Adelic constructions of low discrepancy sequences

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Dedicated to the memory of Professor N.M. Korobov

Abstract

In [Fr2,Skr], Frolov and Skriganov showed that low discrepancy point sets in the multidimensional unit cube $[0,1)^s$ can be obtained from admissible lattices in $\mathbb{R}^s$. In this paper,we get a similar result for the case of $(\mathbb{F}_q((x^{-1})))^s$. Then we combine this approach with Halton's construction of low discrepancy sequences.

Key words: low discrepancy sequences, (t,s) sequences, global function field.

1 Introduction.

1.1. Let $(\beta_n)_{n\geq 0}$ be an infinite sequence of points in an s-dimensional unit cube $[0,1)^s$. The sequence $(\beta_n)_{n\geq 0}$ is said to be *uniformly distributed* in $[0,1)^s$ if for every box $V = [0,v_1)\times\cdots\times[0,v_s)\subseteq[0,1)^s$

$$\Delta(V,(\beta_n)_{n=0}^{N-1}) = \#\{0\leq n<N \mid \beta_n\in V\} - Nv_1\dots v_s = o(N),\ N\to\infty.$$

We define the L_∞ and L_2 *discrepancy* of a N-point set $(\beta_{n,N})_{n=0}^{N-1}$ as

$$D((\beta_{n,N})_{n=0}^{N-1}) = \sup_{0<v_1,\dots,v_s\leq 1} |\frac{1}{N}\Delta(V,(\beta_{n,N})_{n=0}^{N-1})|,$$

$$D_2((\beta_{n,N})_{n=0}^{N-1}) = \Big(\int_{[0,1]^s} |\frac{1}{N}\Delta(V,(\beta_{n,N})_{n=0}^{N-1})|^2 dv_1\dots dv_s\Big)^{1/2}.$$

It is known that a sequence $(\beta_n)_{n\geq 0}$ is uniformly distributed if and only if $D((\beta_n)_{n=0}^{N-1})\to 0$ for $N\ \to\infty$.

In 1954, Roth proved that there exists a constant $C_1 > 0$, such that

$$ND_2((\beta_{n,N})_{n=0}^{N-1}) > C_1(\ln N)^{\frac{s-1}{2}}, \quad \text{and} \quad \overline{\lim} \frac{ND_2((\beta_n)_{n=0}^{N-1})}{(\ln N)^{s/2}} > 0$$

for all N-point sets $(\beta_{n,N})_{n=0}^{N-1}$ and all sequences $(\beta_n)_{n\geq 0}$. According to the well-known conjecture (see, for example, [BC, p.283] and [Ni, p.32]), there exists a constant $C_2 > 0$, such that

$$ND((\beta_{n,N})_{n=0}^{N-1}) > C_2(\ln N)^{s-1}, \quad \text{and} \quad \overline{\lim} \frac{ND((\beta_n)_{n=0}^{N-1})}{(\ln N)^{s}} > 0$$

for all N-point sets $(\beta_{n,N})_{n=0}^{N-1}$ and all sequences $(\beta_n)_{n\geq 0}$.

Definition 1. *A sequence $(\beta_n)_{n\geq 0}$ is of* low discrepancy *(abbreviated l.d.s.) if* $\mathrm{D}((\beta_n)_{n=0}^{N-1}) = O(N^{-1}(\ln N)^s)$ *for* $N \to \infty$.

Definition 2. *A sequence of point sets $((\beta_{n,N})_{n=0}^{N-1})_{N=1}^{\infty}$ is of low discrepancy (abbreviated l.d.p.s.) if* $\mathrm{D}((\beta_{n,N})_{n=0}^{N-1}) = O(N^{-1}(\ln N)^{s-1})$, *for* $N \to \infty$.

1.2. Brief review of multidimensional ($s \geq 2$) **low discrepancy sequences** (for a complete review, see [BC], [DrTi], [Mat], and [Ni]).

1.2.1. Halton's sequences. The existence of multidimensional l.d.s. was discovered by Halton in 1960: Let $b \geq 2$ be an integer,

$$n = \sum_{i\geq 0} e_{i,b}(n)b^i, \text{ with } e_{i,b}(n) \in \{0, 1, \ldots, b-1\} \tag{1.1}$$

the b-expansion of the integer n, and

$$\varphi_b(n) = \sum_{i\geq 0} e_{i,b}(n)b^{-i-1}$$

the radical inverse function. Let $b_1, \ldots, b_s \geq 2$ be pairwise coprime integers. Then $(\varphi_{b_1}(n), \ldots, \varphi_{b_s}(n))_{n\geq 0}$ is a l.d.s. The main tool here is the Chinese Remainder Theorem. In 1960, Hammersley proved that $(\varphi_{b_1}(n), \ldots, \varphi_{b_s}(n), \frac{n}{N})_{n=0}^{N-1}$ is an $s{+}1$-dimensional l.d.p.s.

1.2.2. (t, s) **sequences, and** (t, m, s) **point sets.** A subinterval E of $[0, 1)^s$ of the form

$$E = \prod_{i=1}^{s} [a_i b^{-d_i}, (a_i + 1)b^{-d_i}),$$

with $a_i, d_i \in Z,\ d_i \geq 0,\ 0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$ is called an *elementary interval in base* $b \geq 2$.

Definition 3. *Let $0 \leq t \leq m$ be an integer. A (t, m, s)*-net in base b *is a point set $x_1, ..., x_{b^m}$ in $[0, 1)^s$ such that $\#\{n \in [1, b^m] | x_n \in E\} = b^t$ for every elementary interval E in base b with* $\mathrm{vol}(E) = b^{t-m}$.

Let $t \geq 0$ be an integer. A sequence $x_0, x_1, ...$ of points in $[0, 1)^s$ is a (t, s)-sequence in base b *if, for all integers $k \geq 0$ and $m \geq t$, the point set consisting of x_n, $(n \in [kb^m, (k + 1)b^m)$ is a (t, m, s)-net in base b.*

The theory of (t,s)-sequences was developed by Sobol [So1], [So2] for the case of $b=2$. In 1981, Faure constructed (t,s)-sequences for prime $p>2$. The general case was considered by Niederreiter (see [Ni], [NiXi]). For the proof of low discrepancy property of (t,s) sequences, see e.g., [Ni, pp. 54-60].

Let q be an arbitrary prime power, $\mathbb{F}_q$ a finite field with q elements, $\mathbb{F}_q[x]$ a polynomial ring, $\mathbb{F}_q(x)$ the quotient field of $\mathbb{F}_q[x]$ (i.e. the field of all formal rational functions of x over $\mathbb{F}_q$), $\mathsf{K}/\mathbb{F}_q(x)$ a finite extension of $\mathbb{F}_q(x)$, and let $\mathcal{N}(\mathsf{K})$ be the number of rational places of K. By a rational place of K we mean a place of K of degree 1.

In [Te], Tezuka proved that the above constructions of (t,s)-sequences can be obtained by Halton's (Chinese Remainder Theorem) method, applied to $\mathbb{F}_q(x)$. Niederreiter and Xing use a similar approach, applied to the field K. In this way, they obtained a (t,s)-sequence with smallest parameter t for $s \leq \mathcal{N}(\mathsf{K})$ (see [NiXi, p. 204]):

$$t = g \tag{1.2}$$

where g is the genus of K. Niederreiter and Xing [NiXi] used s distinct places (instead of s coprime integers as in Halton's construction) and also some nonspecial divisor. In this paper, we obtain the same estimate (1.2). But we do not use an additional nonspecial divisor.

1.2.3. Lattice nets. In this subsection, we consider l.d.p.s. in $[0,1)^{s+1}$ and l.d.s. in $[0,1)^s$ based on lattices in $\mathbb{R}^{s+1}$. Let K be a totally real algebraic number field of degree $s+1$, and σ the canonical embedding of K in the Euclidean space $\mathbb{R}^{s+1}$, $\sigma : \mathsf{K} \ni \xi \to \sigma(\xi) = (\sigma_1(\xi), \ldots, \sigma_{s+1}(\xi)) \in \mathbb{R}^{s+1}$, where $\{\sigma_j\}_{j=1}^{s+1}$ are $s+1$ distinct embeddings of K in the field $\mathbb{R}$ of real numbers. Let $\lambda \in \mathsf{K}$ be an algebraic integer, $\lambda_i = \sigma_i(\lambda)$ $(i=1,\ldots,s+1)$, $f(x)$ the minimal polynomial of λ; λ is of degree $s+1$ over $\mathbb{Q}$; $E = (\lambda_i^{j-1})_{i,j=1}^{s+1}$; $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_{s+1})$; and $H = E\Lambda E^{-1}$ the companion matrix of $f(x)$.

In 1976, Frolov introduced the point set $Fr(s+1,t) = \frac{1}{t}E\mathbb{Z}^{s+1} \cap [0,1)^{s+1}$ $(t \to \infty)$ with the best possible estimate for the order of magnitude of the integration error on the Sobolev and Korobov class functions (see [Fr1],[By1],[By2]). In 1980, Frolov [Fr2] proved that $Fr(s+1,t)$ is a L_2 low discrepancy point set (i.e., $D_2(Fr(s+1,t)) = O(t^{-1}(\ln t)^{s/2})$ for $t \to \infty$).

In 1994, Skriganov [Skr] proved that $Fr(s+1,t)$ is a l.d.p.s. He also proved the following more general result:

Let $V \subset \mathbb{R}^{s+1}$ be a compact region, $\mathrm{vol}(V)$ the volume of V, tV the dilatation of V by a factor $t>0$, and let $tV+X$ be the translation of tV by a vector $X \in \mathbb{R}^{s+1}$. Let $\Gamma \subset \mathbb{R}^{s+1}$ be a lattice, i.e., a discrete subgroup of $\mathbb{R}^{s+1}$ with a compact fundamental set $\mathsf{F}(\Gamma) = \mathbb{R}^{s+1}/\Gamma$, $\det\Gamma = \mathrm{vol}(\mathsf{F}(\Gamma))$. Let

$$N(V,\Gamma) = \mathrm{card}(V \cap \Gamma) = \sum_{\gamma \in \Gamma} \chi(V,\gamma)$$

be the number of points of the lattice Γ lying inside the region V, where we denote by $\chi(V,X)$, $X \in \mathbb{R}^{s+1}$, the characteristic function of V. We define the error $R(V+X,\Gamma)$ by setting

$$N(V+X,\Gamma) = \frac{\mathrm{vol}(V)}{\det\Gamma} \ + \ R(V+X,\Gamma). \tag{1.3}$$

Definition 4. *The lattice* $\Gamma \subset \mathbb{R}^{s+1}$ *is an* admissible *if*

$$\mathrm{Nm}\Gamma = \inf_{\gamma \in \Gamma \setminus \{0\}} |\mathrm{Nm}\gamma| > 0, \tag{1.4}$$

where $\mathrm{Nm}x = x_1 x_2 \dots x_{s+1},\ x = (x_1, \dots, x_{s+1})$.

For example, $\Gamma = E\mathbb{Z}^{s+1}$ (in Frolov's net) is the admissible lattice. The set of all admissible lattices is dense in $SL(s+1,\mathbb{R})/SL(s+1,\mathbb{Z})$, but its invariant measure is equal to zero. Let $\mathbb{K}^{s+1} = [-\frac{1}{2}, \frac{1}{2}]^{s+1}$, $T = (t_1, \dots, t_{s+1})$ and $T \centerdot V = \{(t_1 x_1, \dots, t_{s+1} x_{s+1}) \mid (x_1, ..., x_{s+1}) \in V\}$.

Theorem A. (see [Skr, Theorem 1.1]) *If* $\Gamma \subset \mathbb{R}^{s+1}$ *is an admissible lattice, then for all* $T \in \mathbb{R}^{s+1}$*, one has the bound*

$$\sup_{X \in \mathbb{R}^{s+1}} |R(T \centerdot \mathbb{K}^{s+1} + X, \Gamma)| < C(\Gamma)(\ln(2 + |\mathrm{Nm}T|))^s. \tag{1.5}$$

The constant in (1.5) *depends upon the lattice* Γ *only by means of the invariants* $\det\Gamma$ *and* $\mathrm{Nm}\Gamma$.

In [L], we constructed l.d.s. based on Frolov-Skriganov's approach. In this paper, we show that a similar approach can be applied to admissible lattices in $(\mathbb{F}_q((x^{-1})))^{s+1}$.

Now we describe the structure of the paper. In §2, we construct l.d.s. applying Halton's (adelic) method to the case of admissible lattices in $\mathbb{R}^{s+1}$. In §3, we obtain a similar result for the case of $(\mathbb{F}_q((x^{-1})))^{s+1}$. In §4, we give examples of (t,s)-sequences obtained from a global function field over $\mathbb{F}_q(x)$ without additional nonspecial divisors.

2 Admissible lattices in $\mathbb{R}^{s+1}$.

2.1. The general case.

In [L], we proposed the following constructions of l.d.s. based on Frolov's and Skriganov's nets.

Let $s \geq 1$ be an integer, $\Gamma = H\mathbb{Z}^{s+1}$ an admissible lattice, where H is an $(s+1)\times(s+1)$ nonsingular matrix with real coefficients. Let

$$W = \Gamma \cap [0,1)^s \times (0, +\infty).$$

By Theorem A and (1.3), the set W is infinite. Let $(u_i, u_{i,s+1}) \in W$ with $u_i \in \mathbb{R}^s$ and $u_{i,s+1} \in \mathbb{R},\ i = 1, 2$. Applying (1.4) to the lattice point $(u_1 - u_2, u_{1,s+1} - u_{2,s+1})$, we have that $u_{1,s+1} \neq u_{2,s+1}$. Hence W can be enumerated by a sequence $(z(n), z_{s+1}(n))_{n=0}^{\infty}$ in the following way:

$$\begin{aligned} z(0) = (0, ..., 0), \quad z_{s+1}(0) = 0, \quad z(n) \in [0,1)^s \quad \text{and} \\ z_{s+1}(n) < z_{s+1}(n+1) \in \mathbb{R}, \quad \text{for} \ \ n = 0, 1, ... \end{aligned} \tag{2.1}$$

According to [L] $(z(n))_{n=0}^{\infty}$ is a l.d.s. in $[0,1)^s$ and $(z(n), z_{s+1}(n)/z_{s+1}(N))_{n=0}^{N-1})$ is a l.d.p.s. in $[0,1)^{s+1}$. By Theorem A and (1.3),

$$|N - z_{s+1}(N-1)/\det(\Gamma)| < C(\Gamma)(\ln(2 + z_{s+1}(N-1)))^s.$$

Hence there exists a real N_1 such that

$$|N - z_{s+1}(N-1)/\det(\Gamma)| < 2C(\Gamma)(\ln(N))^s < N, \quad \text{for} \quad N > N_1. \tag{2.2}$$

Thus

$$z_{s+1}(N-1) = N\det\Gamma + O((\ln(N))^s). \tag{2.3}$$

By definition of the lattice Γ, there exists $y(n) = (y_1(n), ..., y_{s+1}(n)) \in \mathbb{Z}^{s+1}$ such that $(z(n), z_{s+1}(n)) = Hy(n)$.

Let $b_1, \dots, b_d \geq 2$ be pairwise coprime integers. Using notations from (1.1), we define

$$\phi_{b_j}(n) = \sum_{i \geq 0} \sum_{1 \leq m \leq s+1} e_{i,b_j}(y_m(n)) b_j^{-(s+1)(i+1)+m-1} \tag{2.4}$$

and

$$\zeta(n) = (\phi_{b_1}(n), ..., \phi_{b_d}(n), z(n)).$$

Theorem 2.1. *With the above notations, $(\zeta(n))_{n \geq 0}$ is a l.d.s. in $[0,1)^{s+d}$, and $(\zeta(n), z_{s+1}(n)/z_{s+1}(N))_{n=0}^{N-1}$ is a l.d.p.s. in $[0,1)^{s+d+1}$.*

Proof. We will prove the low discrepancy properties of the sequence $(\zeta(n))_{n \geq 0}$. The proof of the low discrepancy properties of the set $(\zeta(n)), z_{s+1}(n)/z_{s+1}(N))_{n=0}^{N-1}$ is completely similar. Let

$$S = [0, v_1) \times ... \times [0, v_{d+s}) \qquad \text{with} \qquad v_i \in (0,1], \ i = 1, ..., d+s.$$

We need to prove that

$$\#\{0 \leq n < N \mid \zeta(n) \in S\} = Nv_1....v_{s+d} + O((\ln(N))^{s+d}). \tag{2.5}$$

Let

$$S_1 = I_1 \times ... \times I_d \times [0, v_{d+1}) \times ... \times [0, v_{d+s}),$$

where

$$I_j = [a_j/b_j^{(s+1)k_j}, (a_j+1)/b_j^{(s+1)k_j}), \quad \text{with} \quad k_j \geq 0, \ a_j \in \mathbb{Z}, \ \ j = 1, ..., d,$$

and let

$$I_j^{'}(m) = [0, d_j/b_j^{(s+1)m}), \quad I_j^{''}(m) = [d_j/b_j^{(s+1)m}, v_j] \quad \text{with} \quad d_j = [v_j b_j^{(s+1)m}], \quad j \leq s,$$

$$V_j = I_1^{'}(m) \times ... \times I_{j-1}^{'}(m) \times I_j^{''}(m) \times [0, v_{j+1}) \times ... \times [0, v_{d+s}), \tag{2.6}$$

with $m = \max_{1 \leq j \leq d} [3 + 2\det\Gamma + (s+1)^{-1}\log_{b_j}(N/\mathrm{Nm}(\Gamma))]$.
Suppose

$$\exists n_1, n_2 \in [0, N-1], \ j \in [1, d] \quad \text{with} \quad (\zeta(n_i), z_{s+1}(n_i)/z_{s+1}(N)) \in V_j \times [0,1)$$

for $i = 1, 2, \ N > N_1$. By (2.4) we have

$$\gamma = (z_1(n_1) - z_1(n_2), ..., z_{s+1}(n_1) - z_{s+1}(n_2)) \in b_j^m \Gamma \qquad \text{and} \qquad |\mathrm{Nm}\gamma| \leq z_{s+1}(N-1).$$

Bearing in mind (2.2) and that

$$|\mathrm{Nm}\gamma| \geq \mathrm{Nm} b_j^m \Gamma = b_j^{(s+1)m} \mathrm{Nm}\Gamma \geq 2N(1 + \det\Gamma),$$

we have a contradiction. Hence the box $V_j \times [0, 1)$ contains at most one point of the sequence $(\zeta(n), z_{s+1}(n_i)/z_{s+1}(N))_{n=0}^{N-1}$ for $N > N_1$. Similarly to the proof of Halton's theorem (see [BC], [Mat] or [Ni]), we obtain from here that the box S can be expressed as a disjoint union of at most $(b_1...b_d)^{s+1}[3 + 2\det\Gamma + \log_2(N/\mathrm{Nm}(\Gamma))]^d$ boxes of the kind S_1, plus a set

$$V = V_1 \cup ... \cup V_d \in [0, 1)^{s+d} \quad \text{ with } \quad \# V \cap (\cup_{0 \leq n < N} \zeta(n)) \leq d.$$

From (2.6) we get

$$\mathrm{vol(V_j)} \leq |I_j^{''}(m)| < \mathrm{Nm}(\Gamma)/N \quad \text{and} \quad \mathrm{vol(V)} \leq d\mathrm{Nm}(\Gamma)/N.$$

Hence to obtain (2.5), it is sufficient to prove that

$$\#\{0 \leq n < N \mid \zeta(n) \in S_1\} = N b_1^{-(s+1)k_1} b_d^{-(s+1)k_d} v_{d+1}....v_{d+s} + O((\ln(N))^s). \qquad (2.7)$$

By (2.4), we have

$$\phi_{b_j}(n) \in I_j \Longleftrightarrow \ y(n) \equiv w_j \ (\mathrm{mod}\ b_j^{k_j}\mathbb{Z}^{s+1}) \quad j = 1, ..., d$$

for some $w_j \in \mathbb{Z}^{s+1}, \ j = 1, ..., d$.
By the Chinese Remainder Theorem, there exists $w_0 \in \mathbb{Z}^{s+1}$ such that

$$(\phi_{b_1}(n), ..., \phi_{b_d}(n)) \in I_1 \times ... \times I_d \Longleftrightarrow \ y(n) \equiv w_0 \ (\mathrm{mod}\ b_1^{k_1}...b_d^{k_d}\mathbb{Z}^{s+1}).$$

Thus

$$(\phi_{b_1}(n), ..., \phi_{b_d}(n)) \in I_1 \times ... \times I_d \Longleftrightarrow \ (z(n), z_{s+1}(n)) \equiv Hw_0 \ (\mathrm{mod}\ b_1^{k_1}...b_d^{k_d}\Gamma).$$

Hence

$$\zeta(n) \in S_1 \Longleftrightarrow (z(n), z_{s+1}(n)) \equiv Hw_0 \ (\mathrm{mod}\ b_1^{k_1}...b_d^{k_d}\Gamma)$$
$$\text{and} \quad z(n) \in [0, v_{d+1}) \times ... \times [0, v_{d+s}).$$

Applying (2.1), we obtain

$$\#\{0 \leq n < N \mid \zeta(n) \in S_1\} = \#\big\{(\gamma_1, ..., \gamma_{s+1}) \in b_1^{k_1}...b_d^{k_d}\Gamma \mid \gamma_i \in [-(Hw_0)_i,$$

$$v_i - (Hw_0)_i), \ i = 1, ..., s, \qquad \gamma_{s+1} \in [-(Hw_0)_{s+1}, z_{s+1}(N-1) - (Hw_0)_{s+1}]\big\}$$
$$= \#\big\{(\gamma_1, ..., \gamma_{s+1}) \in \Gamma \mid \gamma_i \in [-b_1^{-k_1}...b_d^{-k_d}(Hw_0)_i, b_1^{-k_1}...b_d^{-k_d}(v_i - (Hw_0)_i)),$$

$i=1,...,s \quad \gamma_{s+1} \in [-b_1^{-k_1}...b_d^{-k_d}(Hw_0)_{s+1}, b_1^{-k_1}...b_d^{-k_d}(z_{s+1}(N-1)-(Hw_0)_{s+1})]\big\}.$

Now by Theorem A and (2.2), we obtain the assertion (2.7), hence Theorem 2.1 is proved. ■

2.2. The case of algebraic lattices.

Let K be a totally real algebraic number field of degree $s+1$, $\mathcal{O}$ the ring of integers in K. Denote by $\mathcal{A}$ the set of integer divisors of K. For $\mathfrak{b} \in \mathcal{A}$, we denote by $L(\mathfrak{b}) = \{\alpha \in \mathcal{O} \mid \alpha \equiv 0 \pmod{\mathfrak{b}}\}$ the $\mathcal{O}$-ideal associated with $\mathfrak{b}$.

Let $\mathfrak{M} \subset \mathsf{K}$ be an arbitrary $\mathbb{Z}$-module of rank $s+1$. Then the image

$$\Gamma(\mathfrak{M}) = \sigma(\mathfrak{M}) \subset \mathbb{R}^{s+1} \tag{2.8}$$

of $\mathfrak{M}$ under the embedding σ (see §1.2.3.) is the admissible lattice in $\mathbb{R}^{s+1}$. Since every ideal of the field K is a $\mathbb{Z}$-module of rank $s+1$, (2.8) determines a lattice $\Gamma(L(\mathfrak{b})) = \sigma(L(\mathfrak{b})) \subset \mathbb{R}^{s+1}$ corresponding to the ideal $L(\mathfrak{b})$.

Now let $\mathfrak{b}_i \in \mathcal{A}$, $i=1,...,d$, be pairwise coprime divisors in K, and let $b_i = \mathsf{N}(\mathfrak{b}_i)$, where N is the norm of the extension $\mathsf{K}/\mathbb{Q}$. It is easy to see that

$$\#\{\mathcal{O}/L(\mathfrak{b}_i^j)\} = b_i^j \qquad \text{and} \qquad \#\{L(\mathfrak{b}_i^j)/L(\mathfrak{b}_i^{j+1})\} = b_i \quad (j=0,1,2,...),$$

where $L(\mathfrak{b}_i^0) = \mathcal{O}$ $(i=1,...,d)$.

Let $i \in [1,d]$, $j \geq 0$. A digit set $\mathcal{D}_{i,j} \in L(\mathfrak{b}_i^j) \in \mathcal{O}$ is any complete set of coset representatives for $L(\mathfrak{b}_i^j)/L(\mathfrak{b}_i^{j+1})$. We have that, for any $\alpha \in \mathcal{O}$, and every $m \geq 1$

$$\alpha = d_{i,0} + d_{i,1} + ... + d_{i,m-1} + x_m$$

where $d_{i,j} \in \mathcal{D}_{i,j}$, $x_m \in L(\mathfrak{b}_i^m)$. So for each $\alpha \in \mathcal{O}$, we can associate a unique sequence $(d_{i,0}, d_{i,1}, d_{i,2}, ...)$. Let $\eta_{i,j}$ be a one to one map from $\mathcal{D}_{i,j}$ to $\{0,1,...,b_i-1\}$, and let

$$\phi_{\mathfrak{b}_i}(\alpha) = \sum_{j\geq 0} \eta_{i,j}(d_{i,j})/b_i^{j+1}. \tag{2.9}$$

Consider the sequences $(z(n)_{n\geq 0}$ defined in (2.1) with $\Gamma = \Gamma(L(\mathcal{O}))$. Let

$$\zeta(n) = (\varphi_{\mathfrak{b}_1}(n), ..., \varphi_{\mathfrak{b}_d}(n), z(n)),$$

where $\varphi_{\mathfrak{b}_i}(n) = \phi_{\mathfrak{b}_i}((z(n), z_{s+1}(n))$.

Theorem 2.2. *With the above notation $(\zeta(n))_{n\geq 0}$ is a l.d.s. in $[0,1)^{s+d}$, and $(\zeta(n)), z_{s+1}(n)/z_{s+1}(N))_{n=0}^{N-1}$ is a l.d.p.s. in $[0,1)^{s+d+1}$.*

Proof. Let

$$S_1 = I_1 \times ... \times I_d \times [0, v_{d+1}) \times ... \times [0, v_{d+s}), \qquad \text{where} \quad v_{d+i} \in (0,1], \quad i=1,...,s,$$
$$\text{and} \qquad I_j = [a_j/b_j^{l_j}, (a_j+1)/b_j^{l_j}), \qquad l_j \geq 0,\ a_j \in \mathbb{Z},\ \ j=1,...,d.$$

Similarly to (2.5)-(2.7), it is sufficient to prove that

$$\#\{0 \leq n < N \mid \zeta(n) \in S_1\} = N b_1^{-l_1} b_d^{-l_d} v_{d+1} .... v_{d+s} + O((\ln(N))^s). \tag{2.10}$$

The lattice $\Gamma = \Gamma(L(\mathcal{O}))$ is admissible. By (2.9) and (2.3), we have

$$\varphi_{\mathfrak{b}_j}(n) \in I_j \iff \sigma^{-1}((z(n), z_{s+1}(n))) \equiv a^{(j)} \pmod{b_j^{l_j}}$$

for some $a^{(j)} \in \mathcal{O},\ j = 1, ..., d$.
Applying the Chinese Remainder Theorem, we conclude that there exists $r \in \mathcal{O}$ such that

$$(\varphi_{\mathfrak{b}_1}(n), ..., \varphi_{\mathfrak{b}_d}(n)) \in I_1 \times ... \times I_d \iff \sigma^{-1}(z(n), z_{s+1}(n)) \equiv r \pmod{\mathfrak{b}_1^{l_1} ... \mathfrak{b}_d^{l_d}}$$

or

$$(\varphi_{\mathfrak{b}_1}(n), ..., \varphi_{\mathfrak{b}_d}(n)) \in I_1 \times ... \times I_d \iff (z(n), z_{s+1}(n)) - \sigma(r) \in \Gamma(L(\mathfrak{b}_1^{l_1} ... \mathfrak{b}_d^{l_d})).$$

Therefore

$$\begin{aligned} \{(z(n), z_{s+1}(n)) \mid \zeta(n) \in S_1,\ 0 \leq n < N\} = \{\gamma \in \Gamma(L(\mathfrak{b}_1^{l_1} ... \mathfrak{b}_d^{l_d})) \mid \\ \gamma \in [-r_1, -r_1 + v_{d+1}) \times ... \times [-r_s, -r_s + v_{d+s}) \times [-r_{s+1}, -r_{s+1} + z_{s+1}(N-1)]\}, \end{aligned} \tag{2.11}$$

where $r_i = \sigma_i(r),\ i = 1, ..., s+1$.

We cannot apply Theorem A directly to prove (2.10) because the constant in (1.5) depends on the lattice $\Gamma(L(\mathfrak{b}_1^{l_1} ... \mathfrak{b}_d^{l_d}))$. To prove (2.10), we will use the following idea from [NiSkr]: Let $\{\mathfrak{M}_1, ..., \mathfrak{M}_h\}$ be a fixed set of representatives of the ideal class group, and let h be the class number of the field K. Hence there exists an element $\theta \in \mathsf{K}$ such that $\theta L(\mathfrak{b}_1^{l_1} ... \mathfrak{b}_d^{l_d}) = \mathfrak{M}_j$ for some $j \in [1, h]$. Therefore

$$\det(\Gamma(\mathfrak{M}_j)) = \theta_1 ... \theta_{s+1} \det(\Gamma(L(\mathfrak{b}_1^{l_1} ... \mathfrak{b}_d^{l_d}))) = \theta_1 ... \theta_{s+1} b_1^{l_1} ... b_d^{l_d} \det(\Gamma), \tag{2.12}$$

with $\theta_i = \sigma_i(\theta),\ i = 1, ..., s+1$.
By (2.11), we get

$$\{0 \leq n < N \mid \zeta(n) \in S_1\} = \Gamma(\mathfrak{M}_j) \bigcap V, \tag{2.13}$$

where

$$\begin{aligned} V = [-\theta_1 r_1, \theta_1(-r_1 + v_{d+1})) \times ... \times [-\theta_s r_s, \theta_s(-r_s + v_{d+s})) \\ \times [-\theta_{s+1} r_{s+1}, \theta_{s+1}(-r_{s+1} + z_{s+1}(N-1))]. \end{aligned}$$

Using (2.12), we have

$$\mathrm{vol}(V)/\det(\Gamma(\mathfrak{M}_j)) = v_1 ... v_s z_{s+1}(N-1) b_1^{-l_1} ... b_d^{-l_d} / \det(\Gamma).$$

According to (1.3), we obtain

$$R(V, \Gamma(\mathfrak{M}_j)) = \#\Gamma(\mathfrak{M}_j) - v_1 ... v_s b_1^{-l_1} ... b_d^{-l_d} z_{s+1}(N-1)/\det(\Gamma). \tag{2.14}$$

By Theorem A, we obtain

$$|R(V, \Gamma(\mathfrak{M}_j))| < \max_{1 \leq j \leq h} C(\Gamma(\mathfrak{M}_j))(\ln(2 + z_{s+1}(N-1)))^s.$$

Now by (2.3), (2.13) and (2.14), we obtain the assertion (2.10). Theorem 2.2 is proved.
■

3 Uniformly distributed sequences obtained from lattices in $(\mathbb{F}_q((x^{-1})))^{s+1}$.

First, we describe Mahler's variant of Minkowski's theorem on a convex body in a field of series for the following special case:

3.1. Mahler's theorem. Let q be an arbitrary prime power, $\mathbb{F}_q$ a finite field with q elements, $\mathsf{k} = \mathsf{k}(x) = \mathbb{F}_q(x)$ the rational function field over $\mathbb{F}_q$, and $\mathsf{k}[x] = \mathbb{F}_q[x]$ the polynomial ring over $\mathbb{F}_q$. For $\alpha = f/g,\ f, g \in \mathsf{k}[x]$, let

$$\nu(\alpha) = \deg g - \deg f \tag{3.1}$$

be the degree valuation of $\mathsf{k}(x)$. We define an absolute value $\|.\|$ of $\mathsf{k}(x)$ by

$$\|\alpha\| = q^{-\nu(\alpha)}. \tag{3.2}$$

We denote by $\hat{\mathsf{k}} = \mathbb{F}_q((x^{-1}))$ the perfect completion of k with respect to this valuation. Every element α of $\hat{\mathsf{k}}$ has a unique expansion into *the field of formal Laurent series with coefficients from* $\mathbb{F}_q$

$$\alpha = \sum_{k=-w}^{\infty} a_k x^{-k} \tag{3.3}$$

with an integer w and all $a_k \in \mathbb{F}_q$. The degree valuation ν on $\hat{\mathsf{k}}$ is defined by $\nu(\alpha) = -\infty$ if $\alpha = 0$ and $\nu(\alpha) = w$ if $\alpha \neq 0$ and (3.3) is written in such a way that $a_w \neq 0$.

We will be working in the $s+1$ dimensional vector space over $\hat{\mathsf{k}}$. A *lattice* Γ in $\hat{\mathsf{k}}^{s+1}$ is the image of $(\mathsf{k}[x])^{s+1}$ under an invertible $\hat{\mathsf{k}}$-linear mapping A of the vector space $\hat{\mathsf{k}}^{s+1}$ into itself. The points of Γ will be called lattice points. The absolute value (in the sense of (3.2)) of the determinant of A will be denoted by $\det(\Gamma)$. We introduce on $\hat{\mathsf{k}}^{s+1}$ the Haar measure μ such that the set $\{x = (x_1, ..., x_{s+1}) \mid \|x_i\| \leq 1\}$ has measure 1. A *distance function* in $\hat{\mathsf{k}}^{s+1}$ is a function $F:\ \hat{\mathsf{k}}^{s+1} \to \mathbb{R}$ such that

$$\begin{aligned} F(o) = 0, \quad F(y) \neq 0 \quad \text{if} \quad y \neq o, \\ F(\lambda y) = \|\lambda\|\, F(y) \quad \text{for} \quad \lambda \in \hat{\mathsf{k}}, \\ F(y - z) \leq \max(F(y), F(z)). \end{aligned}$$

An inequality of the form $F(y) \leq q^r$, defines a *convex body*, $\mathcal{V}_{F,r} = \mathcal{V}_r$. Let

$$M_F(r) = \#\{(\mathsf{k}[x])^{s+1} \cap \mathcal{V}_{F,r}\} = \#\{\mathsf{k}[x]^{s+1} \cap x^r \mathcal{V}_{F,0}\}. \tag{3.4}$$

A convex body $\mathcal{V}_0$ has a volume [Ma, eq. 20]

$$\mathrm{vol}(\mathcal{V}_0) = \lim_{r\to\infty} M_F(r) q^{-(s+1)(r+1)}. \tag{3.5}$$

In particular, if $F(y) = \|y\|$, then $\mathrm{vol}(\mathcal{V}_0) = \mu(\mathcal{V}_0) = 1$ (see [Ma, p.505] and [DuLu, p.330]). Let

$$F(c, y) = \max(q^{-c_1}\|y_1\|, ..., q^{-c_{s+1}}\|y_{s+1}\|), \tag{3.6}$$

where $c = (c_1, ..., c_{s+1})$. We define the corresponding convex body by $\mathcal{V}_{F(c),0}$. We see

$$\mathcal{V}(c) := \mathcal{V}_{F(c),0} = \{(y_1, ..., y_{s+1}) \in \hat{\mathsf{k}}^{s+1} \mid \|y_i\| \le q^{c_i}, \ \ i = 1, ..., s+1\}. \tag{3.7}$$

Let A be $(s+1) \times (s+1)$ invertible matrix with elements in $\hat{\mathsf{k}}$. The linear transformation $u = \mathsf{A}^{-1} y$ changes $F(y)$ into the new distance function $F'(u) = F(y) = F(\mathsf{A}u)$. According to [Ma, eq. 21],

$$\mathrm{vol}(\mathcal{V}_{F',r}) = \mathrm{vol}(\mathcal{V}_{F,r})(\det \mathsf{A})^{-1}. \tag{3.8}$$

In particular,

$$\mathrm{vol}(\mathcal{V}(c)) = q^{c_1 + ... + c_{s+1}}. \tag{3.9}$$

Let $\Gamma = \mathsf{A}(\mathsf{k}[x])^{s+1}$. Consider the distance function (3.6). Using (3.4), we obtain

$$\begin{aligned} \#\{\Gamma \cap x^r \mathcal{V}_{F(c),0}\} &= \#\{\gamma \in \Gamma \mid \|\gamma_i\| \le q^{r+c_i}\} \\ = \#\{u \in (\mathsf{k}[x])^{s+1} \mid \|(Au)_i\| \le q^{r+c_i}\} &= \#\{(\mathsf{k}[x])^{s+1} \cap x^r \mathcal{V}_{F'(c),0}\} = M_{F'(c)}(r). \end{aligned} \tag{3.10}$$

By (3.5) and (3.8), we get

$$\lim_{r\to\infty} M_{F'(c)}(r) q^{-(s+1)(r+1)} = \mathrm{vol}(\mathcal{V}_{F',0}) = \mathrm{vol}(\mathcal{V}_{F,0})(\det \Gamma)^{-1}.$$

Hence by (3.9) and (3.10), we have

$$\lim_{r\to\infty} \#\{\Gamma \cap x^r \mathcal{V}(c)\} q^{-(s+1)(r+1)} = q^{c_1+...+c_{s+1}}/\det \Gamma. \tag{3.11}$$

Mahler [Ma] proved that there exists $s+1$ $\hat{\mathsf{k}}$-independent lattice points $\gamma_1, ..., \gamma_{s+1} \in \Gamma$ such that:

a) $F(\gamma_1)$ is the minimum of $F(\gamma)$ in all lattice points $\gamma \neq o$;

b) for $j \ge 2$, $F(\gamma_j)$ is the minimum of $F(\gamma)$ in all lattice points independent on $\gamma_1, ..., \gamma_{j-1}$;

c) the points $\gamma_1, ..., \gamma_{s+1}$ are a basis for Γ over $\mathsf{k}[x]$;

d) the number $\sigma_j = F(\gamma_j)$, $1 \le j \le s+1$, (the *successive minima* of $\mathcal{V}_0$) depend only on $F(y)$ and Γ, and satisfy

$$0 < \sigma_1 \le \sigma_2 \le ... \le \sigma_{s+1}, \quad \text{and} \quad \sigma_1 \sigma_2 ... \sigma_{s+1} = \det(\Gamma)/\mathrm{vol}(V_0). \tag{3.12}$$

Now let $< y, z >$ be a standard inner product ($< y, z >= y_1 z_1 + ... + y_{s+1} z_{s+1}$ for $y = (y_1, ..., y_{s+1})$ and $z = (z_1, ..., z_{s+1})$). If Γ is a lattice with basis $\beta_1, ..., \beta_{s+1}$, then the polar body $\mathcal{V}_0^\perp$ and the polar (dual) lattice $\Gamma^\perp$ are defined exactly as in the $\mathbb{R}^{s+1}$ case. Thus $\Gamma^\perp$ is the lattice with basis $\beta_1^\perp, ..., \beta_{s+1}^\perp$, where $< \beta_i, \beta_i^\perp >= 1$ and $< \beta_i, \beta_j^\perp >= 0$ if $i \neq j$. We define the polar function to $F(y)$ by $G(o) = 0$ and for $z \neq o$ by

$$G(z) = \sup_{y \neq o} \frac{\|< y, z >\|}{F(y)}.$$

Then $G(z)$ is a distance function and $\mathcal{V}_0^\perp$ is the convex body defined by $G(z) \le 1$. It is easy to see that $\mathcal{V}_0^\perp$ consists of all points z of $\hat{\mathsf{k}}^{s+1}$ for with $\|< y, z >\| \le 1$ for all $y \in \mathcal{V}_0$. Moreover

$$\det(\Gamma)\det(\Gamma^\perp) = 1, \quad \mathrm{vol}(\mathcal{V}_0^\perp) = (\mathrm{vol}(\mathcal{V}_0))^{-1}, \tag{3.13}$$

and if τ_j are the corresponding successive minima with respect to polar lattice $\Gamma^\perp$, then

$$\sigma_j \tau_{s-j+2} = 1 \quad (1 \le j \le s+1). \tag{3.14}$$

By (3.7), we have

$$\mathcal{V}(c)^\perp = \{(y_1, ..., y_{s+1}) \in \hat{\mathsf{k}}^{s+1} \mid \|y_i\| \le q^{-c_i}, \;\; i = 1, ..., s+1\}. \tag{3.15}$$

3.2. Construction of uniformly distributed sequences. We will consider latices in $s+1$-dimensional space $\hat{\mathsf{k}}^{s+1} = (\mathbb{F}_q((x^{-1})))^{s+1}$ to construct uniformly distributed sequences in $[0,1)^s$.

Let $\ddot{\mathcal{A}} \subset \hat{\mathsf{k}}^{s+1}$, $r \in \mathbb{Z}$ and $z \in \hat{\mathsf{k}}^{s+1}$. We define $\ddot{\mathcal{A}} + z = \{y + z \mid y \in \ddot{\mathcal{A}}\}$ and $c - r = (c_1 - r, ..., c_{s+1} - r)$.

Lemma 3.1. *Let $c_0, c_1, ..., c_{s+1}$ be integers, $c = (c_1, ..., c_{s+1})$, $\Gamma \subset \hat{\mathsf{k}}^{s+1}$ an arbitrary lattice with $\det(\Gamma) = q^{c_0}$, let $z = (z_1, ..., z_{s+1}) \in \hat{\mathsf{k}}^{s+1}$, and let $\mathcal{V}(c)$ contain a basis $\beta_i = (\beta_{i,1}, ..., \beta_{i,s+1})$, $i = 1, ..., s+1$ of Γ. Then the shifted box $\mathcal{V}(c-1) + z$ contains exactly $q^{c_1+...+c_{s+1}-c_0-s-1}$ lattice points.*

Proof. We see that there exists $\alpha_i \in \hat{\mathsf{k}}$ with

$$z = \alpha_1 \beta_1 + ... + \alpha_{s+1}\beta_{s+1}.$$

We consider expansions of α_i of the form (3.3). Let $a_{i,j}$ $(i = 1, ..., s+1)$ be corresponding elements,

$$Q_i = \sum_{j \le 0} a_{i,j} x^{-j} \quad \in \mathsf{k}[x], \quad i = 1, ..., s+1,$$

and let

$$z' = (z'_1, ..., z'_{s+1}) = Q_1 \beta_1 + ... + Q_{s+1}\beta_{s+1}.$$

By (3.7), we have

$$\|z_i - z'_i\| \le \max_{j=1,...,s+1} \|(\alpha_i - Q_i)\beta_{i,j}\| \le q^{-1} \max_{j=1,...,s+1} \|\beta_{i,j}\| \le q^{c_i - 1}.$$

Now let $y = (y_1, ..., y_{s+1}) \in \mathcal{V}(c-1) + z$. We see that

$$\|y_i - z'_i\| = \|y_i - z_i + z_i - z'_i\| \le \max(\|y_i - z_i\|, \|z_i - z'_i\|) \le q^{c_i - 1}.$$

Hence $y \in \mathcal{V}(c-1) + z'$. Similarly, we get that if $y \in \mathcal{V}(c-1) + z'$, then $y \in \mathcal{V}(c-1) + z$. Thus the box $\mathcal{V}(c-1) + z$ coincides with the box $\mathcal{V}(c-1) + z'$. Bearing in mind that $z' \in \Gamma$, we obtain

$$\#\{\Gamma \cap (\mathcal{V}(c-1) + z)\} = \#\{\Gamma \cap \mathcal{V}(c-1)\}.$$

By (3.3) and (3.7), we get that $x^r \mathcal{V}(c-1)$ can be decomposed as follows:

$$x^r \mathcal{V}(c-1) = \bigcup_{\|Q_i\| \le q^{r-1},\, Q_i \in \mathsf{k}[x],\, 1 \le i \le s+1} (\mathcal{V}(c-1) + (x^{c_1} Q_1, ..., x^{c_{s+1}} Q_{s+1})).$$

Therefore

$$\#\{\Gamma \cap x^r \mathcal{V}(c-1)\} = q^{r(s+1)} \#\{\Gamma \cap \mathcal{V}(c-1)\}.$$

We have from (3.11) that

$$\#\Gamma \cap \mathcal{V}(c-1) = \lim_{r\to\infty} \#\{\Gamma \cap x^r \mathcal{V}(c-1)\} q^{-r(s+1)} = q^{c_1+...+c_{s+1}-s-1}/\mathrm{det}\Gamma,$$

and Lemma 3.1 is proved. ■

Let $y = (y_1, ..., y_{s+1}) \in \hat{\mathsf{k}}^{s+1}$,

$$y_i = \sum_{k=-w_i}^{\infty} y_{i,k} x^{-k}$$

with $y_{i,k} \in \mathbb{F}_q$, $\eta_{i,k}$ be a one to one map from $\mathbb{F}_q$ to $\{0, 1, ..., q-1\}$, and let

$$\xi(y) = (\xi(y_1), ..., \xi(y_{s+1}))$$

with

$$\xi(y_i) = \sum_{k\geq -w_i} \eta_{i,k}(y_{i,k}) q^{-k}.$$

Let $\xi(\Gamma) = \{\xi(\gamma) \mid \gamma \in \Gamma\}$,

$$W = \xi(\Gamma) \cap [0, 1)^s \times [0, +\infty).$$

By the definition of a lattice Γ it follows that for all $v \in \mathbb{R}^{s+1}$, the set $\xi(\Gamma) \cap ([0, 1]^{s+1} + v)$ is finite. The set W can be finite or infinite. We see that $(0, ..., 0) \in W$, and $\#W \geq 1$. Hence the set W can be enumerated by a sequence $(z_1(n), ..., z_{s+1}(n))_{0\leq n<\#W}$ in the following way:

$$z_i(n) \in \mathbb{R}, \quad z_i(0) = 0, \quad i = 1, ..., s+1, \qquad z_{s+1}(n) \leq z_{s+1}(n+1),$$

and $(z(n), z_{s+1}(n)) \neq (z(j), z_{s+1}(j))$ for $n \neq j$, where $z(n) = (z_1(n), ..., z_s(n))$.

Theorem 3.1. *Let $\Gamma \subset \hat{\mathsf{k}}^{s+1}$ be an arbitrary lattice. Then the sequence $(z(n))_{n\geq 0}$ is uniformly distributed in $[0, 1)^s$ if and only if*

$$\nexists \gamma^{\perp} = (\gamma_1^{\perp}, ..., \gamma_{s+1}^{\perp}) \in \Gamma^{\perp} \setminus \{0\} \quad \text{with} \quad \gamma_{s+1}^{\perp} = 0. \tag{3.16}$$

Proof. First, we consider the case that (3.16) is not valid. Hence there exists $\gamma_0^{\perp} = (\gamma_{0,1}^{\perp}, ..., \gamma_{0,s+1}^{\perp}) \in \Gamma^{\perp} \setminus \{0\}$ with $\gamma_{0,s+1}^{\perp} = 0$. Let

$$q^m = \max_{1\leq i\leq s} \left\|\gamma_{0,i}^{\perp}\right\|, \quad r = \max(0, m), \qquad \left\|\gamma_{0,j}^{\perp}\right\| = q^m, \quad \text{for some } j \in [1, s], \tag{3.17}$$

and let

$$V = [0, q^{-r-2})^{j-1} \times [q^{-r-1}, q^{-r}) \times [0, q^{-r-2})^{s-j} \times [0, \infty).$$

Suppose that there exist $n \geq 1$ with $(z(n), z_{s+1}(n)) \in V$. Let

$$\alpha := < \xi^{-1}(z(n), z_{s+1}(n)), \gamma_0^{\perp} > = \xi^{-1}(z_1(n))\gamma_{0,1}^{\perp} + ... + \xi^{-1}(z_{s+1}(n))\gamma_{0,s+1}^{\perp}.$$

We see that $\|\xi^{-1}(z_i(n))\| \leq q^{-r-2}$ for $i \in [1,s]$, $i \neq j$, and $\|\xi^{-1}(z_j(n))\| = q^{-r-1}$. Bearing in mind that $\gamma_{0,s+1}^{\perp} = 0$, we obtain from (3.17) $\|\alpha\| = q^{m-r-1} < 1$. On the other hand, $\alpha \in \mathsf{k}[x]$, and by (3.1), (3.2), $\|\alpha\| \geq 1$. Thus there are no points $(z(n), z_{s+1}(n))$ in V for $n \geq 1$. We have that the sequence $(z(n))_{n\geq 0}$ is not uniformly distributed.

Now let (3.16) be valid. Take any $\epsilon > 0$, and choose $m \geq 1$ such that $q^{-m} < \epsilon$. Consider the convex body $\mathcal{V}(c)^{\perp}$ with $c = (-m, ..., -m, r)$. By (3.15) and (3.16), there exists r such that there are no lattice points of $\Gamma^{\perp} \setminus \{o\}$ in $\mathcal{V}(c)^{\perp}$. Using (3.12)-(3.14), we get $\tau_1 > 1$. From (3.14), we obtain $\sigma_{s+1} < 1$. Therefore, $\mathcal{V}(c)$ contains a basis of Γ. According to Lemma 3.1 for every $z \in \hat{\mathsf{k}}^{s+1}$ the box $\mathcal{V}(c-1) + z$ contains exactly $q^{r-ms-s-1}(\det(\Gamma))^{-1}$ lattice points.

Let

$$V = \prod_{i=1}^{s} [G_i q^{-m}, (G_i+1)q^{-m}) \times [Bq^r, (B+1)q^r) = [0, q^{-m})^s \times [0, q^r) + y$$

with integers $G_1, ..., G_s, B$, and $y = (G_1 q^{-m}, ..., G_s q^{-m}, Bq^r) \in [0,1)^s \times [0, \infty)$.
It is easy to see that

$$\xi^{-1}(V) = \mathcal{V}(c-1) + z$$

for some z. Hence the box V contains exactly $q^{r-ms-s-1}(\det(\Gamma))^{-1}$ points of the sequence $(z(n), z_{s+1}(n))_{n\geq 0}$. In particular, for every integer $B \geq 0$

$$\#\{n \geq 0 \mid z_{s+1}(n) \in [Bq^r, (B+1)q^r)\} = q^{r-s-1}(\det(\Gamma))^{-1} =: q^l.$$

Hence

$$z_{s+1}(n) \in [Bq^r, (B+1)q^r) \Longleftrightarrow n \in [Bq^l, (B+1)q^l).$$

We see that

$$\#\{Bq^l \leq n < (B+1)q^l \mid z(n) \in \prod_{i=1}^{s} [G_i q^{-m}, (G_i+1)q^{-m}) = q^{l-ms}.$$

We now consider a subinterval V' of $[0,1)^s$ of the form

$$V' = \prod_{i=1}^{s} [G_i q^{-m}, (G_i + H_i) q^{-m})$$

with integers G_i, H_i satisfying $0 \leq G_i < G_i + H_i \leq q^m$ for $1 \leq i \leq s$. Let $Mq^l \leq N < (M+1)q^l$ for some integer $M \geq 1$. Then

$$Mq^{l-ms} H_1 ... H_s \leq |\#\{0 \leq n < N \mid z(n) \in V'\}| \leq (M+1) q^{l-ms} H_1 ... H_s.$$

Therefore

$$|\#\{0 \leq n < N \mid z(n) \in V'\}/N - \mathrm{vol}(V')| \leq H_1 ... H_s q^{-ms} M^{-1} \leq M^{-1} < \epsilon$$

if N is large enough. Since for every subinterval V of $[0,1)^s$ we can find subinterval V_1, V_2 of the above type with $V_1 \subseteq V \subseteq V_2$ and $\mathrm{vol}(V_2 \setminus V_1) \leq 2s\epsilon$, it follows that $(z(n))_{n\geq 0}$ is uniformly distributed in $[0,1)^s$. Theorem 3.1 is proved. ■

Remark. For the case of $\Gamma = \{(Q\alpha_1 - Q_1, ..., Q\alpha_s - Q_s, Q)) \mid (Q_1, ..., Q_s, Q) \in \mathsf{k}[x]^{s+1}\}$, we obtain a Kronecker lattice (and a Kronecker sequence: $(z(k))_{k\geq 1}$ (see [LaNi], [La])). It is proved in [La] that $D((z(n))_{n=1}^N) = O(N^{-1}(\ln(N))^{s-1})(\ln\ln(N))^{2+\epsilon})$ for almost all $(\alpha_1, ..., \alpha_s) \in \hat{\mathsf{k}}^s$.

Conjecture. We conjecture that this estimate is also true for almost all lattices Γ with respect to the Haar measure on $SL(s, \hat{\mathsf{k}})/SL(s, \mathsf{k}[x])$.

3.3. Admissible lattices in $(\mathbb{F}_q((x^{-1})))^{s+1}$ and (t, s) sequences. We will consider the $s+1$-dimensional space $\hat{\mathsf{k}}^{s+1} = (\mathbb{F}_q((x^{-1})))^{s+1}$ to construct (t, s) sequences.

Lemma 3.2. *Let $c_0, c_1, ..., c_{s+1}$ be integers, $c = (c_1, ..., c_{s+1})$, $\Gamma \subset \hat{\mathsf{k}}^{s+1}$ be an arbitrary lattice with $\det(\Gamma) = q^{c_0}$, $z = (z_1, ..., z_{s+1}) \in \hat{\mathsf{k}}^{s+1}$, and let $\mathcal{V}(c)^\perp \cap \Gamma^\perp \setminus \{o\} = \emptyset$. Then*

$$\#\{\Gamma \cap (\mathcal{V}(c-2) + z)\} = q^{c_1 + ... + c_{s+1} - c_0 - 2s - 2}.$$

Proof. Consider the box $\mathcal{V}(c)^\perp$ and the lattice $\Gamma^\perp$. We see that $\tau_1 > 1$, and by (3.14) $\sigma_{s+1} < 1$. Therefore, $\mathcal{V}(c-1)$ contains a basis of the lattice Γ. Now applying Lemma 3.1, we get the assertion of the lemma. ■

Definition 5. *The lattice $\Gamma \subset \hat{\mathsf{k}}^{s+1}$ is* admissible *if*

$$\mathrm{Nm}\Gamma = \inf_{\gamma \in \Gamma \setminus \{o\}} \|\mathrm{Nm}\gamma\| > 0, \tag{3.18}$$

where $\mathrm{Nm}\gamma = \gamma_1\gamma_2 \ldots \gamma_{s+1}$, $\gamma = (\gamma_1, \ldots, \gamma_{s+1})$.

Examples of such lattices are proposed by Armitage [Arm1, Arm2] (see §4).

Let $s_1 \in \{1, ..., s\}$, $s_2 = s + 1 - s_1$, $H_1, ..., H_{s_2}, r_1, ..., r_{s_2} \geq 0$ be integers, and let

$$W(H, r) = \xi(\Gamma) \cap [0, 1)^{s_1} \times [H_1 q^{r_1}, (H_1 + 1)q^{r_1}) \times ... \times [H_{s_2} q^{r_{s_2}}, (H_{s_2} + 1)q^{r_{s_2}}).$$

Theorem 3.2. *Let $\Gamma \subset \hat{\mathsf{k}}^{s+1}$ be an admissible lattice with*

$$\det(\Gamma^\perp) = q^{-c_0} \qquad \text{and} \qquad \mathrm{Nm}(\Gamma^\perp)/\det(\Gamma^\perp) = q^{-u-s}. \tag{3.19}$$

Then $(z(n))_{n\geq 0}$ is a (t, s) sequence with $t = u$, and $W(H, r)$ is a (t, m, s_1) net with $t = u$ and $m = r_1 + ... + r_{s_2} - c_0$.

Proof. Let $G_1, ..., G_{s_1}, l_1, ..., l_{s_1} \geq 0$ be integers, $G_i < q^{l_i}$ $(1 \leq i \leq s_1)$, and let

$$S = [\frac{G_1}{q^{l_1}}, \frac{G_1 + 1}{q^{l_1}}) \times ... \times [\frac{G_{s_1}}{q^{l_{s_1}}}, \frac{G_{s_1} + 1}{q^{l_{s_1}}}) \times [H_1 q^{r_1}, (H_1 + 1)q^{r_1}) \times ... \times [H_{s_2} q^{r_{s_2}}, (H_{s_2} + 1)q^{r_{s_2}}).$$

To obtain the (t, m, s_1) property of the set $W(H, r)$, we need to prove

$$\#W(H, r) = q^m \qquad \text{and} \qquad \#\{\xi(\Gamma) \cap S\} = q^t \tag{3.20}$$

for $l_1 + ... + l_{s_1} = m - t$ with $t = u$. For the case of $s_1 = s$, we obtain from here the (t, s) property of the sequence $(\zeta(n))_{n\geq 0}$.

Let $c = (-l_1 + 1, ..., -l_{s_1} + 1, r_1 + 1, ..., r_{s_2} + 1)$. It is easy to see that

$$\xi^{-1}(S) = \mathcal{V}(c - 2) + z \tag{3.21}$$

for some z.

Let $\gamma \in \Gamma^{\perp} \setminus \{o\}$. By (3.18) and (3.19), we have $\|\mathrm{Nm}\gamma\| \geq q^{-u-c_0-s}$. If $\gamma \in \mathcal{V}(c)^{\perp}$, then

$$\|\mathrm{Nm}\gamma\| \leq q^{l_1+\ldots+l_{s_1}-r_1-\ldots-r_{s_2}-s-1} = q^{m-u-(m+c_0)-s-1} = q^{-u-c_0-s-1}.$$

Hence $\gamma \notin \mathcal{V}(c)^{\perp}$. Applying Lemma 3.2, we obtain

$$\#\Gamma \cap (\mathcal{V}(c-2)+z) = q^{(-l_1-\ldots-l_{s_1}+r_1+\ldots+r_{s_2}+s+1)-c_0-2s-2} = q^{(u+c_0+2s+2)-c_0-2s-2} = q^t. \quad (3.22)$$

Taking $c = (1, ..., 1, r_1+1, ..., r_{s_2}+1)$, we obtain similarly that

$$\#\xi^{-1}(W(H,r)) = q^{(r_1+\ldots+r_{s_2}+s+1)-c_0-2s-2} = q^{(m+c_0+2s+2)-c_0-2s-2} = q^m.$$

Now by (3.21), we obtain (3.20). Theorem 3.2 is proved. ■

Using lattices from [Arm1] (see Example 1 below), we obtain $(0, s)$ sequences.

Now let $(\beta_1, ..., \beta_{s+1})$ be a basis of Γ. For all $\gamma \in \Gamma$, there exists polynomials $Q_1, ..., Q_{s+1} \in \mathsf{k}[x]$ with

$$\gamma = Q_1\beta_1 + \ldots + Q_{s+1}\beta_{s+1}.$$

Let $\mathsf{b} \in \mathsf{k}[x]$ with $\deg(\mathsf{b}) \geq 1$, $\mathcal{D}$ any complete set of coset representatives for $\mathsf{k}[x]/\mathsf{b}\mathsf{k}[x]$,

$$Q = \sum_{i \geq 0} e_{i,\mathsf{b}}(Q)\mathsf{b}^i, \text{ with } e_{i,\mathsf{b}}(Q) \in \mathcal{D},$$

the b-expansion of the integer polynomial Q, $\eta_{i,j,\mathsf{b}}$ a one-to-one map from $\mathcal{D}$ to $\{0, 1, ..., q^{\deg(\mathsf{b})}-1\}$ and let

$$\phi_{\mathsf{b}}(\gamma) = \sum_{i \geq 0} \sum_{1 \leq j \leq s+1} \eta_{i,j,\mathsf{b}}(e_{i,\mathsf{b}}(Q_j)) q^{(-(s+1)(i+1)+j-1)\deg(\mathsf{b})}. \quad (3.23)$$

Let $\mathsf{b}_1, \ldots, \mathsf{b}_d \in \mathsf{k}[x]$ be pairwise coprime polynomials with $b_i = \deg(\mathsf{b}_i) \geq 1$ $(i = 1, ..., d)$ and let

$$\zeta(n) = (\varphi_{\mathsf{b}_1}(n), ..., \varphi_{\mathsf{b}_d}(n), z(n)),$$

where $\varphi_{\mathsf{b}_i}(n) = \phi_{\mathsf{b}_i}(\xi^{-1}(z(n), z_{s+1}(n)))$.

Theorem 3.3. *With the notation above and the assumptions made in Theorem 3.2* $(\zeta(n))_{n \geq 0}$ *is a* $(t, s+d)$ *sequence with* $t = u + s(b_1 + \ldots + b_d) - d$.

Proof. Let $G_1, ..., G_{d+s+1}, l_1, ..., l_{d+s+1} \geq 0$ be integers, $G_i < q^{l_i}$ $(1 \leq i \leq d+s)$, $l_{d+s+1} = l_1 + \ldots + l_{d+s} + t$, and let

$$S = [\frac{G_1}{q^{l_1}}, \frac{G_1+1}{q^{l_1}}) \times \ldots \times [\frac{G_{d+s}}{q^{l_{d+s}}}, \frac{G_{d+s}+1}{q^{l_{d+s}}}) \times [G_{d+s+1}q^{l_{d+s+1}}, (G_{d+s+1}+1)q^{l_{d+s+1}}).$$

To obtain the assertion of the theorem, we need to prove

$$\#\{n \geq 0 \mid (\zeta(n), n) \in S\} = q^t. \quad (3.24)$$

Let

$$l_i = (s+1)b_ik_i - r_i, \quad \text{with} \quad 0 \le r_i < (s+1)b_i, \quad 1 \le i \le d,$$

$$G_i^{'} = G_iq^{r_i},\ G_i^{''} = (G_i+1)q^{r_i} \quad 1 \le i \le d,$$

and let

$$S(H) = I_1 \times ... \times I_d \times S_1 \times [G_{d+s+1}q^{l_{d+s+1}}, (G_{d+s+1}+1)q^{l_{d+s+1}}),$$

where

$$I_j = [\frac{H_j}{q^{(s+1)b_jk_j}}, \frac{H_j+1}{q^{(s+1)b_jk_j}}), \quad 1 \le j \le d, \quad S_1 = [\frac{G_{d+1}}{q^{l_{d+1}}}, \frac{G_{d+1}+1}{q^{l_{d+1}}}) \times ... \times [\frac{G_{d+s}}{q^{l_{d+s}}}, \frac{G_{d+s}+1}{q^{l_{d+s}}}).$$

We see that

$$S = \bigcup_{G_1^{'} \le H_1 < G_1^{''}} ... \bigcup_{G_{d+h}^{'} \le H_d < G_d^{''}} S(H). \tag{3.25}$$

Hence to obtain (3.24), it is sufficient to prove that

$$\#\{n \ge 0 \mid (\zeta(n), n) \in S(H)\} = q^{t_1} \tag{3.26}$$

with $t_1 = t - r_1 - ... - r_d$. Let

$$S^{'}(H) = I_1 \times ... \times I_d \times S_1 \times [G_{d+s+1}q^r, (G_{d+s+1}+1)q^r), \tag{3.27}$$

where $r = l_{d+s+1} + c_0 + s + 1$.

It is easy to see that (3.26) follows from the following assertion

$$\#\{n \ge 0 \mid (z_{s+1}(n) \in [Bq^r, (B+1)q^r)\} = q^{l_{d+s+1}}, \quad B = 0,1,... \tag{3.28}$$
$$\text{and} \qquad \#\{(\zeta(n), z_{s+1}(n)) \in S^{'}(H)\} = q^{t_1}.$$

According to (3.23),

$$\varphi_{\mathsf{b}_i}(n) \in I_j \iff \xi^{-1}((z(n), z_{s+1}(n))) \equiv w_j \ (\text{mod } \mathsf{b}_j^{k_j}\Gamma)$$

for somes $w_j \in \Gamma$, $j = 1, ..., d$. By the Chinese Remainder Theorem there exists $w_0 \in \Gamma$ such that

$$(\varphi_{\mathsf{b}_1}(n), ..., \varphi_{\mathsf{b}_d}(n)) \in I_1 \times ... \times I_d \iff \xi^{-1}((z(n), z_{s+1}(n))) \equiv w_0 \ (\text{mod } \mathsf{b}_1^{k_1}...\mathsf{b}_d^{k_d}\Gamma).$$

Using (3.27), we get

$$(\zeta(n), z_{s+1}(n)) \in S^{'}(H) \iff \xi^{-1}((z(n), z_{s+1}(n))) \equiv w_0 \ (\text{mod } \mathsf{b}_1^{k_1}...\mathsf{b}_d^{k_d}\Gamma) \tag{3.29}$$
$$\text{and} \quad \xi^{-1}((z(n), z_{s+1}(n))) \in \mathcal{V}(c-2) + \xi^{-1}(G_{d+1}/q^{l_{d+1}}, ..., G_{d+s}/q^{l_{d+s}}, G_{d+s+1}q^r),$$

where $c = (-l_{d+1}+1, ..., -l_{d+s}+1, r+1)$. By the assumptions made in (3.19) we have

$$\det(\mathsf{b}_1^{k_1}...\mathsf{b}_d^{k_d}\Gamma) = q^{(s+1)(b_1k_1+...+b_dk_d)}\det(\Gamma) = q^{c_0+(s+1)(b_1k_1+...+b_dk_d)}$$
$$\text{and} \quad \mathrm{Nm}((\mathsf{b}_1^{k_1}...\mathsf{b}_d^{k_d}\Gamma)^{\perp}) = q^{-(s+1)(b_1k_1+...+b_dk_d)}\mathrm{Nm}(\Gamma)^{\perp} = q^{-u-s-c_0-(s+1)(b_1k_1+...+b_dk_d)}.$$

Hence

$$\mathrm{Nm}((\mathsf{b}_1^{k_1}...\mathsf{b}_d^{k_d}\Gamma)^{\perp})/\det((\mathsf{b}_1^{k_1}...\mathsf{b}_d^{k_d}\Gamma)^{\perp}) = q^{-u-s}.$$

Similarly to (3.22), from (3.29) we get

$$\begin{aligned}\#\{(\zeta(n), z_{s+1}(n)) \in S^{'}(H)\} &= q^{(-l_{d+1}...-l_{d+s}+r+s+1)-c_0-(s+1)(b_1k_1+...+b_dk_d)-2s-2} \\ &= q^{(c_0+t+s+1+l_1+...+l_d)-c_0-(s+1)(b_1k_1+...+b_dk_d)-s-1} = q^{t-r_1-...-r_d} = q^{t_1}. \qquad (3.30)\end{aligned}$$

Taking $c = (1, ..., 1, r+1)$, we obtain

$$\begin{aligned}\#\{n \geq 0 \mid (z_{s+1}(n) \in [Bq^r, (B+1)q^r)\} &= \#(\Gamma \cap (\mathcal{V}(c-2) + \xi^{-1}((0, ..., 0, Bq^r))) \\ &= q^{(r+s+1)-c_0-2s-2} = q^{(l_{d+s+1}+c_0+2s+2)-c_0-2s-2} = q^{l_{d+s+1}},\end{aligned}$$

hence the assertion (3.28) and Theorem 3.3 are proved. ■

4 Constructions of (t, s) sequences from global function fields.

In [Arm1], [Arm2], Armitage gave examples of admissible lattices by constructing a special algebraic extension K of $\mathbb{F}_q(x)$ (see Example 1 and Example 2 below). According to §3.3 we get $(0, s)$ sequences from the lattices described in Example 1, and (g, s) sequences from the lattices described in Example 2, where g is the genus of K.

In [Arm3], Armitage constructed a lattice Γ from an arbitrary algebraic extension of $\mathbb{F}_q(x)$ (see Example 3). In this section, we use this lattice Γ to obtain a (t, s) sequence without additional nonspecial divisors (compare with [NiXi, p. 204, 213]).

4.1. Armitage's examples:

Example 1. [Arm1] Case $s \leq q$. The field $\mathbb{F}_q$ contains at least s distinct elements, say $\beta_1, ..., \beta_s$. Let $f(y) = (y-x)(y-\beta_1)...(y-\beta_s) - 1$. It is proved in [Arm1] that the polynomial $f(y)$ is irreducible over $\mathsf{k}(x)$, and the equation $f(y) = 0$ has $s+1$ roots in $\hat{\mathsf{k}}$, say $\lambda_1, ..., \lambda_{s+1}$. We consider linear forms $L_i = u_1 + u_2\lambda_i + ... + u_{s+1}\lambda_i^s$ $(i = 1, ..., s+1)$ with $u_i \in \mathsf{k}[x]$. Let D be the determinant of these forms. Then $\|D\| = q^s$, and $\|L_1...L_{s+1}\| \geq 1$ for all $u_1, ..., u_{s+1}$ not all 0 in $\mathsf{k}[x]$ (see [Arm1]). Hence $\Gamma = (L_1, ..., L_{s+1})$ is the admissible lattice with $u = 0$ (see (3.19)). We note that in [Arm1] the algorithm how to find the roots $\lambda_1, ..., \lambda_{s+1}$ is described.

Example 2. [Arm2] Case $s > q$. Let K be a finite algebraic extension of $\mathsf{k}(x)$ with genus g, and let $s+1$ denote the number of places of K of degree 1. It follows from Riemann-Roch's theorem that there exists $y \in \mathsf{K}$ that has simple poles at the places of degree 1 and no other singularities. Thus K is a "totally reel" extension of $\mathsf{k}(x)$ of degree $s+1$; that is, K has an imbedding $\theta \; : \; \mathsf{K} \to \hat{\mathsf{k}} \times ... \times \hat{\mathsf{k}}$ along the diagonal, where at each infinite place K is to be viewed as contained in $\hat{\mathsf{k}}$. If the integral closure $\mathcal{O}$ of $\mathsf{k}[x]$ in K has an $\mathsf{k}[x]$-basis $(\alpha_1, ..., \alpha_{s+1})$ and if $\theta(\alpha_i) = (a_{i,1}, ..., a_{i,s+1})$ then the matrix $\mathsf{A} = (a_{ij})$ gives rise to a lattice Γ and a corresponding set of linear forms ($\Gamma = (L_1, ..., L_{s+1})$ with $L_i = u_1a_{i,1} + ... + u_{s+1}a_{i,s+1}$). The determinant $\det\mathsf{A}$ is D with $\|D\| = q^{g+s}$, and

$\|L_1...L_{s+1}\| \geq 1$ for all $u_1, ..., u_{s+1}$ not all 0 in $\mathsf{k}[x]$. The proof of these assertions follows easily from [Arm3]. See also Example 3 below. By (3.19), Γ is the admissible lattice with $u = g$.

Example 3. [Arm3] Let $\mathsf{k} = \mathsf{k}(x) = \mathbb{F}_q(x)$, $\mathsf{k}[x]$ be defined as above and let K be a finite algebraic extension of k of degree $s+1$. Let ν be the valuation of k defined in (3.1) and let $\mathfrak{d}$ be the prime divisor of k corresponding to ν. Let $\mathcal{S} = \{\mathfrak{B}_1, ..., \mathfrak{B}_h\}$ be the set of extensions of $\mathfrak{d}$ to K. The corresponding normalized exponential valuations of K will be denoted by $\nu_1, ..., \nu_h$. Let $e_i,\ f_i$ denote the ramification index and residue class degree, respectively, of $\mathfrak{B}_i$ over $\mathfrak{d}$. Let $\hat{\mathsf{k}} = \mathbb{F}_q((x^{-1}))$, and let $\hat{\mathsf{K}}_i$ denote the perfect completion of K with respect to ν_i. The unique extensions of $\mathfrak{B}_i$ and ν_i to $\hat{\mathsf{K}}_i$ will be denoted by $\mathfrak{B}_i$ and ν_i. Set $\mathsf{K}_{\mathfrak{d}} = \hat{\mathsf{k}} \otimes_{\mathsf{k}} \mathsf{K}$. Then one has a canonical homomorphism, ρ, of $\hat{\mathsf{k}}$-algebras

$$\rho : \mathsf{K}_{\mathfrak{d}} \to \prod_{i=1}^{h} \hat{\mathsf{K}}_i$$

defined by a continuous extension of the canonical diagonal embedding $\psi = (\psi_1, ..., \psi_h)$

$$\psi_i : \mathsf{K} \to \hat{\mathsf{K}}_i, \quad 1 \leq i \leq h \qquad \text{and} \qquad \psi : \mathsf{K} \to \prod_{i=1}^{h} \hat{\mathsf{K}}_i, \tag{4.1}$$

([Bou], Chap. 6, §8, No. 2). By ([VS], p.137, or [Bou, Chap. 6, §8, No. 5, Th. 2, Cor. 2]) ρ is an isomorphism of $\hat{\mathsf{k}}$-algebras.

Write $[\hat{\mathsf{K}}_i : \hat{\mathsf{k}}] = n_i$. Then [VS, p.137] we have $e_i f_i = n_i, \quad n_1 + ... + n_h = s+1$.

As is known, there exists a $\mathfrak{B}_i$-integral basis for $\hat{\mathsf{K}}_i/\hat{\mathsf{k}}$ ([We], p. 52, Th. 2.3.2). In particular, such a basis is given by

$$\omega_{ij}\pi_i^l \quad (1 \leq j \leq f_i;\ 0 \leq l \leq e_i - 1)$$

where ω_{ij} are integral elements at $\mathfrak{B}_i$, whose residue class mod $\mathfrak{B}_i$ are linearly independent over the residue class field of k mod $\mathfrak{d}$, and π_i is a prime element for $\mathfrak{B}_i$ that is, $\nu_i(\pi_i) = 1$.

Then for $\alpha \in \mathsf{K}$, we have

$$\psi_i(\alpha) = \sum_{j=1}^{f_i} \sum_{l=0}^{e_i-1} \omega_{ij}\pi_i^l \alpha_{lf_i+j}^{(i)} \quad \text{with} \quad \alpha_{lf_i+j}^{(i)} \in \hat{\mathsf{k}} \tag{4.2}$$

and we define a k-linear injection

$$\theta_i : \mathsf{K} \to \hat{\mathsf{k}}^{n_i} \tag{4.3}$$

by

$$\theta_i(\alpha) = (\alpha_1^{(i)}, ..., \alpha_{n_i}^{(i)}) \qquad\qquad (\alpha_j^{(i)} \in \hat{\mathsf{k}}).$$

These maps define a k-linear injection $\theta = (\theta_1, ..., \theta_h)$

$$\theta : \mathsf{K} \to \hat{\mathsf{k}}^{s+1}. \tag{4.4}$$

At the same time, one has the $\hat{\mathsf{k}}$-linear injection

$$\vartheta : \prod_{i=1}^{h} \hat{\mathsf{K}}_i \to \hat{\mathsf{k}}^{s+1}. \tag{4.5}$$

For $\alpha \in \mathsf{K}$, we have

$$\theta(\alpha) = \vartheta(\psi(\alpha)). \tag{4.6}$$

Let $(\beta_1, ..., \beta_{s+1})$ be a basis of K. By [Bou, Chap. 6, §7, No.2, Th.1; §8, No.2, Prop.2], the set $\psi(\mathsf{K})$ is everywhere dense in $\mathsf{K}_{\mathfrak{d}} = \prod_{i=1}^{h} \hat{\mathsf{K}}_i$. Hence the set $\psi(\beta_1), ..., \psi(\beta_{s+1})$ generates $\mathsf{K}_{\mathfrak{d}}$ as a $\hat{\mathsf{k}}$ vector space. Bearing in mind that $\dim_{\hat{\mathsf{k}}}(\mathsf{K}_{\mathfrak{d}}) = s+1$, we obtain $\psi(\beta_1), ..., \psi(\beta_{s+1})$ is a basis of $\mathsf{K}_{\mathfrak{d}}$, and $\theta(\beta_1), ..., \theta(\beta_{s+1})$ is a basis of $\hat{\mathsf{k}}^{s+1}$. In particular, ϑ is a $\hat{\mathsf{k}}$-linear isomorphism. Let $\mathcal{O}$ denote the integral closure of $\mathsf{k}[x]$ in K. Denote by $\mathfrak{D}(\mathsf{K})$ the group of divisor of K. The group $\mathfrak{D}(\mathsf{K})$ can be written as a direct sum $\mathfrak{D}(\mathsf{K}) = \mathfrak{S} \oplus \mathcal{S}$, where $\mathfrak{S}$ and $\mathcal{S}$ are the groups of "finite" and "infinite" divisors respectively. A given divisor $\mathfrak{U} = \prod \mathfrak{B}^{\kappa(\mathfrak{B},\mathfrak{U})}$ (with $\kappa(\mathfrak{B},\mathfrak{U}) = \nu_{\mathfrak{B}}(\mathfrak{U})$) of K can be written in the form $\mathfrak{U} = \mathfrak{U}_e\mathfrak{U}_u$ with

$$\mathfrak{U}_e = \prod_{\mathfrak{B}\in\mathfrak{S}} \mathfrak{B}^{\kappa(\mathfrak{B},\mathfrak{U})}, \quad \mathfrak{U}_u = \prod_{\mathfrak{B}\in\mathcal{S}} \mathfrak{B}^{\kappa(\mathfrak{B},\mathfrak{U})}. \tag{4.7}$$

We set

$$\begin{aligned} L(\mathfrak{U}_e) &= L(\mathfrak{U}_e, \mathfrak{S}) = \{\alpha \in \mathsf{K} \mid \nu_{\mathfrak{B}}(\alpha) \geq \nu_{\mathfrak{B}}(\mathfrak{U}), \;\; \mathfrak{B} \in \mathfrak{S}\}, \\ L(\mathfrak{U}_u) &= L(\mathfrak{U}_u, \mathcal{S}) = \{\alpha \in \mathsf{K} \mid \nu_{\mathfrak{B}}(\alpha) \geq \nu_{\mathfrak{B}}(\mathfrak{U}), \;\; \mathfrak{B} \in \mathcal{S}\}. \end{aligned} \tag{4.8}$$

Now $L(\mathfrak{U}_e)$ is an $\mathcal{O}$-ideal. By ([ZS], p. 267, Th.9), $L(\mathfrak{U}_e)$ has an $\mathsf{k}[x]$-basis of $s+1$ elements. Hence $\Gamma(\mathfrak{U}) = \theta(L(\mathfrak{U}_e))$ is a lattice in $\hat{\mathsf{k}}^{s+1}$. In particular, $\Gamma_{\mathcal{O}} = \theta(\mathcal{O})$ is a lattice in $\hat{\mathsf{k}}^{s+1}$. Let $\Gamma(\mathfrak{U})$ be the lattice defined by $L(\mathfrak{U}_e)$.

By ([Arm3], eq. (38)-(40) and (44)), we have

$$\|\det\Gamma(\mathfrak{U})\| = q^{g+s+\delta(\mathfrak{U})} \quad \text{with} \quad \delta(\mathfrak{U}) = \sum_{\mathfrak{B}\in\mathfrak{S}} \deg(\mathfrak{B})\nu_{\mathfrak{B}}(\mathfrak{U}), \tag{4.9}$$

where g is the genus of K. In particular,

$$\|\det\Gamma_{\mathcal{O}}\| = q^{g+s}.$$

Now let $\mathfrak{U} = \mathfrak{B}_1^{a_1}...\mathfrak{B}_d^{a_h}$. We define

$$\hat{L}(\mathfrak{U}, \mathcal{S}) := \{\tilde{\alpha} = (\tilde{\alpha}_1, ..., \tilde{\alpha}_h) \in \prod_{i=1}^{h} \hat{\mathsf{K}}_i \mid \nu_{\mathfrak{B}_i}(\tilde{\alpha}_i) \geq a_i = \nu_{\mathfrak{B}_i}(\mathfrak{U}), \quad i = 1, ..., h\} \tag{4.10}$$

and

$$\vartheta(\hat{L}(\mathfrak{U}), \mathcal{S}) := \tilde{\mathcal{V}}(a_1, ..., a_h). \tag{4.11}$$

Let $y = (y_1^{(1)}, ..., y_{n_1}^{(1)}, ..., y_1^{(h)}, ..., y_{n_h}^{(h)}) \in \hat{\mathsf{k}}^{s+1}$. We consider the isomorphism (4.5) and the representation (4.2). We see that

$$\breve{y}_i = \sum_{j=1}^{f_i} \sum_{l=0}^{e_i-1} \omega_{ij}\pi_i^l y_{lf_i+j}^{(i)} \quad 1 \leq i \leq h, \qquad \breve{y} = (\breve{y}_1, ..., \breve{y}_h) = \vartheta^{-1}(y).$$

By (4.10) and (4.11), we have

$$y \in \tilde{\mathcal{V}}(a_1, ..., a_h) \Longleftrightarrow \nu_i(\breve{y}_i) = \nu_i(\sum_{j=1}^{f_i}\sum_{l=0}^{e_i-1} \omega_{ij}\pi_i^l y^{(i)}_{lf_i+j}) \geq a_i, \qquad 1 \leq i \leq h. \tag{4.12}$$

For some integer m_i, we have $a_i = m_i e_i + r_i,\ 0 \leq r_i < e_i\ \ (1 \leq i \leq h)$.
Let $\mathsf{a} = (\mathsf{a}_1^{(1)}, ..., \mathsf{a}_{n_1}^{(1)}, ..., \mathsf{a}_1^{(h)}, ..., \mathsf{a}_{n_h}^{(h)}) \in \mathbb{Z}^{s+1}$ with

$$\mathsf{a}^{(i)}_{lf_i+j} = \begin{cases} m_i + 1, & \text{for} \quad 0 \leq l \leq r_i - 1 \\ m_i, & \text{for} \quad r_i \leq l \leq e_i - 1, \quad 1 \leq j \leq f_i,\ 1 \leq i \leq h. \end{cases} \tag{4.13}$$

According to [Arm3, eq. (27),(28)], (4.12) is equivalent to

$$y \in \tilde{\mathcal{V}}(a_1, ..., a_h) \Longleftrightarrow \nu(y^{(i)}_{lf_i+j}) \geq \mathsf{a}^{(i)}_{lf_i+j} \quad 0 \leq l \leq e_i - 1,\ 1 \leq j \leq f_i,\ 1 \leq i \leq h.$$

Using (3.2) and (3.15), we see that

$$f_1 a_1 + ... + f_h a_h = \sum_{1 \leq i \leq h}\sum_{1 \leq j \leq f_i}\sum_{0 \leq l < e_i} \mathsf{a}^{(i)}_{lf_i+j} \quad \text{and} \quad \mathcal{V}(\mathsf{a})^{\perp} = \tilde{\mathcal{V}}(a_1, ..., a_h). \tag{4.14}$$

4.2. Construction of (t, s) sequences. Let

$$\gamma = (\gamma_1^{(1)} ..., \gamma_{n_1}^{(1)}, ..., \gamma_1^{(h)}, ..., \gamma_{n_h}^{(h)}) \in \Gamma_{\mathcal{O}}^{\perp}$$

with

$$\gamma_j^{(i)} = \sum_{k \geq -w_{i,m}(\gamma)} \gamma^{(i)}_{m,k} x^{-k}, \quad \text{and} \quad \gamma^{(i)}_{m,k} \in \mathbb{F}_q, \quad 1 \leq m \leq n_i. \tag{4.15}$$

Let $\eta^{(i)}_{m,k}$ be a one-to-one map from $\mathbb{F}_q$ to $\{0, 1, ..., q-1\}$ with $\eta^{(i)}_{m,k}(0) = 0$, and let

$$\xi(\gamma) = (\xi(\gamma)_1, ..., \xi(\gamma)_h)$$

$$\text{with} \quad \xi(\gamma)_i = \sum_{k \leq w^{(i)}(\gamma)}\sum_{1 \leq j \leq f_i}\sum_{0 \leq l < e_i} \eta^{(i)}_{f_i l+j,k}(\gamma^{(i)}_{f_i l+j,k}) q^{e_i f_i k + f_i l + j - 1}.$$

where $w^{(i)}(\gamma) = \max_{1 \leq m \leq e_i f_i} w_{i,m}(\gamma)$.

Let $\xi(\Gamma_{\mathcal{O}}^{\perp}) = \{\xi(\gamma) \mid \gamma \in \Gamma_{\mathcal{O}}^{\perp}\}$

$$W = \xi(\Gamma_{\mathcal{O}}^{\perp}) \cap [0, 1)^{h-1} \times [0, +\infty).$$

We have that for all $v \in \mathbb{R}^h$ the set $\xi(\Gamma_{\mathcal{O}}^{\perp}) \cap ([0, 1]^h + v)$ is finite. We see that $(0, ..., 0) \in W$, and $\#W \geq 1$. Let $(u_i, u_{i,h}) \in W$ with $u_i \in \mathbb{R}^{h-1}$ and $u_{i,h} \in \mathbb{R}$, $i = 1, 2$, and $u_{1,h} = u_{2,h}$. Hence $\theta_h^{-1}(\xi^{-1}((u_1, u_{1,h})) = \theta_h^{-1}(\xi^{-1}((u_2, u_{2,h})) \in \mathsf{K}$. Applying (4.3)-(4.4), we have that $u_1 = u_2$. Thus W can be enumerated by a sequence $(z(n), z_h(n))_{0 \leq n < \#W}$ in the following way:

$$z(n) = (z_1(n), ..., z_{h-1}(n)), \quad z_i(n) \in \mathbb{R}, \quad z_i(0) = 0, \ \ i = 1, ..., h,$$
$$\text{and} \qquad z_h(n) < z_h(n+1) \in \mathbb{R}, \quad \text{for} \ \ n = 0, 1, ... \tag{4.16}$$

Now let $\mathfrak{b}_1, ..., \mathfrak{b}_d$ be pairwise coprime integer divisors with $\mathfrak{b}_i = \mathfrak{b}_{e,i}$ (see 4.7), and $\mathsf{f}_{\mathfrak{b}_i} = \deg(\mathfrak{b}_i) \geq 2$ $(i = 1, ..., d)$. Let $i \in [1, d]$. A digit set $\mathcal{D}_{i,k} \subset \Gamma(\mathfrak{b}_i^{-k})^\perp$ associated with $\mathfrak{b}_i$ is any complete set of coset representatives for $\Gamma(\mathfrak{b}_i^{-k})^\perp / \Gamma(\mathfrak{b}_i^{-k-1})^\perp$ $k \geq 0$, where $\Gamma(\mathfrak{b}_i^0)^\perp = \Gamma_{\mathcal{O}}^\perp$. By (4.9), we get

$$\#\mathcal{D}_{i,k} = q^{\mathsf{f}_{\mathfrak{b}_i}}, \;\; k \geq 0.$$

We have that, for any $\gamma \in \Gamma_{\mathcal{O}}^\perp$ and every $m \geq 1$,

$$\gamma = d_0 + d_1 + ... + d_{m-1} + x_m \tag{4.17}$$

where $d_{i,k} \in \mathcal{D}_{i,k}, \;\; k \in [0, m-1]$ and $x_m \in \Gamma(\mathfrak{b}_i^{-m})^\perp$. So for each $\gamma \in \Gamma_{\mathcal{O}}^\perp$, we can associate a unique sequence $(d_{i,0}, d_{i,1}, d_{i,2}, ...)$. Let $\eta_{i,k}$ be a one-to-one map from $\mathcal{D}_{i,k}$ to $\{0, 1, ..., q^{\mathsf{f}_{\mathfrak{b}_i}} - 1\}$,

$$\phi_{\mathfrak{b}_i}(\gamma) = \sum_{j \geq 0} \eta_{i,j}(d_{i,j}) / q^{(j+1)\mathsf{f}_{\mathfrak{b}_i}}, \tag{4.18}$$

and let

$$\zeta(n) = (\varphi_{\mathfrak{b}_1}(n), ..., \varphi_{\mathfrak{b}_d}(n), z(n)) \tag{4.19}$$

where $\varphi_{\mathfrak{b}_i}(n) = \phi_{\mathfrak{b}_i}(\xi^{-1}(z(n), z_h(n)))$.

Theorem 4.1. *With the above notation, $(\zeta(n))_{n \geq 0}$ is a $(t, h+d-1)$ sequence with $t = g + f_1 + ... + f_h + \mathsf{f}_{\mathfrak{b}_1} + ... + \mathsf{f}_{\mathfrak{b}_d} - h - d$.*

4.3. Proof Theorem 4.1. First, we need the following variant of the Chinese Remainder Theorem :

Lemma 4.1. *Let $\mathfrak{N}_1, \mathfrak{N}_2$ be pairwise coprime integer divisors, and let $m_i = \deg(\mathfrak{N}_i)$, $\Gamma_i = \Gamma(\mathfrak{N}_i^{-1})$, $i = 1, 2$. Then for all $\alpha_1, \alpha_2 \in \Gamma_{\mathcal{O}}^\perp$, there exists $\alpha \in \Gamma_{\mathcal{O}}^\perp$ with $\alpha \equiv \alpha_i (\mathrm{mod}\ \Gamma_i^\perp)$, and*

$$\{\gamma \in \Gamma_{\mathcal{O}}^\perp \mid \gamma \equiv \alpha_i \ (\mathrm{mod}\ (\Gamma_i^\perp)), \ i = 1, 2\} = \{\gamma \in \Gamma_{\mathcal{O}}^\perp \mid \gamma \equiv \alpha \ (\mathrm{mod}\ \Gamma(\mathfrak{N}_1^{-1}\mathfrak{N}_2^{-1})^\perp)\}.$$

Proof. By the Chinese Remainder Theorem, we have $L(\mathfrak{N}_1^{-1}\mathfrak{N}_2^{-1})$=$L(\mathfrak{N}_1^{-1}) \cup L(\mathfrak{N}_2^{-1})$. Hence $\Gamma(\mathfrak{N}_1^{-1}\mathfrak{N}_2^{-1})$ =$\Gamma(\mathfrak{N}_1^{-1}) \cup \Gamma(\mathfrak{N}_2^{-1})$. By (4.9), we get

$$\begin{aligned} \#\{\Gamma_i / \Gamma_{\mathcal{O}}\} &= \|\det(\Gamma_i)/\det(\Gamma_{\mathcal{O}})\| = q^{m_i}, \quad i = 1, 2, \quad \text{and} \\ \#\{(\Gamma_1 \cup \Gamma_2)/\Gamma_{\mathcal{O}}\} &= \left\|\det(\Gamma(\mathfrak{N}_1^{-1}\mathfrak{N}_2^{-1})/\det(\Gamma_{\mathcal{O}})\right\| = q^{m_1+m_2}. \end{aligned} \tag{4.20}$$

It is easy to prove that

$$(\Gamma_1 \cup \Gamma_2)^\perp = \Gamma_1^\perp \cap \Gamma_2^\perp. \tag{4.21}$$

In fact, let $\beta \in (\Gamma_1 \cup \Gamma_2)^\perp$. Then for all $y \in \Gamma_1 \cup \Gamma_2$ we have $< \beta, y > \in \mathsf{k}[x]$. Hence $\beta \in \Gamma_i^\perp$ for $i = 1, 2$. Now let $\beta \in \Gamma_1^\perp \cap \Gamma_2^\perp$. Then $< \beta, y > \in \mathsf{k}[x]$ for all $y \in \Gamma_i, \ i = 1, 2$. Thus $\beta \in (\Gamma_1 \cup \Gamma_2)^\perp$.

By (4.20), (4.21) and (3.13), we get

$$\#\{\Gamma_{\mathcal{O}}^\perp / \Gamma_i^\perp\} = q^{m_i}, \quad i = 1, 2 \quad \text{and} \quad \#\{\Gamma_{\mathcal{O}}^\perp / (\Gamma_1^\perp \cap \Gamma_2^\perp)\} = q^{m_1+m_2}.$$

Let $\Gamma_3 = \Gamma_1^\perp \cap \Gamma_2^\perp$. Bearing in mind that $\Gamma_{\mathcal{O}}^\perp \supset \Gamma_1^\perp \supset \Gamma_3$, we obtain

$$(\Gamma_{\mathcal{O}}^\perp/\Gamma_3)/(\Gamma_1^\perp/\Gamma_3) \cong \Gamma_{\mathcal{O}}^\perp/\Gamma_1^\perp.$$

Therefore $\#\{\Gamma_1^\perp/\Gamma_3\} = q^{m_2}$. Now let $\beta_1, ..., \beta_l \in \Gamma_1^\perp$ be any complete set of coset representatives for $\Gamma_1^\perp/\Gamma_3$ with $l = q^{m_2}$. Suppose that $\alpha_1 + \beta_k \equiv \alpha_1 + \beta_j \pmod{\Gamma_2^\perp}$ for some $k, j \in [1, l],\ k \neq j$. Then $\beta_k \equiv \beta_j \pmod{\Gamma_i^\perp}$ for $i = 1, 2$. So $\beta_k \equiv \beta_j \pmod{\Gamma_3}$. We have a contradiction. Hence $\alpha_1 + \beta_1, ..., \alpha_1 + \beta_l$ is the complete set of coset representatives for $\Gamma_{\mathcal{O}}^\perp/\Gamma_2^\perp$. Thus there exists $j \in [1, l]$ with $\alpha_2 \equiv \alpha_1 + \beta_j \pmod{\Gamma_2^\perp}$. Lemma 4.1 is proved $\blacksquare$

We obtain immediately by induction the following assertion:

Corollary 4.1. *Let $k_1, ..., k_d \geq 0$ be integers. Then for all $\alpha_1, ..., \alpha_d \in \Gamma_{\mathcal{O}}^\perp$, there exists $\alpha \in \Gamma_{\mathcal{O}}^\perp$ with $\alpha \equiv \alpha_i \pmod{\Gamma(\mathfrak{b}_i^{-k_i})^\perp}$, and*

$$\{\gamma \in \Gamma_{\mathcal{O}}^\perp \mid \gamma \equiv \alpha_i \pmod{\Gamma^\perp(\mathfrak{b}_i^{-k_i})},\ i = 1, ..., d\} = \{\gamma \in \Gamma_{\mathcal{O}}^\perp \mid \gamma \equiv \alpha \pmod{\Gamma(\mathfrak{b}_1^{-k_1} ... \mathfrak{b}_d^{-k_d})^\perp}\},$$

where $\mathfrak{b}_1, ..., \mathfrak{b}_d$ are pairwise coprime integer divisors.

Lemma 4.2. *Let $\mathfrak{N}$ be an integer divisor with $\mathfrak{N} = \mathfrak{N}_e$, $\mathsf{f} = \deg(\mathfrak{N})$, $z = (z_1, ..., z_{s+1}) \in \hat{\mathsf{k}}^{s+1}$, a_i integers $1 \leq i \leq h$, $\mathsf{a} \in \mathbb{Z}^{s+1}$ defined in (4.13), $c = (c_1, ..., c_{s+1})$, $c_{n_1+...+n_{i-1}+j} \geq \mathsf{a}_j^{(i)}$ $(1 \leq j \leq n_i,\ 1 \leq i \leq h)$, and let $f_1 a_1 + ... + f_h a_h - \mathsf{f} > 0$. Then*

$$\Gamma(\mathfrak{N}^{-1})^\perp \cap \{\mathcal{V}(c-2) + z\} = q^{c_1+...+c_{s+1}-c_0-2s-2},$$

where $c_0 = -g - s + \delta(\mathfrak{N})$.

Proof. Suppose that there exists

$$\gamma \in \mathcal{V}(\mathsf{a})^\perp \cap \Gamma(\mathfrak{N}^{-1}) \setminus \{o\}. \tag{4.22}$$

By (4.14), we obtain $\gamma \in \tilde{\mathcal{V}}(a_1, ..., a_h)$. Let $\breve{\gamma} = (\breve{\gamma}_1, ..., \breve{\gamma}_h) = \vartheta^{-1}(\gamma)$. According to (4.11)-(4.13), we get

$$\nu_i(\breve{\gamma}_i) \geq a_i, \quad 1 \leq i \leq h.$$

We have $\theta^{-1}(\gamma) \in \mathsf{K}$. Using (4.1) and (4.6), we obtain

$$\psi(\theta^{-1}(\gamma)) = \breve{\gamma}, \qquad \text{and} \quad \psi_i(\theta^{-1}(\gamma)) = \breve{\gamma}_i, \quad 1 \leq i \leq h.$$

Hence

$$\nu_i(\psi_i(\theta^{-1}(\gamma))) \geq a_i, \quad 1 \leq i \leq h \tag{4.23}$$

and

$$\nu_i(\theta^{-1}(\gamma)) \geq a_i, \quad 1 \leq i \leq h. \tag{4.24}$$

Using (4.22) and (4.8), we get

$$\nu_{\mathfrak{B}}(\theta^{-1}(\gamma)) \geq \nu_{\mathfrak{B}}(\mathfrak{N}^{-1}) \quad \text{for all} \quad \mathfrak{B} \in \mathfrak{S}. \tag{4.25}$$

Let $\mathfrak{U}_1 = \mathfrak{N}\mathfrak{B}^{-a_1} ... \mathfrak{B}^{-a_h}$. By (4.24) and(4.25), we have

$$\nu_{\mathfrak{B}}(\theta^{-1}(\gamma)) + \nu_{\mathfrak{B}}(\mathfrak{U}_1) \geq 0 \quad \forall \mathfrak{B} \in \mathfrak{D}.$$

Thus $\theta^{-1}(\gamma)$ belong the Riemann-Roch space of the divisor $\mathfrak{U}_1$ (see, for example, [NiXi, p. 5]). Bearing in mind that

$$\deg(\mathfrak{N}\mathfrak{B}^{-a_1}...\mathfrak{B}^{-a_h}) = \mathsf{f} - f_1 a_1 - ... - f_h a_h < 0,$$

we get that the Riemann-Roch space of the divisor $\mathfrak{U}_1$ is empty. Hence supposition (4.22) is false: $\mathcal{V}(\mathsf{a})^{\perp} \cap \Gamma(\mathfrak{N}^{-1}) \setminus \{o\} = \emptyset$. Taking into account that $c_{n_1+...+n_{i-1}+j} \geq \mathsf{a}_j^{(i)}$ $(1 \leq j \leq n_i,\ 1 \leq i \leq h)$, we obtain $\mathcal{V}(c)^{\perp} \subseteq \mathcal{V}(\mathsf{a})^{\perp}$. Therefore

$$\mathcal{V}()^{\perp} \cap \Gamma(\mathfrak{N}^{-1}) \setminus \{\mathfrak{o}\} = \emptyset.$$

According to (4.9), $\left\|\det\Gamma(\mathfrak{N}^{-1})^{\perp}\right\| = q^{-g-s+\delta(\mathfrak{N})}$. Now using Lemma 3.2 with $\Gamma = \Gamma(\mathfrak{N}^{-1})^{\perp}$, we obtain the assertion of Lemma 4.2. ■

End of Proof of Theorem 4.1. Let $G_1, ..., G_{d+h}, l_1, ..., l_{d+h} \geq 0$ be integers, $G_i < q^{l_i}$ $(1 \leq i \leq d+h-1)$, $l_{d+h} = l_1 + ... + l_{d+h-1} + t$, and let

$$S = [\frac{G_1}{q^{l_1}}, \frac{G_1+1}{q^{l_1}}) \times ... \times [\frac{G_{d+h-1}}{q^{l_{d+h-1}}}, \frac{G_{d+h-1}+1}{q^{l_{d+h-1}}}) \times [G_{d+h}q^{l_{d+h}}, (G_{d+h}+1)q^{l_{d+h}}).$$

We need to prove

$$\#\{n \geq 0 \mid (\zeta(n), n) \in S\} = q^t. \tag{4.26}$$

Let

$$l_i = \mathsf{f}_{\mathfrak{b}_i} k_i - p_i, \quad \text{with} \quad 0 \leq p_i < \mathsf{f}_{\mathfrak{b}_i}\ 1 \leq i \leq d,$$

and let

$$G_i^{'} = G_i q^{p_i},\ G_i^{''} = (G_i+1)q^{p_i} \quad 1 \leq i \leq d.$$

Now let

$$S(H) = I_1 \times ... \times I_d \times S_1 \times I_{d+h}, \qquad \text{and} \qquad S_1 = I_{d+1} \times ... \times I_{d+h-1},$$

where

$$I_j = [\frac{H_j}{q^{\mathsf{f}_{\mathfrak{b}_j} k_j}}, \frac{H_j+1}{q^{\mathsf{f}_{\mathfrak{b}_j} k_j}}), \quad I_{d+i} = [\frac{G_{d+i}}{q^{l_{d+i}}}, \frac{G_{d+i}+1}{q^{l_{d+i}}}), \quad I_{d+h} = [G_{d+h}q^{l_{d+h}}, (G_{d+h}+1)q^{l_{d+h}}),$$

with $1 \leq j \leq d,\ 1 \leq i < h$. We see that

$$S = \bigcup_{G_1^{'} \leq H_1 < G_1^{''}} ... \bigcup_{G_{d+h}^{'} \leq H_{d+h} < G_{d+h}^{''}} S(H).$$

Hence to obtain (4.26), it is sufficient to prove that

$$\#\{n \geq 0 \mid (\zeta(n), n) \in S(H)\} = q^{t_1} \tag{4.27}$$

with $t_1 = t - p_1 - ... - p_d$.

Let

$$-l_{d+i}-1=f_i(v_{i,1}e_i+v_{i,2})+v_{i,3}, \qquad l_{d+h}-g=f_h(v_{h,1}e_h+v_{h,2})+v_{h,3}$$

with $0\leq v_{i,2}<e_i,\ 0\leq v_{i,3}<f_i,\ 1\leq i\leq h$. We see that $v_{i,1}<0$ for $1\leq i<h$. We define $c=(c_1,...,c_{s+1})$ and $\mathsf{a}=(\mathsf{a}_1^{(1)}...,\mathsf{a}_{n_1}^{(1)},...,\mathsf{a}_1^{(h)},...,\mathsf{a}_{n_h}^{(h)})$ as follows:

$$c_{n_1+...+n_{i-1}+lf_i+j}=\begin{cases} v_{i,1}+2, & \text{for } \ 0\leq l\leq v_{i,2}-1 \quad \text{or} \ \ l=v_{i,2} \ \text{ and } \ j\leq v_{i,3}+1 \\ v_{i,1}+1, & \textit{otherwise}, \qquad\quad 1\leq j\leq f_i,\ 1\leq i\leq h \end{cases}$$

and

$$\mathsf{a}_{lf_i+j}^{(i)}=\begin{cases} v_{i,1}+2, & \text{for} \quad 0\leq l\leq v_{i,2}-1 \\ v_{i,1}+1, & \textit{otherwise}, \qquad\quad 1\leq j\leq f_i,\ 1\leq i\leq h. \end{cases}$$

It is easy to see that

$$\mathsf{a}_{lf_i+j}^{(i)}\leq c_{n_1+...+n_{i-1}+lf_i+j} \qquad \text{for} \quad 1\leq l\leq e_i,\ 1\leq j\leq f_i,\ 1\leq i\leq h \tag{4.28}$$

and

$$\sum_{1\leq j\leq f_i}\sum_{0\leq l<e_i} c_{n_1+...+n_{i-1}+lf_i+j}=(v_{i,1}+1)f_ie_i+f_iv_{i,2}+v_{i,3}+1, \qquad 1\leq i\leq h.$$

Hence

$$c_1+...+c_{s+1}=l_{d+h}-l_{d+1}-...-l_{d+h-1}-g+s+2. \tag{4.29}$$

We have similarly that

$$\sum_{1\leq j\leq f_i}\sum_{0\leq l<e_i}\mathsf{a}_{lf_i+j}^{(i)}=(v_{i,1}+1)f_ie_i+f_iv_{i,2}.$$

Now we define $a_1,...,a_h$ according to (4.13). By (4.13), we have

$$\begin{aligned} f_1a_1+...+f_ha_h&=\textstyle\sum_{1\leq i\leq h}\sum_{1\leq j\leq f_i}\sum_{0\leq l<e_i}\mathsf{a}_{lf_i+j}^{(i)}=\sum_{1\leq i\leq h}(v_{i,1}+1)f_ie_i+f_iv_{i,2} \\ &=l_{d+h}-l_{d+1}-...-l_{d+h-1}-g+s+2-h-v_{1,3}-...-v_{h,3} \end{aligned}$$

Hence

$$\begin{aligned} f_1a_1+...+f_ha_h+\deg(\mathfrak{b}_1^{-k_1}....\mathfrak{b}_d^{-k_d})&=f_1a_1+...+f_ha_h-k_1\mathfrak{f}_{\mathfrak{b}_1}-...-k_d\mathfrak{f}_{\mathfrak{b}_d} \\ =l_{d+h}-l_{d+1}-...-l_{d+h-1}-g+s+2-h&-v_{1,3}-...-v_{h,3}-l_1-...-l_d-p_1-...-p_d \\ =t-g+s+2-h-v_{1,3}-...&-v_{h,3}-p_1-...-p_d\geq s+2-h\geq 1. \end{aligned} \tag{4.30}$$

Consider the decomposition (4.15). Let

$$\begin{aligned} \gamma(n)&=(\gamma_1^{(1)}(n),...,\gamma_{n_1}^{(1)}(n),...,\gamma_1^{(h)}(n),...,\gamma_{n_h}^{(h)}(n))=\xi^{-1}(z(n),z_h(n))\in\Gamma_{\mathcal{O}}^{\perp}, \\ z&=(z_1^{(1)}...,z_{n_1}^{(1)},...,z_1^{(h)},...,z_{n_h}^{(h)})=\xi^{-1}(G_{d+1}q^{-l_1},...,G_{d+h-1}q^{-l_{d+h-1}},G_{d+h}q^{l_{d+h}}). \end{aligned}$$

It is easy to verify that

$$(z(n), z_h(n)) \in S_1 \times I'_{d+h} \Longleftrightarrow \nu(\gamma^{(i)}_{lf_i+j}(n) - z^{(i)}_{lf_i+j}) \geq c_{n_1+...+n_{i-1}+lf_i+j} - 2$$

for all $0 \leq l \leq e_i - 1,\ 1 \leq j \leq f_i,\ 1 \leq i \leq h$, where

$$I'_{d+h} = q^{-g+1} I_{d+h} = [G_{d+h} q^{l_{d+h}-g+1}, (G_{d+h}+1) q^{l_{d+h}-g+1})$$

(we need the factor q^{-g+1} to prove (4.33)). Hence

$$(z(n), z_h(n)) \in S_1 \times I'_{d+h} \Longleftrightarrow \gamma(n) \in \mathcal{V}(c-2) + z.$$

By (4.17)-(4.19), we have

$$\varphi_{\mathfrak{b}_j}(n) \in I_j \Longleftrightarrow \xi^{-1}(z(n), z_h(n)) \equiv w_j (\bmod\ \Gamma(\mathfrak{b}_i^{-k_i})^{\perp})$$

for some $w_j \in \Gamma^{\perp}_{\mathcal{O}},\ j = 1, ..., d$.
Using Corollary 4.1, we get

$$(\varphi_{\mathfrak{b}_1}(n), ..., \varphi_{\mathfrak{b}_d}(n)) \in I_1 \times ... \times I_d \Longleftrightarrow \xi^{-1}(z(n), z_h(n)) \equiv w_0\ (\bmod\ \Gamma(\mathfrak{b}_1^{-k_1} .... \mathfrak{b}_d^{-k_d})^{\perp}) \tag{4.31}$$

for some $w_0 \in \Gamma^{\perp}_{\mathcal{O}},\ 1 \leq i \leq d$. By (4.31), we have

$$\begin{aligned}(z(n), z_h(n)) \in I_1 \times ... \times I_d \times S_1 \times I'_{d+h} \Longleftrightarrow \xi^{-1}((z(n), z_{s+1}(n))) \equiv w_0 \\ (\bmod\ \Gamma(\mathfrak{b}_1^{-k_1} .... \mathfrak{b}_d^{-k_d})^{\perp}) \quad \text{and} \quad \xi^{-1}((z(n), z_{s+1}(n))) \in \mathcal{V}(c-2) + z.\end{aligned} \tag{4.32}$$

Therefore

$$\begin{aligned} q^{\rho_1} &:= \#\{n \geq 0 \mid (\zeta(n), z_h(n)) \in I_1 \times ... \times I_d \times S_1 \times I'_{d+h}\} \\ &= \#\{\gamma \in \Gamma(\mathfrak{b}_1^{-k_1} .... \mathfrak{b}_d^{-k_d})^{\perp} \mid \gamma - w_0 \in \mathcal{V}(c-2) + z)\}. \end{aligned}$$

Bearing in mind (4.28) and (4.30), we get that the suppositions of Lemma 4.2 are true. Thus

$$\rho_1 = c_1 + ... + c_{s+1} - c_0 - 2s - 2,$$

where $c_0 = \left\| \det \Gamma(\mathfrak{b}_1^{-k_1} .... \mathfrak{b}_d^{-k_d})^{\perp} \right\|$. By (4.9), we get

$$c_0 = -g - s + k_1 \mathfrak{f}_{\mathfrak{b}_1} + ... + k_d \mathfrak{f}_{\mathfrak{b}_d} = l_1 + ... + l_d + p_1 + ... + p_d - g - s.$$

According to (4.29), we have

$$c_1 + ... + c_{s+1} - c_0 - 2s - 2 = l_{d+h} - l_1 - ... - l_{d+h-1} - g + s + 2 - p_1 - ... - p_d + g + s - 2s - 2 = t_1.$$

Therefore the assertion

$$\#\{n \geq 0 \mid (\zeta(n), z_h(n)) \in I_1 \times ... \times I_d \times S_1 \times I'_{d+h}\} = q^{t_1}$$

is true for all $l_1, ..., l_{d+h} \geq 0$ with $l_{d+h} = l_1 + ... + l_{d+h-1} + t$. In particular, for $l_i = 0,\ i = 1, ...d + h - 1$ and $l_{d+h} = t$, we obtain

$$\#\{n \geq 0 \mid z_h(n) \in [Bq^{t-g+1}, (B+1)q^{t-g+1})\} = q^t. \tag{4.33}$$

for all $B \geq 0$. Hence

$$\#\{n \geq 0 \mid z_h(n) \in [Bq^{l_{d+h}-g+1}, (B+1)q^{l_{d+h}-g+1})\} = q^{l_{d+h}}$$

for all $B \geq 0$ and $l_{d+h} \geq t$. Thus

$$\#\{n \geq 0 \mid (\zeta(n), n) \in S(H)\} = q^{t_1}.$$

Hence assertion (4.27) and Theorem 4.1 are proved. ■

Acknowledgment. I am very gratiful to twe referee for many corrections and suggestions which improved this paper.

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2000 Mathematics Subject Classification. Primary 11K38.