega)!} \leq 2^{4 p_1} \left(\frac{\rho}{p_1} \right)^{p_1 (1 - \zeta)} w^{\zeta p_1} m^{p_1 (1 - \zeta)} s_2^{\zeta p_1} \tag{68}$$

(we used the identity $1/(\omega!(p-\omega)!) = \binom{p}{w}/p!$ in the last inequality). To check (67) we need to verify inequality

$$m_3 \geq 2^7 u_0 K_1 w^\zeta p_1 m^{1-\zeta} s_1^{1/p_1} s_2^\zeta \geq 32 \left(\frac{\rho^{p_1 (1-\zeta)} p_1^{\zeta p_1}}{l_1! \dots l_\rho!} \right)^{1/p_1} u_0 K_1 w^\zeta m^{1-\zeta} s_1^{1/p_1} s_2^\zeta \,. \tag{69}$$

But the last inequality easily follows from $\varepsilon \geq 16/(\eta p)$, $m_3 \geq m/(8K^9)$ and

$$m \geq \mathbf{M} \geq 2^{44} (K^*)^4 p |\Lambda|^{1+\eta} \log |\Lambda| (2^{27} (K^*)^{11} p)^{\eta p} \geq |\Lambda| w p \cdot (s^{1/p_1} \log |\Lambda| \, 2^{27} (K^*)^{11} p^2)^{1/\zeta} \,.$$

Applying Lemma 4.7 to the sets A_i, A_i^*, we get new sets $S_1^*, \dots, S_w^* \in \mathcal{S}$ such that for all $l = 2, 3, \dots, w$, we have $|(\bigcup_{i=1}^{l-1} S_i^*) \bigcap S_l^*| \leq \zeta p_1$. Note that $|G_{S_i^*, \alpha}| \geq \varepsilon |D_\alpha|$, $i = 1, \dots, w$ and $|D_\alpha| \geq \varepsilon m_3 / s_2$. Applying Lemma 4.8 with parameter $t = [\varepsilon w/2]$ to the sets $G_{S_1^*, \alpha}, \dots, G_{S_w^*, \alpha} \subseteq D_\alpha$, we get a set of indices $i_1 < \cdots < i_t$ from $[w]$ such that if $G^* = G_{S_{i_1}^*, \alpha} \cap \cdots \cap G_{S_{i_t}^*, \alpha}$ then $|G^*| \geq \varepsilon \binom{w}{t}^{-1} |D_\alpha|/2$. Let x be an arbitrary element of D_α, and $\Gamma_i(x) = \{\beta \in S_i^* \ : \ x \in D_\beta\}$. Clearly, for any $x \in G_{S_i^*, \alpha}$, we have $|\Gamma_i(x)| \geq \varepsilon p_1$. Let $E = \bigcup_{i=1}^w S_i^*$ and $I = \{i_1, \dots, i_t\}$. Obviously, $|E| \leq w p_1$. Consider the set

$$Z = \{ \ x \in D_\alpha \ : \ x \text{ belongs to at least } \frac{\varepsilon p_1 t}{2} \textit{ different} \text{ sets } D_\beta,\ \beta \in E \ \} \,.$$

Let us prove that $G^* \subseteq Z$. Let $x \in G^*$. Then x belongs to the sets D_β, $\beta \in \bigcup_{i \in I} \Gamma_i(x)$. Let us estimate the cardinality of $\bigcup_{i \in I} \Gamma_i(x)$. We have

$$|\bigcup_{i \in I} \Gamma_i(x)| = |\bigcup_{i \in I \setminus \{i_t\}} \Gamma_i(x)| + |\Gamma_{i_t}(x)| - |(\bigcup_{i \in I \setminus \{i_t\}} \Gamma_i(x)) \bigcap \Gamma_{i_t}(x)| \geq$$

$$\geq |\bigcup_{i \in I \setminus \{i_t\}} \Gamma_i(x)| + \varepsilon p_1 - |(\bigcup_{i \in I \setminus \{i_t\}} S_i^*) \bigcap S_{i_t}^*| \geq |\bigcup_{i \in I \setminus \{i_t\}} \Gamma_i(x)| + \varepsilon p_1 - \zeta p_1 \geq \cdots \geq \frac{\varepsilon p_1 t}{2} \,.$$

Whence $G^* \subseteq Z$ and, consequently, $|Z| \geq |G^*| \geq \varepsilon \binom{w}{t}^{-1} |D_\alpha|/2$. Let $l = [\varepsilon p_1 t/4]$. Then

$$Z \subseteq \bigcup_{r_1, \dots, r_l \in E \text{ — different}} \left(D_{r_1} \bigcap \cdots \bigcap D_{r_l} \right) \,.$$

Thus there is a tuple of indices $r_1 < \cdots < r_l$ from E such that

$$|D_{r_1} \bigcap \cdots \bigcap D_{r_l}| \geq \binom{p_1 w}{l}^{-1} |Z| \geq \binom{p_1 w}{l}^{-1} \binom{w}{t}^{-1} \frac{\varepsilon^2 m_3}{2 s_2} \geq \frac{1}{2^8 p^2} \left(\frac{m}{(K^*)^9 |\Lambda|} \right)^\eta \,.$$

Put $\mathcal{L}_1 = \{\lambda_{r_1}, \dots, \lambda_{r_l}\}$ and $\mathcal{L}_1' = D_{r_1} \bigcap \dots \bigcap D_{r_l}$. Then $\mathcal{L}_1 \bigcap \mathcal{L}_1' = \emptyset$, $\mathcal{L}_1 + \mathcal{L}_1' \subseteq Q_3 \subseteq Q$ and

$$|\mathcal{L}_1| = l \ge \frac{\log(\frac{m_3}{s_2})}{32 \log(\frac{2^8}{\varepsilon^2})} \ge \frac{\log(\frac{m}{8(K^*)^9|\Lambda|})}{2^{10}\log(2^{20}K^*)} \,.$$

Now we can use an iterative procedure. If $|\mathcal{L}_1 + \mathcal{L}_1'| \ge |Q_3|/2$ then we finish our procedure. Otherwise consider the set $Q_3' = Q_3 \setminus (\mathcal{L}_1 + \mathcal{L}_1')$ and use our previous arguments. We find sets $\mathcal{L}_2 \subseteq \Lambda_1$, $\mathcal{L}_2' \subseteq \Lambda_2$ such that $\mathcal{L}_2 \bigcap \mathcal{L}_2' = \emptyset$, $\mathcal{L}_2 + \mathcal{L}_2' \subseteq Q_3' \subseteq Q$ and such that

$$|\mathcal{L}_2| \ge \frac{\log(\frac{|Q|}{16(K^*)^9|\Lambda|})}{2^8 \log(2^{12}K^*)}, \quad |\mathcal{L}_2'| \ge \frac{1}{2^{10}p^2}\left(\frac{|Q|}{(K^*)^9|\Lambda|}\right)^{\eta} \,.$$

By dissociativity of the sets Λ and $\Lambda_1 \bigcap \Lambda_2 = \emptyset$, we get $(\mathcal{L}_1 + \mathcal{L}_1') \bigcap (\mathcal{L}_2 + \mathcal{L}_2') = \emptyset$. If $|\mathcal{L}_1 + \mathcal{L}_1'| + |\mathcal{L}_2 + \mathcal{L}_2'| \ge |Q_3|/2$ then we finish our algorithm. At the end we construct sets $\mathcal{L}_1, \mathcal{L}_1', \dots, \mathcal{L}_h, \mathcal{L}_h'$ such that inequality (57) holds.

We need to consider the case when (58) holds but either estimete $p \le \log|\Lambda| / \log\log|\Lambda|$ is not true or $m < \mathcal{M}$. If inequality (58) holds then $X = 1$. Using (65) and a simple bound $|D_\alpha| \le s_2$, we obtain that the number of sets in the family $\mathcal{S}$ is at least $m_3^{p_1}/(2K_1^{p_1}p_1^{p_1}s_2^{p_1})$. Applying condition (58), we see that the last quantity is at least 1. Hence there is a set S and a number $\alpha \in [s_1]$ such that $|G_{S,\alpha}| \ge \varepsilon |D_\alpha|$ and $|D_\alpha| \ge \varepsilon m_3/s_2$. Let

$$Z = \{\ x \in D_\alpha \ :\ x \text{ belongs to at least } \varepsilon p_1 \textit{ different} \text{ sets } D_\beta,\ \beta \in S\ \} \,.$$

Then $G_{S,\alpha} \subseteq Z$. We have $[\varepsilon p_1/2] \ge 2^{-18}\frac{p}{K^*} \ge 1$. Put $l = \min\{[\varepsilon p_1/2], [\log(m_3/s_2)/8]\}$. We have

$$Z \subseteq \bigcup_{r_1,\dots,r_l \in S \text{ — different}} \left(D_{r_1} \bigcap \dots \bigcap D_{r_l} \right) \,.$$

Whence there is a tuple of indices $r_1 < \dots < r_l$ from S such that

$$|D_{r_1} \bigcap \dots \bigcap D_{r_l}| \ge \binom{\lceil \varepsilon p_1 \rceil}{l}^{-1} |G_{S,\alpha}| \ge \frac{\varepsilon^2}{16^l}\frac{m_3}{s_2} \ge \frac{1}{16p^2}\left(\frac{m}{(K^*)^9|\Lambda|}\right)^{1/2} \,.$$

Put $\mathcal{L}_1 = \{\lambda_{r_1}, \dots, \lambda_{r_l}\}$ and $\mathcal{L}_1' = D_{r_1} \bigcap \dots \bigcap D_{r_l}$. Using the arguments as above, we get the required result. This concludes the proof.

Note 4.11 It is easy to see that the bound for the cardinalities of $\mathcal{L}_i$ from inequality (56) is best possible. We give a scheme of the proof of the last statement. Let us preserve all notations of Theorem 4.9. Let $K > 1$ be a fixed constant, $\Lambda_1, \Lambda_2 \subseteq \Lambda$, $\Lambda_1 \bigcap \Lambda_2 = \emptyset$. Let $Q \subseteq \Lambda_1 + \Lambda_2$ be a set which we will describe later, and $m := |Q|$. Let also $|\Lambda_1| := s$, $|\Lambda_2| = [mK/s]$. Suppose that sets $D_\alpha \subseteq \Lambda_2$, $\alpha = 1, \dots, s$ are random sets. It means that for any $\alpha \in [s]$ an arbitrary element from Λ_2, belongs to set D_α with probability $1/K$. Clearly, with positive probability, we have $|D_\alpha| \approx m/s$, $\alpha = 1, \dots, s$ $|D_\alpha \bigcap D_\beta| \approx m/(sK)$, $\alpha \ne \beta$, $\alpha, \beta = 1, \dots, s$, and

$$T_p(Q) \gg p^{2p} \sum_{S \subseteq [s],\, |S| = p} \prod_{\alpha \in S} \left(\sum_{\beta \in S} |D_\alpha \bigcap D_\beta| \right) \gg \frac{p^{2p}m^p}{K^p} \,.$$

Thus inequality (55) holds. Nevertheless if $\mathcal{L}_1 \subseteq \Lambda_1$, $\mathcal{L}_2 \subseteq \Lambda_2$, $|\mathcal{L}_2| = l > 0$, $\mathcal{L}_1 + \mathcal{L}_2 \subseteq Q$ then $|\mathcal{L}_2| \le |D_{\alpha_1} \bigcap \dots \bigcap D_{\alpha_l}| \ll m/(sK^l)$ and we get a bound $l \ll \log(m/s)/\log K$.

Theorem 4.9 has a simple corollary.

Proposition 4.12 *Let $K, \eta > 0$ be real numbers, $\eta \in (0, 1/2]$, $K \ge 1$, p, d be positive integers, $d \ge 3$, and $\Lambda \subseteq \mathbf{F}_2^n$ be an arbitrary set, $\Lambda \in \mathbf{\Lambda}(2dp)$. Let also Q be a subset of $d\dot{\Lambda}$, $|\Lambda| \ge 8d^2$, $p \ge 2^{50+8d} K^d / \eta$ and*

$$T_p(Q) \ge \frac{p^{dp} |Q|^p}{K^{(d-1)p}} \,. \tag{70}$$

Suppose that $p \le \log |\Lambda| / \log\log |\Lambda|$ and

$$|Q| \ge 2^{60+50d+\frac{2}{\eta}} M^{17} K^{2d} p^3 d^{-d} |\Lambda|^{d-1} \times$$

$$\times \max \left\{ (2^{30} M^{11} p)^{\eta p} |\Lambda|^{\eta} \log |\Lambda|, \exp \left(\frac{\log(2^{30} M^{20}) \log(\frac{p \log M}{\log p})}{\log(\frac{2^{-25} \eta p}{M})} \right) \right\} ,$$

where $M = 2^{13} (8K)^{d-1}$. Then there are sets $\mathcal{L}, \mathcal{L}' \subseteq \Lambda$ and elements $\lambda_1 + \dots + \lambda_{d-2}$ from Λ such that $\mathcal{L}_i \bigcap \mathcal{L}'_j = \emptyset$, $\lambda_i \notin \mathcal{L}, \mathcal{L}'$,

$$|\mathcal{L}| \ge \frac{\log(\frac{|Q| d^d}{2^{140+80d} K^{3d} |\Lambda|^{d-1}})}{2^{10} \log(2^{40} 8^d K^d)}, \quad |\mathcal{L}'| \ge \frac{1}{2^{10} p^2} \left(\frac{|Q| d^d}{2^{140+80d} K^{3d} |\Lambda|^{d-1}} \right)^{\eta} , \tag{71}$$

and

$$\lambda_1 + \dots + \lambda_{d-2} + \mathcal{L} + \mathcal{L}' \subseteq Q \,. \tag{72}$$

Note 4.13 The proposition above is a very simple inverse theorem for subsets of sums of more than two dissociated sets. In Theorem 4.9 we proved that our set Q contains a sum of two dissociated sets whereas in Proposition 4.12 we have just inclusion $\lambda_1 + \dots + \lambda_{d-2} + \mathcal{L} + \mathcal{L}' \subseteq Q$. One could expect that, actually, condition (70) implies that there are some sets $\mathcal{L}_1, \dots, \mathcal{L}_d \subseteq \Lambda$ such that $\mathcal{L}_1 + \dots + \mathcal{L}_d \subseteq Q$. The author is going to prove such an analog of Theorem 4.9 for subsets of $d\dot{\Lambda}$, Λ is a dissociated set, $d \ge 3$ in forthcoming papers.

Proof. Let $m = |Q|$, $\beta_1 = 4^{-d}$, $\beta_2 = 4^{-d} + 1/\sqrt{m}$, and $a = \lceil |\Lambda|/d \rceil$. Since $T_p(Q) \ge p^{dp} |Q|^p / K^{(d-1)p}$, it follows that $D_p(Q) \ge (d-1) p \log(p/K)$. Using Theorem 3.3 with parameters d and $C = 2^{-6}$, we get (β_1, β_2)–connected set $Q_1 \subseteq Q$ of degree p such that $m_1 := |Q_1| \ge m/(dK^{2(d-1)})$ and $T_p(Q_1) \ge p^{dp} m_1^p / K^{(d-1)p}$. Let also $a_i = a$, $i = 1, \dots, d-1$ and $a_d = |\Lambda| - \sum_{i=1}^{d-2} a_i$. Since $|\Lambda| \ge 8d^2$, it follows that $|\Lambda|/(2d) \le a_d \le |\Lambda|/d$. It is easy to see that

$$Q(x) = \left(\frac{d! (|\Lambda| - d)!}{(a_1 - 1)! \dots (a_d - 1)!} \right)^{-1} \sum_{S_1, \dots, S_d,\, |S_i| = a_i,\, \bigsqcup_{i=1}^d S_i = \Lambda} \left(Q \bigcap (S_1 + \dots + S_d) \right) (x) \,. \tag{73}$$

Using formula (73), we obtain that there is a tuple of disjoint sets $S_1, \dots, S_d \subseteq \Lambda$ such that

$$|Q_1 \bigcap (S_1 + \dots + S_d)| \ge m_1 d! \frac{(|\Lambda| - d)!}{(a_1 - 1)! \dots (a_d - 1)!} \left(\frac{|\Lambda|!}{a_1! \dots a_d!} \right)^{-1} =$$

$$= m_1 d! \frac{a_1 \dots a_d}{|\Lambda| (|\Lambda| - 1) \dots (|\Lambda| - d + 1)} \ge \frac{1}{2} e^{-d} m_1 \,. \tag{74}$$

Put $Q_2 = Q_1 \bigcap (S_1 + \dots + S_d)$.

Let d_1 be a positive integer, $d_1 \le d$, $l_1, \dots, l_{d_1}$ be different numbers from $[d]$, $L = \{l_1, \dots, l_{d_1}\}$, $\overline{L} = [d] \setminus L$. Let also $w_{l_i} \in S_{l_i}$ be arbitrary elements, $i \in [d_1]$, $\vec{w} = (w_{l_1}, \dots, w_{l_{d_1}})$ be a vector, and $W = \{w_{l_1}, \dots, w_{l_{d_1}}\}$. Define the sets $D(W)$, $Q(W)$

$$D(W) = \{\sum_{i \in \overline{L}} \lambda_i \ : \ \sum_{i \in \overline{L}} \lambda_i + \sum_{i \in L} w_i \in Q'\}, \quad Q(W) = \{q \in Q' \ : \ q = \sum_{i \in \overline{L}} \lambda_i + \sum_{i \in L} w_i\}$$

Clearly, $D(W) = Q(W) + \sum_{i \in L} w_i$. We shall write $D(\vec{w})$, $Q(\vec{w})$ instead of $D(W)$, $Q(W)$. By assumption the set Λ belongs to the family $\mathbf{\Lambda}(2dp)$. Using this, it is easy to see that the sets $Q(W \bigcup \{l_1\})$ and $Q(W \bigcup \{l_2\})$, $\lambda_1 \ne \lambda_2$ are disjoint. Besides, $Q(W \bigcup \{l\}) \subseteq Q(W)$. Whence, for all $x \ge 1$, we have

$$\sum_{\lambda} |Q(W \bigcup \{\lambda\})|^x \le |Q(W)|^{x-1} \sum_{\lambda} |Q(W \bigcup \{\lambda\})| = |Q(W)|^x \,. \tag{75}$$

Let $x_1, x_2 \ge 1$ be arbitrary numbers. Using bound (75) and Cauchy–Schwartz, we get

$$\sum_{\lambda} |Q(W_1 \bigcup \{\lambda\})|^{x_1/2} |Q(W_2 \bigcup \{\lambda\})|^{x_2/2} \le |Q(W_1)|^{x_1/2} |Q(W_1)|^{x_2/2} \,. \tag{76}$$

Clearly, there is an analog of formula (76) for larger number of sets $Q(W_i \bigcup \{\lambda\})$.

By Q_2' denote the union of the sets $Q_2(\vec{a})$, $\vec{a} \in S_1 \times \ldots \times S_{d-2}$ such that $|Q_2(\vec{a})| \ge |Q_2|/(4|S_1| \ldots |S_{d-2}|)$. Then $|Q_2'| \ge |Q_2|/2$. Certainly, we can find a set $Q' \subseteq Q_2'$ such that $|Q'| = \lceil 4^{-d} m_1 \rceil$ and such that $Q' = \bigsqcup \tilde{Q}_2(\vec{a})$, $\tilde{Q}_2(\vec{a}) \ge |Q_2|/(16|S_1| \ldots |S_{d-2}|)$. Let $m' = |Q'|$. Since the set Q_1 is (β_1, β_2)–connected of degree p and $C = 2^{-6}$, it follows that

$$T_p(Q') \ge 2^{-12p} \left(\frac{m'}{m_1} \right)^{2p} T_p(Q_1) \ge \frac{p^{dp} m'^p}{2^{12p} 4^{dp} K^{(d-1)p}} \,. \tag{77}$$

Consider the equation

$$q_1 + \dots + q_{2p} = 0 \,, \tag{78}$$

where $q_i \in Q'$, $i = 1, \dots, 2p$. By σ' denote the number of solutions of (78). Let $\vec{a}_1, \dots, \vec{a}_{d-2}$ be arbitrary vectors from $S_1 \times \ldots \times S_{d-2}$, and let $\vec{v} = (\vec{a}_1, \dots, \vec{a}_{2p})$. Denote by $\sigma(\vec{v}) = \sigma(\vec{a}_1, \dots, \vec{a}_{2p})$ the set of solutions of equation (78) such that $q_i \in Q(\vec{a}_i)$, $i \in [2p]$. Further by $\mathcal{M}$ denote the family of partitions of the segment $[2p]$ onto p sets $\{C_1, \dots, C_p\}$, $|C_j| = 2$, $j \in [p]$. Let also $\mathcal{V}$ be the collection of all partitions $\{\mathcal{M}_1, \dots, \mathcal{M}_{d-2}\}$, $\mathcal{M}_i \in \mathcal{M}$, $i \in [d-2]$. Clearly, the total number of different tuples $\{C_1, \dots, C_p\}$ in $\mathcal{V}$ does not exceed $p^{p(d-2)}$. By definition a vector $\vec{v} = (\vec{a}_1, \dots, \vec{a}_{2p})$ belongs to $\mathcal{V}$ if for any $j = 1, \dots, d-2$ and for all set C of partition $\mathcal{M}_j$, $C = \{\alpha, \beta\}$, we have $\lambda_\alpha = \lambda_\beta$. Obviously

$$\sigma' \le \sum_{\mathcal{V}} \sum_{(\vec{a}_1, \dots, \vec{a}_{d-2}) \in \mathcal{V}} |\sigma((\vec{a}_1, \dots, \vec{a}_{d-2}))| = \sum_{\mathcal{V}} \sum_{\vec{v} \in \mathcal{V}} |\sigma(\vec{v})| \,. \tag{79}$$

Using Lemma 2.2, we get

$$|\sigma(\vec{v})| \le \left(\prod_{i=1}^{2p} T_p(\vec{a}_i) \right)^{1/2p} \,. \tag{80}$$

Suppose that for all vectors $\vec{a}$ from $S_1 \times \ldots \times S_{d-2}$, we have

$$T_p(\vec{a}) \le \frac{p^{2p} |Q(\vec{a})|^p}{M^p} \,, \tag{81}$$

where $M = 2^{13}(8K)^{(d-1)}$. By the last inequality and (79), (80), we obtain

$$\sigma' \le \frac{p^{2p}}{M^p} \sum_{\mathcal{V}} \sum_{(\vec{a}_1,\dots,\vec{a}_{d-2}) \in \mathcal{V}} \prod_{i=1}^{2p} |Q(\vec{a}_i)|^{1/2} \,. \tag{82}$$

Using formulas (75), (76) several times, we get

$$\sigma' \le \frac{p^{2p}}{M^p} \sum_{\mathcal{V}} m^p \le \frac{p^{dp} m^p}{M^p} \,.$$

We obtain a contradiction with (77). Hence there is vector $\vec{a}$ from $S_1 \times \ldots \times S_{d-2}$ such that inequality (81) does not hold and

$$|Q(\vec{a})| \ge \frac{|Q_2|}{4|S_1| \ldots |S_{d-2}|} \ge \frac{m}{64de^d(4K)^{2(d-1)}|S_1| \ldots |S_{d-2}|} \ge \frac{md^d}{2^{50d}K^{2d}|\Lambda|^{d-2}} \,.$$

Let $\vec{a} = (a_1, \dots, a_{d-2})$. Put $\lambda_i = a_i$, $i \in [d-2]$. Applying Theorem 4.9 to the set $Q(\vec{a}) \subseteq S_{d-1} + S_d$, we get sets $\mathcal{L}$, $\mathcal{L}'$ such that inequalitiy (71) holds and inclusion (72) is true. This completes the proof.

5. Appendix.

In the section we prove an analog of Theorem 1.3 for an arbitrary Abelian group G.

Theorem 5.1 *Let δ, α be real numbers, $0 < \alpha \le \delta$, A be a subset of G, $|A| = \delta|G|$, $k \ge 2$ be a positive integer, and the set $\mathcal{R}_\alpha$ be as in (3). Suppose that $B \subseteq \mathcal{R}_\alpha$ be an arbitrary set. Then*

$$T_k(B) \ge \frac{\delta\alpha^{2k}}{\delta^{2k}} |B|^{2k} \,.$$

Proof. Let $r \in \widehat{G}$. Define the quantity $\theta(r) \in \mathbf{S}^1$ by the formula $\widehat{A}(r) = |\widehat{A}(r)|\theta(r)$. We have

$$\alpha N|B| \le \sum_{r \in B} |\widehat{A}(r)| = \sum_x \sum_r B(r)\theta^{-1}(r)e(-r \cdot x) \,.$$

Using Hölder's inequality, we obtain

$$(\alpha N|B|)^{2k} \le \sum_x \left| \sum_r B(r)\theta^{-1}(r)e(-r \cdot x) \right|^{2k} \cdot \left(\sum_x A(x) \right)^{2k-1} = NT_k(B \cdot \theta^{-1})(\delta N)^{2k-1} \,. \tag{83}$$

Let us prove a simple lemma.

Lemma 5.2 *Let $f : G \to \mathbb{C}$ be an arbitrary function, and $k \ge 2$ be a positive integer. Then $T_k(f) \le T_k(|f|)$.*

Proof of the lemma. Using the triangle inequality, we get for any functions $g, h : G \to \mathbb{C}$ the following holds $|(g * h)(x)| \le (|g| * |h|)(x)$, $x \in G$. By definition of $T_k(f)$ we obtain the required result.

Using the last lemma and (83), we get $T_k(B) \ge T_k(B \cdot \theta^{-1}) \ge \frac{\delta\alpha^{2k}}{\delta^{2k}}|B|^{2k}$. This concludes the proof.

References

[1] *Gowers W. T.* Rough structure and classification // Geom. Funct. Anal., Special Volume - GAFA2000 "Visions in Mathematics", Tel Aviv, (1999) Part I, 79–117.

[2] *Nathanson M.* Additive number theory. Inverse problems and the geometry of sumsets / Graduate Texts in Mathematics 165, Springer–Verlag, New York, 1996.

[3] *Freiman G. A.* Foundations of a Structural Theory of Set Addition / Kazanskii Gos. Ped. Inst., Kazan, 1966. Translations of Mathimatical Monographs **37**, AMS, Providence, R.I., USA.

[4] *Bilu Y.* Structure of sets with small sumset // Structure Theory of Sets Addition, Astérisque, Soc. Math. France, Montrouge **258** (1999), 77–108.

[5] *Ruzsa I.* Generalized arithmetic progressions and sumsets // Acta Math. Hungar. **65** (1994), 379–388.

[6] *Chang M.– C.,* A polynomial bound in Freiman's theorem // Duke Math. J. **113** (2002) no. 3, 399–419.

[7] *Green B.* Arithmetic Progressions in Sumsets // Geom. Funct. Anal., **12** (2002) no. 3, 584–597.

[8] *György E., Ruzsa I.* The structure of sets with few sums along a graph // http://www.cs.elte.hu/ elekes/Abstracts/alag.ps, submitted for publication.

[9] *Green B., Ruzsa I.* An analoge of Freiman's theorem in an arbitrary abelian group // J. London Math. Soc., submitted for publication.

[10] *Green B.* Structure Theory of Set Addition // ICMS Instructional Conference in Combinatorial Aspects of Mathematical Analysis, Edinburgh March 25 April 5 2002.

[11] *Green B.* Finite field model in additive combinatorics // Surveys in Combinatorics 2005, LMS Lecture Notes **329**, 1–29.

[12] *Sanders T.* An application of a local version of a Chang's theorem // http://www.arXiv:math.CA/0607668.

[13] *Sanders T.* Three terms arithmetic progressions in sumsets // http://www.arXiv:math.NT/0611304.

[14] *Sanders T.* The Littlewood–Gowers problem // http://www.arXiv:math.CA/0605522.

[15] *Sanders T.* Notes on Bourgain's refinement of Chang's quantitative version of Ruzsa's proof of Freiman's theorem // Preprint, 2007.

[16] *Sanders T.* Notes on a preprint of Shkredov's // Prepint, 2007.

[17] *Yudin A. A.* On the measure of large values of a trigonametric sum // Number Theory (under the edition of G.A. Freiman, A.M. Rubinov, E.V. Novosyolov), Kalinin State Univ., Moscow (1973), 163–174.

[18] *Besser A.* Sets of integers with large trigonometric sums // Astérisque **258** (1999), 35–76.

[19] *Lev V. F.* Linear Equations over $\mathbb{F}_p$ and Moments of Exponential Sums // Duke Mathematical Journal **107** (2001), 239–263.

[20] *Konyagin S. V., Lev V. F.* On the distribution of exponential sums // Integers: Electronic Journal of Combinatorial Number Theory **0** # A01, (2000).

[21] *Rudin W.* Fourier analysis on groups / Wiley 1990 (reprint of the 1962 original).

[22] *Rudin W.* Trigonometric series with gaps // J. Math. Mech. **9** (1960), 203–227.

[23] *Schoen T.* Linear equations in $\mathbb{Z}_p$ // LMS, submitted for publication.

[24] *Tao T., Vu V.* Additive combinatorics / Cambridge University Press 2006.

[25] *Mink H.* Permanents / Moscow: "Nauka", 1981.

[26] *Bourgain J.* Roth's Theorem on Progressions Revisited // Preprint, 2007.

[27] *Vaughan R.* The Hardy–Littlewood method / Moscow: "Mir", 1985.

[28] *Shkredov I. D.* On sets of large exponential sums // Doklady of Russian Academy of Sciences, 411, N 4, 455–459, 2006.

[29] *Shkredov I. D.* On sets of large exponential sums // Izvestiya of Russian Academy of Sciences, 72, N 1, 2008.

[30] *Shkredov I. D.* Some examples of sets of large exponential sums // Matematicheskii Sbornik, 198, N 12, 105–140, 2007.

[31] *Shkredov I. D.* On sets with small doubling // Mathematical Notes, v. 73, N 4, 577–596, 2008.