Hankel Operators plus Orthogonal Polynomials

Yield Combinatorial Identities

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Abstract: A Hankel operator $H = [h_{i+j}]$ can be factored as $H = MM^*$, where M maps a space of L^2 functions to the corresponding moment sequences. Furthermore, a necessary and sufficient condition for a sequence to be in the range of M can be expressed in terms of an expansion in orthogonal polynomials. Combining these two results yields a wealth of combinatorial identities that incorporate both the matrix entries h_{i+j} and the coefficients of the orthogonal polynomials.

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1. Introduction

The context throughout this paper will be the Hilbert spaces l^2 of square-summable sequences and $L^2_\alpha(-1,1)$ of square-integrable functions on $[-1,1]$ with the distribution $d\alpha$, α nondecreasing. Their inner products are the usual dot product for the former and the usual integral inner product for the latter.

The Hankel operators in Section 2 are exactly as in [9]. That is, we study the infinite matrices $H = [h_{i+j}]_{i,j\geq 0}$, where

$$h_j = \int_{-1}^{1} x^j \, d\alpha(x), \qquad j = 0, 1, 2, \ldots$$

The following result in [9] will be essential:

Widom's Theorem: The following are equivalent:

(a) H represents a bounded operator on l^2.

(b) $h_j = O(1/j)$ as $j \to \infty$.

(c) $\alpha(1) - \alpha(x) = O(1-x)$ as $x \to 1^-$, $\quad \alpha(x) - \alpha(-1) = O(1+x)$ as $x \to -1^+$.

In Theorem 1 we show that $H = MM^*$, where $M : L^2_\alpha(-1,1) \to l^2$ is the bounded operator that maps each function $f \in L^2_\alpha(-1,1)$ to its sequence of moments:

$$M(f)_k = \int_{-1}^{1} x^k f(x)\, d\alpha(x), \qquad k = 0, 1, 2, \ldots$$

We also show that, for $\mathbf{u} = \langle u_k \rangle \in l^2$, we have the formula

$$M^*\mathbf{u}(x) = \sum_{k=0}^{\infty} u_k x^k$$

In Section 3, we explore the space $\mathcal{M}_\alpha$ of Hausdorff moment sequences given by the range of the operator M. Extending an idea in [2], we derive in Theorem 2 the following necessary and sufficient condition for a sequence to belong to $\mathcal{M}_\alpha$. If $\mathbf{u} = \langle u_j \rangle \in l^2$ and $\langle p_k \rangle$ is an orthogonal sequence of polynomials in $L^2_\alpha(-1,1)$ with

$$p_k(x) = \sum_{j=0}^{k} a_{kj} x^j, \qquad a_{kk} \neq 0, \quad k = 0, 1, 2, \ldots \tag{1}$$

define the sequence $c = \langle c_k \rangle$ by

$$c_k = \sum_{j=0}^{k} \overline{a_{kj}} u_j, \qquad k = 0, 1, 2, \ldots \tag{2}$$

Then $\mathbf{u} = \langle u_j \rangle \in \mathcal{M}_\alpha$ if and only if $\sum_{k=0}^{\infty} |c_k|^2 / \|p_k\|^2 < \infty$.

Finally, in Theorem 3 we use Theorems 1 and 2 to produce combinatorial identities as follows. Suppose $\mathbf{v} = \langle v_k \rangle \in l^2$ and $\mathbf{u} = H\mathbf{v}$. If $\langle p_k \rangle$ and $\langle c_k \rangle$ are the sequences described above in (1) and (2), we obtain one type of identity by equating two ways of expressing the c_k's. On the one hand,

$$c_k = \sum_{j=0}^{k} \overline{a_{kj}} (Hv)_j = \sum_{j=0}^{k} \overline{a_{kj}} \sum_{i=0}^{\infty} h_{j+i} v_i$$

On the other hand, the c_k's are Fourier coefficients with respect to the orthogonal polynomials $\langle p_k \rangle$ for some function $f \in L^2_\alpha(-1,1)$. Since $H\mathbf{v} = MM^*\mathbf{v} = Mf$, where $f(x) = M^*\mathbf{v}(x) = \sum_k v_k x^k$, we obtain a second formula for c_k in terms of the components v_k of $\mathbf{v}$. Other identities come from Parseval's equation.

In Section 4, we explore several examples.

2. A factorization of Hankel Operators

The Hankel operators we investigate are given by the infinite matrices $H = [h_{i+j}]_{i,j\geq 0}$, where

$$h_j = \int_{-1}^{1} x^j \, d\alpha(x), \qquad j = 0, 1, 2, \ldots$$

and α is nondecreasing on $[-1, 1]$. We assume H represents a bounded operator on l^2, which means we have the equivalent conditions on $\langle h_j \rangle$ and α in Widom's theorem.

Theorem 1: For all $f \in L^2_\alpha(-1, 1)$, let

$$M(f) = \mathbf{u}^f = \langle u^f_k \rangle, \quad \text{where } u^f_k = \int_{-1}^{1} x^k f(x) \, d\alpha(x), \quad k = 0, 1, 2, \ldots \tag{3}$$

For all $\mathbf{v} = \langle v_k \rangle \in l^2$, let

$$N(\mathbf{v}) = f_{\mathbf{v}}, \quad \text{where } f_{\mathbf{v}}(x) = \sum_{k=0}^{\infty} v_k x^k, \quad -1 < x < 1 \tag{4}$$

(a) N is a bounded linear operator from l^2 into $L^2_\alpha(-1, 1)$.

(b) M is a bounded linear operator from $L^2_\alpha(-1, 1)$ into l^2.

(c) $M^* = N$.

(d) $H = MM^*$.

(e) M is injective.

Proof: (a) We use the Cauchy criterion to show that $f_{\mathbf{v}} \in L^2_\alpha(-1, 1)$:

$$\begin{aligned} \left\| \sum_{k=m}^{n} v_k x^k \right\|^2 &= \int_{-1}^{1} \left(\sum_{k=m}^{n} v_k x^k \right) \left(\sum_{j=m}^{n} \overline{v_j} x^j \right) d\alpha(x) = \sum_{k=m}^{n} \sum_{j=m}^{n} v_k \overline{v_j} \int_{-1}^{1} x^{k+j} \, d\alpha(x) \\ &= \sum_{k=m}^{n} \sum_{j=m}^{n} v_k \overline{v_j} h_{j+k} \leq \sum_{k=m}^{n} \sum_{j=m}^{n} |v_k||v_j| \frac{p}{k+j} \end{aligned}$$

for some $p > 0$, where the inequality uses the fact that $h_j = O(1/j)$. Hence, by Hilbert's inequality, $\sum_k v_k x^k$ converges in $L^2_\alpha(-1, 1)$.

To show that N is bounded, we use the change of variables

$$\int_{-1}^{0} g(x)\, d\alpha(x) = \int_{0}^{1} g(-y)\, d\beta(y), \quad \text{where } g \text{ is integrable and } \beta(y) = -\alpha(-y) \tag{5}$$

Thus

$$\|f_{\mathbf{v}}\|^2 = \int_0^1 |f_{\mathbf{v}}(x)|^2\, d\alpha(x) + \int_0^1 |f_{\mathbf{v}}(-y)|^2\, d\beta(y)$$

and the matrices A and B defined below are bounded Hankel operators on l^2 by Widom's Theorem.

$$A = [a_{j+k}],\ B = [b_{j+k}], \quad \text{where } a_j = \int_0^1 x^j\, d\alpha(x),\ b_j = \int_0^1 y^j\, d\beta(y)$$

Therefore

$$\int_0^1 |f_{\mathbf{v}}(x)|^2\, d\alpha(x) = \int_0^1 \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} v_k\overline{v_j}x^{k+j}\, d\alpha(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} v_k\overline{v_j}\, a_{k+j} = A\mathbf{v}\cdot\mathbf{v} \le \|A\|\|\mathbf{v}\|^2$$

and

$$\begin{aligned}\int_0^1 |f_{\mathbf{v}}(-y)|^2\, d\beta(y) &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} v_k\overline{v_j}\int_0^1 (-y)^{k+j}\, d\beta(y)\\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^k v_k\overline{(-1)^j v_j}\, b_{k+j} \le \|B\|\|\mathbf{v}\|^2\end{aligned}$$

The fact that $\sum_k v_k x^k$ converges in $L^2_\alpha(-1,1)$ justifies the above interchanges of limits. Therefore $\|N(\mathbf{v})\|^2 \le (\|A\| + \|B\|)\|\mathbf{v}\|^2$.

(b) Again using the change of variables (5), we have, for any $f \in L^2_\alpha(-1,1)$,

$$u^f_k = \int_{-1}^{1} x^k f(x)\, d\alpha(x) = \int_0^1 x^k f(x)\, d\alpha(x) + \int_0^1 (-1)^k y^k f(-y) d\beta(y)$$

Let a_k denote the first of the two integrals on the right and b_k denote the second. The following argument is an adaptation of the proof in [5, Theorem 324]. Let $\mathbf{v} = \langle v_k\rangle \in l^2$. The interchange of limits below is again justified by the fact that $\sum_k v_k x^k$ converges in $L^2_\alpha(-1,1)$:

$$\sum_{k=0}^{\infty} v_k a_k = \int_0^1 \sum_{k=1}^{\infty} v_k x^k f(x)\, d\alpha(x) = \int_0^1 N\mathbf{v}(x) f(x)\, d\alpha(x)$$

Hence, by the Cauchy-Schwarz inequality and part (a),

$$\left|\sum_{k=0}^{\infty} v_k a_k\right|^2 \leq \|N(\mathbf{v})\|^2 \|f\|^2 \leq \|N\|^2 \|\mathbf{v}\|^2 \|f\|^2$$

Therefore, by the so-called converse of Hölder's inequality [5, Theorem 15], $\|\langle a_k \rangle\| \leq \|N\| \|f\|$. Similarly, $\|\langle b_k \rangle\| \leq \|N\| \|f\|$, and so $\|M(f)\| \leq 2\|N\| \, \|f\|$.

(c) and (d) are straightforward verifications.

(e) Suppose $M(g) = 0$. Then, for all $\mathbf{v} \in l^2$,

$$0 = M(g) \cdot \mathbf{v} = \langle g, M^* \mathbf{v} \rangle = \langle g, f_{\mathbf{v}} \rangle$$

Since the functions $f_{\mathbf{v}}$ include all polynomials, and since the polynomials are dense in $L^2_\alpha(-1, 1)$, it follows that $g = 0$.

$\square$

Although the factorization $H = MM^*$ appears to be new, it is closely related to the well-known fact [7, Chapter 2] that H is unitarily equivalent to the integral transform T on $L^2_\alpha(-1, 1)$ defined by

$$Tf(x) = \int_{-1}^{1} \frac{f(y)}{1 - xy} \, d\alpha(y)$$

Specifically, one can show that $T = M^* M$ and hence that H and T are unitarily equivalent whenever M^* is injective. For M^* to be injective, it suffices to assume that α has infinitely many points of increase with a cluster point in the open interval $(-1, 1)$.

3. Orthogonal polynomials and moment sequences

In this section we use a complete orthogonal sequence $\langle p_k \rangle$ of polynomials in $L^2_\alpha(-1, 1)$ to characterize the elements of $\mathcal{M}_\alpha$, where $\mathcal{M}_\alpha$ denotes the space of moment sequences given by the range of M (see [8]). Although these sequences may be finite or infinite, to simplify notation, the details below will be expressed as if the sequences are infinite. The conclusions are valid in all cases, and the finite case will be treated as one of the examples in Section 4.

The following theorem extends an idea in [2], where it was applied to the shifted Legendre polynomials.

Theorem 2: Let $\langle p_k \rangle$ be an orthogonal sequence of polynomials in $L^2_\alpha(-1,1)$ with

$$p_k(x) = \sum_{j=0}^{k} a_{kj} x^j, \quad a_{kk} \neq 0, \;\; k = 0, 1, 2, \ldots \tag{6}$$

Let $\mathbf{u} = \langle u_j \rangle \in l^2$ and define $c = \langle c_k \rangle$ by

$$c_k = \sum_{j=0}^{k} \overline{a_{kj}}\, u_j, \qquad k = 0, 1, 2, \ldots \tag{7}$$

Then $\mathbf{u} \in \mathcal{M}_\alpha$ if and only if

$$\sum_{k=0}^{\infty} \frac{|c_k|^2}{\|p_k\|^2} < \infty \tag{8}$$

Furthermore, when (8) holds, then $\mathbf{u} = M(f)$, where

$$f(x) = \sum_{k=0}^{\infty} \frac{c_k}{\|p_k\|^2}\, p_k(x) \tag{9}$$

Proof: If $\mathbf{u} = \langle u_k \rangle \in \mathcal{M}_\alpha$, then there exists $f \in L^2_\alpha(-1,1)$ such that $\mathbf{u} = M(f)$; that is,

$$u_k = \int_{-1}^{1} x^k f(x)\, d\alpha(x), \qquad k = 0, 1, 2, \ldots$$

Hence, since $\langle p_k \rangle$ is a complete orthogonal sequence in $L^2_\alpha(-1,1)$ (see [8]),

$$f(x) = \sum_{k=0}^{\infty} b_k p_k(x) \;\; \text{where } b_k = \frac{1}{\|p_k\|^2} \int_{-1}^{1} f(x)\overline{p_k(x)}\, d\alpha(x)$$

Therefore, by equations (7) and (6),

$$c_k = \sum_{j=0}^{k} \overline{a_{kj}} \int_{-1}^{1} x^j f(x)\, d\alpha(x) = \int_{-1}^{1} f(x)\overline{p_k(x)}\, d\alpha(x) = \|p_k\|^2 b_k$$

So, by Parseval's equation,

$$\sum_{k=0}^{\infty} \frac{|c_k|^2}{\|p_k\|^2} = \sum_{k=0}^{\infty} \|p_k\|^2 |b_k|^2 = \|f\|^2 < \infty$$

Furthermore, (9) holds since $b_k = c_k / \|p_k\|^2$.

On the other hand, if $\mathbf{u} = \langle u_k \rangle \in l^2$ and $\sum_k |c_k|^2/\|p_k\|^2 < \infty$, let

$$f(x) = \sum_{k=0}^{\infty} \frac{c_k}{\|p_k\|^2}\, p_k(x)$$

Since $\sum_k |c_k|^2/\|p_k\|^2 < \infty$, the above series converges in $L^2_\alpha(-1,1)$ and

$$c_k = \int_{-1}^{1} f(x)\overline{p_k(x)}\, d\alpha(x), \qquad k = 0,1,2,\ldots$$

This equation can also be written, using (6) and (7), as

$$\sum_{j=0}^{k} \overline{a_{kj}}\, u_j = \int_{-1}^{1} f(x) \sum_{j=0}^{k} \overline{a_{kj}}\, x^j\, d\alpha(x), \qquad k = 0,1,2,\ldots$$

which implies, since $a_{kk} \neq 0$,

$$u_j = \int_{-1}^{1} f(x) x^j\, d\alpha(x), \qquad j = 0,1,2,\ldots$$

Therefore $\mathbf{u} = \langle u_j \rangle = M(f) \in \mathcal{M}_\alpha$. □

Finally, we use Theorems 1 and 2 to produce combinatorial identities corresponding to each choice of α.

Theorem 3: Suppose α is a nondecreasing function on $[-1,1]$ satisfying

$$\alpha(1) - \alpha(x) = O(1-x) \text{ as } x \to 1^-, \quad \alpha(x) - \alpha(-1) = O(1+x) \text{ as } x \to -1^+$$

and let

$$h_j = \int_{-1}^{1} x^j\, d\alpha(x), \qquad j = 0,1,2,\ldots$$

Suppose also that $\langle p_k \rangle$ is an orthogonal sequence of polynomials in $L^2_\alpha(-1,1)$ with

$$p_k(x) = \sum_{j=0}^{k} a_{kj}\, x^j, \quad a_{kk} \neq 0, \quad k = 0,1,2,\ldots$$

and

$$x^k = \sum_{j=0}^{k} b_{kj}\, p_j(x), \qquad k = 0,1,2,\ldots$$

Then, for all non-negative integers k and m,

$$\sum_{j=0}^{k} \overline{a_{kj}}\, h_{j+m} = \begin{cases} \|p_k\|^2\, b_{mk} & \text{if } m \geq k \\ 0 & \text{if } m < k \end{cases} \tag{10}$$

and

$$\sum_{i=k}^{m} \frac{1}{\|p_i\|^2} \sum_{j=0}^{i} \overline{a_{ij}} h_{j+m} a_{ik} = \delta_{mk} \tag{11}$$

Furthermore, for all $\mathbf{v} = \langle v_k \rangle \in l^2$,

$$\sum_{k=0}^{\infty} \frac{1}{\|p_k\|^2} \left| \sum_{i=0}^{\infty} \sum_{j=0}^{k} \overline{a_{kj}}\, h_{j+i} v_i \right|^2 = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} v_k \overline{v_j}\, h_{k+j} \tag{12}$$

and

$$\sum_{k=0}^{\infty} \|p_k\|^2 \left| \sum_{j=k}^{\infty} v_j b_{jk} \right|^2 = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} v_k \overline{v_j}\, h_{k+j} \tag{13}$$

Proof: Let $H = [h_{i+j}]_{i,j \geq 0}$, $\mathbf{v} = \langle v_k \rangle \in l^2$, and $\mathbf{u} = H\mathbf{v} = MM^*\mathbf{v}$. Then $\mathbf{u} \in \mathcal{M}_\alpha$, and we use Theorem 2 to find two expressions for the coefficients c_k in equation (7). On the one hand,

$$c_k = \sum_{j=0}^{k} \overline{a_{kj}}\, u_j = \sum_{j=0}^{k} \overline{a_{kj}}\, (H\mathbf{v})_j = \sum_{j=0}^{k} \overline{a_{kj}} \sum_{i=0}^{\infty} h_{j+i} v_i \tag{14}$$

On the other hand,

$$c_k = \int_{-1}^{1} f(x) \overline{p_k(x)}\, d\alpha(x) \quad \text{where } f(x) = \sum_{k=0}^{\infty} \frac{c_k}{\|p_k\|^2} p_k(x)$$

Using Theorem 1, we may also write the last expression for c_k in terms of the components of $\mathbf{v}$. Note that $H\mathbf{v} = M(f)$ and $H\mathbf{v} = MM^*\mathbf{v} = M(f_\mathbf{v})$. Since M is injective, we have $f = f_\mathbf{v}$ *a. e.* and so

$$\begin{aligned} c_k &= \int_{-1}^{1} \sum_{j=0}^{\infty} v_j x^j\, \overline{p_k(x)}\, d\alpha(x) \\ &= \sum_{j=0}^{\infty} v_j \int_{-1}^{1} \sum_{i=0}^{j} b_{ji} p_i(x) \overline{p_k(x)}\, d\alpha(x) = \|p_k\|^2 \sum_{j=k}^{\infty} v_j b_{jk} \end{aligned} \tag{15}$$

Equating formulas (14) and (15) for c_k and letting $v_k = \delta_{km}$, we obtain the identity (10). Then, solving for b_{mk}, $(m \geq k)$ in (10) and substituting into $\sum_{k=i}^{m} b_{mk} a_{ki} = \delta_{mi}$ $(m \geq i)$, we obtain the identity (11).

Furthermore, by Parseval's equation and $MM^* = H$, we have

$$\sum_{k=0}^{\infty} \frac{|c_k|^2}{\|p_k\|^2} = \|f_{\mathbf{v}}\|^2 = \|M^*\mathbf{v}\|^2 = \langle M^*\mathbf{v}, M^*\mathbf{v}\rangle = H\mathbf{v} \cdot \mathbf{v} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} v_k \overline{v_j} h_{k+j}$$

Substituting formulas (14) and (15) for c_k in this result yields the identities (12) and (13). $\square$

Another perspective on Theorem 3

The analysis below shows that, with a different set of assumptions, the identities in Theorem 3 are quite natural. Thus, some instances of them are likely to be known.

Suppose, instead of a distribution $d\alpha$ as in Theorem 3, we are given the following assumptions about a matrix H and a sequence of polynomials $\langle p_k \rangle$. Let $H = [h_{i+j}]_{i,j\geq 0}$ be any infinite Hankel matrix with real entries; that is, do not assume $h_j = O(1/j)$ as $j \to \infty$. Also let

$$p_k(x) = \sum_{j=0}^{k} a_{kj} x^j, \quad a_{kk} \neq 0, \quad k = 0, 1, 2, \ldots$$

be a sequence of polynomials with real coefficients that are orthogonal in the sense that

$$\sum_{i=0}^{k}\sum_{j=0}^{m} a_{ki} h_{i+j} a_{mj} \begin{cases} = 0 & \text{if } k \neq m \\ \neq 0 & \text{if } k = m \end{cases} \tag{16}$$

Note that, under the assumptions of Theorem 3, property (16) is the statement that $\langle p_k, p_m \rangle = 0$ if $k \neq m$ and $\langle p_k, p_k \rangle \neq 0$. So we will denote the left side of (16) by $\langle p_k, p_m \rangle$ even though the matrix H does not necessarily generate a true inner product. Then (16) can be written as $AHA^* = D$, where A denotes the lower-triangular matrix whose (k, j) entry, for $k \geq j$, is a_{kj} and D denotes the diagonal matrix whose kth diagonal entry is $\langle p_k, p_k \rangle$. Since the diagonal entries of A are nonzero, A has a formal inverse B. Hence $AHA^* = D$ implies $AH = DB^*$, which is the identity (10). Of course, $\|p_k\|^2$ must be interpreted as $\langle p_k, p_k \rangle$, which is nonzero but possibly negative. Identity (11) follows from (10), as in the proof of Theorem 3. The left side of identity (12) can be expressed as $\|D^{-1/2}AH\mathbf{v}\|^2$, where $D^{-1/2}$ is diagonal and the norm is the usual l^2 norm. However, now we must assume that the sequence $\mathbf{v}$ has only finitely many nonzero terms. Then (12) may be derived as follows:

$$\|D^{-1/2}AH\mathbf{v}\|^2 = (D^{-1/2}AH\mathbf{v}) \cdot (D^{-1/2}AH\mathbf{v}) = (D^{-1}DB^*\mathbf{v}) \cdot (AH\mathbf{v}) = \mathbf{v} \cdot H\mathbf{v}$$

Identity (13) may be derived similarly.

4. Examples

In the examples below, some of the instances of identity (10) have been independently confirmed using the WZ methods in [6]. On the other hand, instances of identities (12) and (13) do not yet seem to be susceptible of confirmation by such algorithmic methods.

Example 1 (*Hilbert matrices and binomial coefficients*): For any $r > 0$, let $\alpha(x) = x^r$ if $0 \le x \le 1$ and $\alpha(x) = 0$ if $-1 \le x \le 0$. Then $h_j = r/(j+r)$ and so $H = [r/(i+j+r)]_{i,j\ge 0}$. In [3], Berg shows that the corresponding orthogonal polynomials may be written as

$$p_k(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{k+j+r-1}{j} x^j$$

It follows that $\|p_k\|^2 = r/(2k+r)$ and

$$x^k = \frac{1}{(k+r)\binom{2k+r}{k}} \sum_{j=0}^{k} (-1)^j \binom{2k+r}{k-j} (2j+r) p_j(x)$$

Hence equations (10), (11), (12) and (13) become, respectively,

$$\sum_{j=0}^{k} \binom{k}{j} \binom{k+j+r-1}{k} \frac{(-1)^j}{j+m+r} = \frac{(-1)^k \binom{2m+r}{m-k}}{\binom{2m+r}{m}(m+r)}$$

$$\sum_{i=k}^{m} \sum_{j=0}^{i} \binom{i}{j} \binom{i+j+r-1}{i} \binom{i}{k} \binom{i+k+r-1}{i} \frac{(-1)^{j+k}(2i+1)}{j+m+r} = \delta_{mk}$$

$$\sum_{k=0}^{\infty} (2k+r) \left| \sum_{i=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} \binom{k+j+r-1}{k} \frac{(-1)^j v_i}{j+i+r} \right|^2 = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{v_k \overline{v_j}}{k+j+r}$$

$$\sum_{k=0}^{\infty} (2k+r) \left| \sum_{j=k}^{\infty} \frac{\binom{2j+r}{j-k} v_j}{(j+r)\binom{2j+r}{j}} \right|^2 = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{v_k \overline{v_j}}{k+j+r}$$

where $\langle v_j \rangle \in l^2$.

Example 2 (*Legendre polynomials*): We begin by considering all Jacobi polynomials:

$$p_k^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_k}{k!} \, {}_2F_1(-k, \alpha+\beta+k+1; \alpha+1; (1-x)/2)$$

where ${}_2F_1$ denotes the Gauss hypergeometric series, $(\alpha+1)_k$ denotes a shifted factorial, and $\alpha, \beta > -1$. The Jacobi polynomials are orthogonal on $[-1,1]$ relative to the weight function $w(x) = (1-x)^\alpha (1+x)^\beta$, and

$$\|p_k^{(\alpha,\beta)}\|^2 = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)}{k!\,\Gamma(\alpha+\beta+k+1)(\alpha+\beta+2k+1)}$$

Using Euler's integral representation of Gauss' hypergeometric series [1, Theorem 2.2.1], we find the moments to be

$$\begin{aligned} h_j &= \int_{-1}^{1} x^j (1-x)^\alpha (1+x)^\beta \, dx \\ &= j! \left((-1)^j \, \frac{{}_2F_1(-\alpha, 1+j; 2+\beta+j; -1))}{(1+\beta)_{1+j}} + \frac{{}_2F_1(-\beta, 1+j; 2+\alpha+j; -1)}{(1+\alpha)_{1+j}} \right) \end{aligned}$$

Rather than writing out identities (10), (11), (12), and (13) for this general case, we write the more compact versions for the special case $\alpha = \beta = 0$. Then $h_j = 2/(j+1)$ if j is even, $h_j = 0$ if j is odd, and $p_k = p_k^{(0,0)}$ is the kth Legendre polynomial; hence $\|p_k\|^2 = 2/(2k+1)$ and

$$p_k(x) = \sum_{\substack{j=0 \\ j+k \text{ even}}}^{k} \frac{(-1)^{(k-j)/2}}{2^k} \binom{k}{\frac{k-j}{2}} \binom{k+j}{k} x^j$$

$$x^k = \sum_{\substack{j=0 \\ j+k \text{ even}}}^{k} \frac{k!(2j+1)}{2^{(k-j)/2}(\frac{1}{2}(k-j))!(j+k+1)!!} p_j(x)$$

Note: The double factorial $n!!$ is defined to be 1 when $n = 0$ and, for positive integers n,

$$n!! = n(n-2)(n-4)\cdots(1 \text{ or } 2)$$

where the terminal factor in this product is 1 when n is odd and 2 when n is even.

Hence equations (10), (11), (12), and (13) become, respectively,

$$\sum_{\substack{j=0 \\ j+k \text{ even} \\ j+m \text{ even}}}^{k} \binom{k}{\frac{k-j}{2}} \binom{k+j}{k} \frac{(-1)^{(k-j)/2}}{2^k(j+m+1)} = \begin{cases} \frac{m!}{2^{(m-k)/2}(\frac{1}{2}(m-k))!(k+m+1)!!} & \text{if } m+k \text{ even}, m \geq k \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{\substack{i=k \\ i+k \text{ even}}}^{m} \sum_{\substack{j=0 \\ i+j \text{ even} \\ j+m \text{ even}}}^{i} \frac{(-1)^{i-(j+k)/2}}{2^{2i}(j+m+1)} \binom{i}{\frac{i-j}{2}} \binom{i+j}{i} \binom{i}{\frac{i-k}{2}} \binom{i+k}{i} = \delta_{mk}$$

$$\sum_{k=0}^{\infty}(2k+1)\left|\sum_{i=0}^{\infty}\sum_{\substack{j=0\\ j+k \text{ even}\\ j+i \text{ even}}}^{\infty}\binom{k}{\frac{k-j}{2}}\binom{k+j}{k}\frac{(-1)^{k-j)/2}v_i}{2^k(j+i+1)}\right|^2=\sum_{k=0}^{\infty}\sum_{\substack{j=0\\ j+k \text{ even}}}^{\infty}\frac{v_i\overline{v_j}}{k+j+1}$$

$$\sum_{k=0}^{\infty}(2k+1)\left|\sum_{\substack{j=k\\ j+k \text{ even}}}^{\infty}\frac{j!v_j}{2^{(j-k)/2}(\frac{1}{2}(j-k))!(k+j+1)!!}\right|^2=\sum_{k=0}^{\infty}\sum_{\substack{j=0\\ j+k \text{ even}}}^{\infty}\frac{v_i\overline{v_j}}{k+j+1}$$

where $\langle v_j\rangle \in l^2$.

Example 3 (*Fibonacci numbers and Fibonomial coefficients*): We begin by considering all little q-Jacobi polynomials [4, Section 7.3]:

$$p_k(x;a,b;q)=\sum_{j=0}^{n}\frac{(q^{-k};q)_j(abq^{k+1};q)_j}{(q;q)_j(aq;q)_j}(xq)^j$$

where $(a;q)_j$ denotes the q-shifted factorial. We assume that $|q|<1$, $|aq|<1$, and $|b|\leq 1$, which suffices to show that these polynomials satisfy the orthogonality property

$$\sum_{i=0}^{\infty}p_k(q^i;a,b;q)p_m(q^i;a,b;q)\frac{(bq;q)_i}{(q;q)_i}(aq)^i=\frac{\delta_{km}}{h_m(a,b;q)} \tag{17}$$

where

$$h_m(a,b;q)=\frac{(abq;q)_m(1-abq^{2m+1})(aq;q)_m(aq;q)_\infty}{(q;q)_m(1-abq)(bq;q)_m(abq^2;q)_\infty}(aq)^{-m}$$

We can express the orthogonality property (17) in terms of an integral on $[-1,1]$ as follows. Let $p_k(x)=p_k(\phi x;a,b;q)$, where $\phi>1$, and define the discrete signed measure μ by

$$\mu=\sum_{i=0}^{\infty}\frac{(bq;q)_i}{(q;q)_i}(aq)^i\delta_{q^i/\phi} \tag{18}$$

Then (17) becomes

$$\int_{-1}^{1}p_k(x)p_m(x)\,d\mu(x)=\frac{\delta_{km}}{h_m(a,b;q)}$$

The corresponding moments are

$$h_j=\int_{-1}^{1}x^j\,d\mu(x)=\sum_{i=0}^{\infty}\frac{(bq;q)_i}{(q;q)_i}(aq)^i\left(q^i/\phi\right)^j$$

When $b = 1$, this sum is a geometric series:

$$h_j = \frac{1}{\phi^j} \sum_{i=0}^{\infty} \left(aq^{j+1}\right)^i = \frac{1}{\phi^j(1 - aq^{j+1})}$$

The following interesting special case was found by Berg [3]. For any $\alpha \in \mathbb{N}$, let

$$b = 1,\ \phi = \frac{1+\sqrt{5}}{2},\ q = \frac{1-\sqrt{5}}{1+\sqrt{5}},\ a = q^{\alpha-1},\ \text{and}$$

$$\mu_\alpha = (1 - q^\alpha) \sum_{i=0}^{\infty} q^{i\alpha} \delta_{q^i/\phi}, \quad \alpha = 1, 2, 3, \ldots$$

Note that μ_α is the signed measure (18) multiplied by $1 - q^\alpha$ so it becomes a measure with total mass 1. Berg shows that $h_j = F_\alpha / F_{\alpha+j},\ j = 0, 1, 2, \ldots$, that

$$p_k^{(\alpha)}(x) = p_k(\phi x; q^{\alpha-1}, 1; q) = \sum_{j=0}^{k} (-1)^{jk - \binom{j}{2}} \binom{k}{j}_{\mathbb{F}} \binom{\alpha+k+j-1}{k}_{\mathbb{F}} x^j$$

and

$$\int_{-1}^{1} p_k^{(\alpha)}(x) p_m^{(\alpha)}(x)\, d\mu_\alpha(x) = \delta_{km} (-1)^{k\alpha} \frac{F_\alpha}{F_{\alpha+2k}}$$

where F_j denotes the jth Fibonacci number (with $F_0 = 0, F_1 = 1$) and $\binom{k}{j}_{\mathbb{F}}$ denotes the Fibonomial coefficient $\prod_{i=1}^{j} \frac{F_{k-i+1}}{F_i}$.

When α is even, μ_α is a probability measure that, together with the polynomials $p_k^{(\alpha)}(x)$, satisfies the hypotheses of Theorem 3. Note that $\|p_k^{(\alpha)}\|^2 = F_\alpha / F_{\alpha+2k}$. Hence, equations (11) and (12) become

$$\sum_{i=k}^{m} \sum_{j=0}^{i} (-1)^{i(j+k) - \binom{j}{2} - \binom{k}{2}} \binom{i}{j}_{\mathbb{F}} \binom{i}{k}_{\mathbb{F}} \binom{\alpha+i+j-1}{i}_{\mathbb{F}} \binom{\alpha+i+k-1}{i}_{\mathbb{F}} \frac{F_{\alpha+2i}}{F_{\alpha+j+m}} = \delta_{mk}$$

$$\sum_{k=0}^{\infty} F_{\alpha+2k} \left| \sum_{i=0}^{\infty} \sum_{j=0}^{k} (-1)^{jk - \binom{j}{2}} \binom{k}{j}_{\mathbb{F}} \binom{\alpha+k+j-1}{k} \frac{v_i}{F_{\alpha+j+i}} \right|^2 = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{v_k \overline{v_j}}{F_{\alpha+k+j}}$$

where $\alpha = 2, 4, 6, \ldots$ and $\langle v_j \rangle \in l^2$. Furthermore, the analysis following Theorem 3 shows that both identities also hold for $\alpha = 1, 3, 5, \ldots$, provided $v_j = 0$ for all but finitely many values of j. For such α, note that $\langle p_k^{(\alpha)}, p_k^{(\alpha)} \rangle < 0$ when k is odd.

Example 4 (*finite dimensional spaces*): Let α be a step function with positive jumps J_k at the distinct points $z_k \in (-1,1)$, $k = 0,1,\ldots,r$. Then $L^2_\alpha(-1,1)$ has dimension $r+1$, and

$$h_j = \sum_{k=0}^{r} z_k^j J_k, \qquad j = 0,1,2,\ldots \tag{19}$$

In what follows, we will frequently use the following submatrices of the infinite Hankel matrix $H = [h_{i+j}]_{i,j\geq 0}$: Let H_k denote the upper-left $(k+1)\times(k+1)$ submatrix of H. From equation (19), we see that

$$H_k = V_k D_r V_k^{\mathrm{T}}, \qquad k = 0,1,\ldots,r \tag{20}$$

where V_k is rows 0 through k of the Vandermonde matrix whose row 1 is $[z_0, z_1, \ldots, z_r]$, and where D_r is the diagonal matrix with diagonal entries $J_0, J_1, \ldots, J_r$.

In seeking an orthogonal basis of polynomials for $L^2_\alpha(-1,1)$, we choose their leading coefficients to be 1:

$$p_k(x) = x^k + \sum_{j=0}^{k-1} a_{kj}\, x^j, \qquad k = 0,1,\ldots,r$$

The remaining coefficients are then uniquely determined; we compute them as follows. Let $\mathbf{h}(p,q)$ denote the vector with components $h_p, \ldots, h_q$, and let $\mathbf{a}(k)$ denote the vector with components $a_{k0}, \ldots, a_{k,k-1}$. We claim that

$$H_{k-1}\,\mathbf{a}(k) = -\mathbf{h}(k, 2k-1), \qquad k = 1,\ldots,r \tag{21}$$

which may also be written

$$\sum_{j=0}^{k-1} h_{m+j}\, a_{kj} = -h_{m+k}, \qquad k = 1,\ldots,r, \quad m = 0,\ldots,k-1$$

Since H_k is nonsingular when $k = 0,1,\ldots,r$, equation (21) uniquely determines the coefficients a_{kj}. One may confirm (21) by a straightforward calculation to show that the resulting polynomials are mutually orthogonal. By Cramer's rule, we then have the explicit formula

$$p_k(x) = x^k - \frac{1}{\det H_{k-1}} \sum_{j=0}^{k-1} \det(H_{k-1}(j, \mathbf{h}(k,2k-1)))x^j, \quad k = 0,1,\ldots,r \tag{22}$$

where $M(j,\mathbf{w})$ denotes the matrix M with its jth column replaced by the vector $\mathbf{w}$. A similar calculation shows that

$$\|p_k\|^2 = \frac{\det H_k}{\det H_{k-1}}, \qquad k = 0,1,\ldots,r$$

where we assign the value 1 to $\det H_{-1}$.

Next we express each monomial x^m as a linear combination of the basis polynomials $p_0, p_1, \ldots, p_r$:

$$x^m = \sum_{k=0}^{r} b_{mk}\, p_k(x), \qquad m = 0, 1, 2, \ldots$$

By equations (10) and (22),

$$\begin{aligned} b_{mk} &= \frac{1}{\|p_k\|^2} \sum_{j=0}^{k} \overline{a_{kj}}\, h_{j+m} \\ &= \frac{\det H_{k-1}}{\det H_k} \left(h_{k+m} - \frac{1}{\det H_{k-1}} \sum_{j=0}^{k-1} \det(H_{k-1}(j, \mathbf{h}(k, 2k-1))) h_{j+m} \right) \\ &= \frac{1}{\det H_k} \left((\det H_{k-1}) h_{k+m} - \sum_{j=0}^{k-1} \det(H_{k-1}(j, \mathbf{h}(k, 2k-1))) h_{j+m} \right) \\ &= \frac{\det(H_k(k, \mathbf{h}(m, m+k)))}{\det H_k} \end{aligned}$$

To confirm the last step above, evaluate $\det(H_k(k, \mathbf{h}(m, m+k)))$ by the Laplace expansion down the last column of the matrix. For the jth entry of that column, $j = 0, 1, \ldots, k-1$, the expansion formula assigns the sign $(-1)^{k+j}$. We must also move the vector $\mathbf{h}(k, 2k-1)$ from column j to column $k-1$, and the corresponding transposes change the sign of the determinant by the factor $(-1)^{k-1-j}$. Thus, the overall sign factor is $(-1)^{k+j+k-1-j} = -1$. Having confirmed the above computation of b_{mk}, we conclude

$$x^m = \sum_{k=0}^{r} \frac{\det(H_k(k, \mathbf{h}(m, m+k)))}{\det H_k} p_k(x), \qquad m = 0, 1, 2, \ldots$$

Hence equation (13) becomes

$$\sum_{k=0}^{r} \frac{1}{(\det H_k)(\det H_{k-1})} \left| \sum_{j=k}^{\infty} v_j \det(H_k(k, \mathbf{h}(j, j+k))) \right|^2 = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} v_k \overline{v_j} h_{k+j}$$

where $\langle v_j \rangle \in l^2$.

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