ON FUGLEDE'S CONJECTURE FOR THREE INTERVALS

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ABSTRACT. In this paper, we prove the *Tiling implies Spectral* part of Fuglede's cojecture for the three interval case. Then we prove the converse *Spectral implies Tiling* in the case of three equal intervals and also in the case where the intervals have lengths 1/2, 1/4, 1/4. Next, we consider a set $\Omega \subset \mathbb{R}$, which is a union of n intervals. If Ω is a spectral set, we prove a structure theorem for the spectrum provided the spectrum is assumed to be contained in some lattice. The method of this proof has some implications on the *Spectral implies Tiling* part of Fuglede's conjecture for three intervals. In the final step in the proof, we need a symbolic computation using *Mathematica*. Finally with one additional assumption we can conclude that the *Spectral implies Tiling* holds in this case.

1. Introduction

Let Ω be a Lebesgue measurable subset of $\mathbb{R}$ with finite positive measure. For $\lambda \in \mathbb{R}$, let

$$e_\lambda(x) = \frac{1}{|\Omega|^{1/2}} e^{2\pi i \lambda x} \chi_\Omega(x), \quad x \in \mathbb{R}.$$

Ω is said to be a *spectral set* if there exists a subset $\Lambda \subset \mathbb{R}$, such that the set $E_\Lambda = \{e_\lambda : \lambda \in \Lambda\}$ is an orthonormal basis for the Hilbert space $L^2(\Omega)$, and then the pair (Ω, Λ) is called a *spectral pair.*

We say that Ω as above *tiles* $\mathbb{R}$ by translations if there exists a subset $\mathcal{T} \subset \mathbb{R}$, such that the set $\{\Omega + t : t \in \mathcal{T}\}$, consisting of the translates of Ω by $\mathcal{T}$, forms a partition a.e. of $\mathbb{R}$. The pair $(\Omega, \mathcal{T})$ is called a *tiling pair*. These definitions clearly extend to $\mathbb{R}^d$, $d > 1$.

The study of the relationship between tiling and spectral properties of measurable sets started with a conjecture proposed by Bent Fuglede in 1974.

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Fuglede's Conjecture. Let Ω be a measurable set in $\mathbb{R}^d$ with finite positive measure. Then Ω is spectral if and only if Ω tiles $\mathbb{R}^d$ by translations.

Fuglede proved this conjecture in $\mathbb{R}^d$ under the additional assumption that the spectrum, or the tiling set, is a d-dimensional lattice [F]. In recent years there has been a lot of activity on this problem. It is now known that, in this generality, the conjecture is false in both directions if the dimension $d \geq 3$ ([T], [M], [KM2]) and ([KM1], [FR], [FMM]). However, the conjecture is still open in all dimensions $d \geq 3$ under the additional hypothesis that the set Ω is a convex set. For convex sets, the conjecture is trivial for $d = 1$, and for $d = 2$, it was proved in [IKT1], [IKT2] and [K]. In dimension 1, the problem has been shown to be related to some number-theoretic questions, which are of independent interest and many partial results supporting the conjecture are known (see, for example [LW2], [L2], [K2], [PW]).

In this paper, we restrict ourselves to one dimension and to the case when the set Ω is a union of three intervals. This work was inspired by the paper [L1] by I. Laba, where Fuglede's conjecture is proved for the case that Ω is a union of two intervals. We state Laba's Theorem here in order to put the main result of this paper in perspective.

Theorem [L1]. Let $\Omega = [0, r) \cup [a, a+1-r)$, with $0 < r \leq 1/2,\ a \geq r$. Then the following are equivalent;

(1) Ω is spectral.
(2) Either (i) $a - r \in \mathbb{Z}$, or (ii) $r = 1/2$, $a = n/2$ for some $n \in \mathbb{Z}$.
(3) Ω tiles $\mathbb{R}$.

Further, when Ω is spectral, then $\Lambda = \mathbb{Z}$ if 2(i) holds, and $\Lambda = 2\mathbb{Z} \cup (2\mathbb{Z} + p/n)$ for some odd integer p, if 2(ii) holds.

In section 2, we prove that if Ω is a union of three intervals, then "Tiling implies Spectral". This proof, though somewhat long, uses elementary arguments and some known results. We observe that the occurrence of certain patterns in the tiling imposes restrictions on the lengths of the intervals. This allows us to identify the different cases to be considered, and we prove that Ω is spectral in each case.

In section 3, we consider two particular cases where $\Omega = A \cup B \cup C$ with A, B, C intervals, and either $|A| = |B| = |C| = 1/3$, or $|A| = 1/2$, $|B| = |C| = 1/4$, and prove "Spectral implies Tiling" in these cases. Here, orthogonality conditions impose restrictions on the end-points of the intervals, and we use a powerful theorem on tiling of integers due to Newman [N] to conclude that Ω tiles $\mathbb{R}$. (Newman uses the word tesselation for tiling of integers in his paper).

In Section 4 we briefly digress to the case of n-intervals, and obtain information about the spectrum for such spectral sets. We assume that $\Omega = \cup_{j=1}^{n}[a_j, a_j + r_j)$, with $\sum_{j=1}^{n} r_j = 1$ and $a_j + r_j < a_{j+1}$, is spectral with a spectrum Λ. Note that if Λ is a spectrum for Ω, then any translate of Λ is again a spectrum for Ω. In this paper we always assume that

$$0 \in \Lambda \subset \Lambda - \Lambda.$$

Further, by the orthogonality of the set E_Λ, we have

$$0 \in \Lambda \subset \Lambda - \Lambda \subset \mathbb{Z}_\Omega$$

where $\mathbb{Z}_\Omega$ stands for the zero set of of the Fourier transform of the indicator function χ_Ω, along with the point 0, i.e.

$$\mathbb{Z}_\Omega = \{\xi \in \mathbb{R} : \widehat{\chi_\Omega}(\xi) = 0\} \cup \{0\}.$$

In our investigation, the geometry of the zero set $\mathbb{Z}_\Omega$ will play an important role, as also a deep theorem due to Landau [L] regarding the density of sets of interpolation and sets of sampling. If (Ω, Λ) is a spectral pair, then Landau's theorem applies and says that the asymptotic density $\rho(\Lambda)$ of Λ equals $1/|\Omega|$, where the asymptotic density is given by $\rho(\Lambda) := \lim_{r\to\infty} \frac{card(\Lambda\cap[-r,r])}{2r}$.

In the literature, it is generally assumed that Λ is contained in some lattice $\mathcal{L}$. Since Λ has positive asymptotic density, by Szemerèdi's theorem [S], Λ will contain arbitrarily long arithmetic progressions (APs). In Theorem 4.3, we prove that if Ω is a union of n intervals and Λ contains an AP of length $2n$, then Λ must contain the complete AP. As a consequence, we show that Ω $d-$tiles $\mathbb{R}$, where d is the common difference of the AP. So we need to search for APs of length $2n$ in Λ. In Lemma 4.4, we show that even if $\Lambda - \Lambda$ is just δ-separated (instead of being contained in a lattice), then Λ will contain APs of arbitrary length.

We return to three intervals in Section 5, with the assumption that $\Lambda - \Lambda$ is δ-separated. It then turns out that either (i) $\Omega/\mathbb{Z} \approx [0, 1]$, or (ii) $\Omega/2\mathbb{Z} \approx [0, 1/2] \cup [n/2, (n+1)/2]$, for some n, or (iii) the case is that of equal intervals (not necessarily 3 equal intervals!). In the first two cases "Spectral implies Tiling" follows from [F] and [L1], respectively. The third case is rather complex, and assuming that $\Lambda \subset \mathcal{L}$, we are lead to questions about vanishing sums of roots of unity. Then we use symbolic computation using Mathematica in an attempt to resolve Fuglede's conjecture for three intervals. With one additional assumption on the spectrum, we show "Spectral implies Tiling". The analysis for this computation is given in Section 6.

2. **Tiling implies Spectral**

Let A, B, C be three disjoint intervals in $\mathbb{R}$. In this section we prove the following theorem:

Theorem 2.1. *Let $\Omega = A \cup B \cup C$, $|A| + |B| + |C| = 1$. If Ω tiles $\mathbb{R}$ by translations, then Ω is a spectral set.*

For the proof, we adopt the following notation for convenience: Whenever we need to keep track of the intervals A, B, C as being part of a translate of Ω by t, we will write $\Omega_t = \Omega + t = A_t \cup B_t \cup C_t$.

We begin with a simple lemma.

Lemma 2.2. *Suppose Ω tiles $\mathbb{R}$. Suppose that in some tiling by Ω, AA occurs, then either $|A| = \frac{1}{2}$ or $|A| = |B| = |C| = \frac{1}{3}$.*

Proof. Suppose not all three intervals are of equal length, and there exists a set $\mathcal{T}$ such that $(\Omega, \mathcal{T})$ is a tiling pair. If for some $s, t \in \mathcal{T}$, A_sA_t, occurs, then $|A| \geq |B|, |C|$.

Suppose BB also occurs, then $|B| \geq |A|$, so $|A| = |B| < 1/2$, since $|C| > 0$. Now CC cannot also occur (for then $|A| = |B| = |C|$). Hence there is a gap between C_s and C_t which equals $0 < |A| - |C| < |A|$, $|B|$, so it cannot be filled at all.

Hence if AA occurs, neither BB nor CC can also occur in that tiling. Now the gap between B_s and B_t is of length $|A| - |B| < |A|$, and no B can lie in this gap, so it can be filled only by a C. It follows that $|A| - |B| = |C|$ and so $|A| = \frac{1}{2}$. □

This leads us to consider the following cases:

- **Case 1:** $|A|, |B|, |C| \neq \frac{1}{2}$ and not all are equal:
  - **(1a):** No two of $|A|, |B|, |C|$ are equal.
  - **(1b):** $|A| = |B| \neq |C|$.
- **Case 2:** $|A| = \frac{1}{2}$:
  - **(2a):** $|B| \neq |C|$.
  - **(2b):** $|B| = |C| = \frac{1}{4}$.
- **Case 3:** $|A| = |B| = |C| = \frac{1}{3}$.

Proof of Theorem: We will prove that Tiling implies Spectral, in each of the above cases.

Case(1a) : $|A|, |B|, |C|$ are all distinct and none equals 1/2.
We claim that in this case the tiling pattern has to be of the form

$$---ABC|ABC|ABC---$$

or

$$---ACB|ACB|ACB---$$

Then Ω tiles $\mathbb{R}$ by $\mathbb{Z}$, hence is spectral [F]. Since we already know that XX cannot appear, our claim will be proved if we show that no three consecutive intervals appear as XYX.

Suppose A_tBA_s occurs. No C can appear between C_t and C_s, and the gap between the C_t and C_s is of length $|A|+|B|-|C| < |A|+|B|$ and so cannot be filled by AB, (nor by AA, nor by BB), hence equals either $|A|$ or $|B|$. But then, either $|B| = |C|$ or $|A| = |C|$, a contradiction.

Case(1b): $|A| = |B| \neq |C|,\ |C| \neq 1/2$.

We already know that none of AA, BB, CC can occur in the tiling. We show below that ABA and BAB cannot occur: Suppose A_sBA_t occurs, then C_s and C_t are consecutive C's and the gap between them has length $|A| + |B| - |C| = 2|A| - |C| = 4|A| - 1$. This gap has to be filled by A's and B's. Let $4|A| - 1 = m|A|$. Clearly $m < 4$, and we check easily that this leads to a contradiction ($m = 0 \Rightarrow |C| = 1/2$; $m = 1 \Rightarrow |A| = |B| = |C| = 1/3$, $m = 2 \Rightarrow |A| = |B| = 1/2$ and $m = 3 \Rightarrow |A| = 1$). Hence A_sBA_t cannot occur.

This means that in any tiling, the gap between two consecutive C's is filled by at most two of A and B. In other words, C is translated by at most 1; hence also A and B. But if none of the translates is greater than 1, then CAC, CBC, BCB, and ACA are excluded (for an A_sCA_t occurs iff B_sCB_t occurs, and also A_sCA_t occurs iff either C_sAC_t occurs or C_sBC_t occurs; but an occurrence of A_sCA_t would imply that B is translated by $|B| + 2|A| + |C| > 1$, which is not possible). Hence, the tiling pattern must be either

$$---ABCABCABC---$$

or

$$---ACBACBACB---$$

i.e. Ω tiles $\mathbb{R}$ by $\mathbb{Z}$, and hence is spectral by [F].

Case (2a) : $|A| = 1/2,\ |B| \neq |C|$.

By Lemma 2.2, BB and CC cannot occur in the tiling. Next we show that none of BAB, ABA, CAC, ACA can occur.

First note that BAB occurs iff ABA occurs: for, if BAB occurs as, say $B_sA_rB_t$, where $s, r, t \in \mathcal{T}$, then clearly $t - s = |A| + |B|$. Consider the corresponding tile translates,

$$\Omega_s = A_s \cup B_s \cup C_s, \text{ and } \Omega_t = A_t \cup B_t \cup C_t.$$

Then the gap between A_t and A_s is of length $|B|$. This gap cannot be filled by A, since $|A| > |B|$, nor can it be filled by C, since $|C| \neq |B|$, and two consecutive C's are ruled out anyway. So this gap can be filled up by a single B, resulting in A_sBA_t, an ABA pattern. The proof of the converse is similar.

Now, suppose A_sBA_t occurs. The gap between C_s and C_t is of length $|A|+|B|-|C| < |A|+|B|$. This gap has to be filled by a single A or a single B; and this would imply $|B| = |C|$ or $|A| = |C| = 1/2$. Similarly CAC and ACA cannot occur.

Next, suppose that there is a string $B_sAA...AAB_t$ in the tiling with n consecutive A's between B_s and B_t. The gap between C_s and C_t is then of length $n/2+|B|-|C|$ and has to be filled by A's and B's. But B's can occur just after the C_s and just before the C_t, otherwise ABA will occur. If no B occurs, then $|B| = |C|$ and if a single B occurs then $|C| = 0$ or $1/2$. Hence there exists a string

$$---C_sBAA --- ABC_t --$$

with m consecutive A's with a B on either side, lying between C_s and C_t. Then $\frac{n}{2} + |B| - |C| = 2|B| + \frac{m}{2}$, for some integer m, which means $m = n-1$. We can repeat the argument n times to to get that BAB must occur somewhere, which is not possible. Similarly $CAA---AC$ does not occur.

Thus the only possibilities are tiling patterns of the form :

(1) $---AA --- A\,|BC\,|BC ---|\,BC|\,AA --- ABC ---$
(2) $---AA --- A\,|BC\,|BC ---|\,BC|\,BA --- ACB ---$

and $(1)'$, $(2)'$ with B and C interchanged. We show that (2) and $(2)'$ cannot happen. Suppose (2) occurs; consider the part consisting of a sequence of n BS's followed by a B between A_s and A_t, i.e. $A_sBC---BCBA_t$ and the C_s, C_t corresponding to these A's. The gap is of length $\frac{n}{2} + \frac{1}{2} + |B| - |C|$ and has to be filled by A's and B's. But this leads to a contradiction as shown above. So finally the tiling pattern has to be of the form (1) or $(1)'$.

We write $\Omega = [0, 1/2) \cup [b, b+r) \cup [c, c+1/2-r)$. It is easy to see that the above tiling pattern (namely (1) or $(1)'$) for Ω, say by a tiling set $\mathcal{T}$ implies that both $\Omega_1 = [0, 1/2) \cup [b, b+1/2)$ and $\Omega_2 = [0, 1/2) \cup [c-r, c-r+1/2)$ tile $\mathbb{R}$ by the same tiling set $\mathcal{T}$. (Alternatively we may have to work with the sets $\Omega_1' = [0, 1/2) \cup [b+r-1/2, b+r)$, $\Omega_2' = [0, 1/2) \cup [c, c+1/2))$. Using the result for two intervals [L1], this implies that $b = \frac{n}{2}$, $c - r = \frac{k}{2}$ with $n, k \in \mathbb{Z}$. Then $c - r - b = \frac{k-n}{2}$. Hence $\Omega_1 = [0, 1/2) \cup [n/2, n/2 + 1/2)$. Clearly n is a period for the tiling set $\mathcal{T}$, but $n/2$ is not. So if k_0 is the period of the tiling then $k_0|n$, but $k_0 \not| \frac{n}{2}$. Hence if j is the largest integer such that $2^j|n$, the same is true for k_0. Therefore, $\frac{n}{k_0} \in 2\mathbb{Z}+1$. Also $k_0|k$, and $\frac{k}{k_0} \in 2\mathbb{Z}+1$ and so $k_0|l$, where $l = c - r - b = \frac{k-n}{2}$, and $l \in \mathbb{Z}$.

We show finally that the set $\Lambda = 2\mathbb{Z} \cup (2\mathbb{Z} + \frac{1}{k_0})$ is a spectrum for Ω (we have taken $p = \frac{n}{k_0}$ in Laba's theorem as stated in the Introduction). To check orthogonality, note that if $\lambda \in \Lambda - \Lambda,\ \lambda \neq 0$, then

$$\begin{aligned}
\int_\Omega e^{2\pi i\lambda\xi} d\xi &= \int_0^{1/2} + \int_b^{b+r} + \int_c^{c+1/2-r} e^{2\pi i\lambda\xi} d\xi \\
&= \int_0^{1/2} e^{2\pi i\lambda\xi} d\xi + \int_b^{b+r} e^{2\pi i\lambda\xi} d\xi + \int_{b+r+l}^{b+1/2+l} e^{2\pi i\lambda\xi} d\xi \\
&= \int_0^{1/2} e^{2\pi i\lambda\xi} d\xi + \int_b^{b+r} e^{2\pi i\lambda\xi} d\xi + e^{2\pi i\lambda l} \int_{b+r}^{b+1/2} e^{2\pi i\lambda\xi} d\xi \\
&= \int_{\Omega_1} e^{2\pi i\lambda\xi} d\xi = 0
\end{aligned}$$

since $k_0|l$ and since Λ is a spectrum for Ω_1. As in [L1], it is easy to see that Λ is complete. Alternatively, this follows from [P1] and [LW].

Case 2b is a special case of 4 equal intervals, and **Case 3** is 3 equal intervals. In each of these cases tiling implies spectral follows by more general results [L2]. However, these cases can be handled by using a theorem due to Newman [N], and we do this in the next section.

3. **The Equal Interval Cases**

In this section, we will use the fact that spectral and tiling properties of sets are invariant under translations and dilations. For convenience, we scale the set Ω suitably and prove both implications of Fuglede's conjecture for the two sets $\Omega_3 = [0,1] \cup [a, a+1] \cup [b, b+1]$ and $\Omega_4 = [0,2] \cup [a, a+1] \cup [b, b+1]$.

An essential ingredient of our proofs will be the following theorem on tiling of integers.

Theorem 3.1. (Newman) *Let $A = \{a_1, a_2, ..., a_k\}$ be distinct integers with $k = p^\alpha$, p a prime and α a positive integer. For each pair a_i, a_j, $i \neq j$, let e_{ij} denote the largest integer so that $p^{e_{ij}}|(a_i - a_j)$, and let $S = \{e_{i,j} : i, j = 1, 2, ..., k; i \neq j\}$. Then the set A tiles the set of integers $\mathbb{Z}$ iff the set S has at most α distinct elements.*

3.1. **Tiling implies Spectral.** Without loss of generality we may assume that $0 \in \mathcal{T}$, where $\mathcal{T}$ is the tiling set. Then in the cases under consideration, the end-points of the intervals will be integers and we are in the context of tiling of $\mathbb{Z}$.

Case 3: We apply Newman's Theorem to the set $A = \{0, a, b\}$, $p = 3$, $\alpha = 1$, hence S must be a singleton. If $a = 3^j n$ and $b = 3^k m$ with n, m not divisible by 3, we see that $\{j, k\} \subset S$. It follows that $j = k$ and $m - n$ is not divisible by 3. We may write $a = 3^j(3r + 1)$ and $b = 3^j(3s + 2)$; $r, s \in \mathbb{Z}$. But then, we can check easily that the set $\Lambda = \mathbb{Z} \cup (\mathbb{Z} + \frac{1}{3^{j+1}}) \cup (\mathbb{Z} + \frac{2}{3^{j+1}})$ is a spectrum for Ω_3.

To check completeness, suppose that $f \in L^2(\Omega)$ satisfies $\left\langle f, e^{2\pi i\lambda \cdot} \right\rangle = 0,\ \forall \lambda \in \Lambda$. With $\omega = e^{2\pi i/3}$, define $1-$periodic functions by $f_1^{\#},\ f_2^{\#},\ f_3^{\#}$ by

$$
\begin{pmatrix} f_1^{\#}(x) \\ f_2^{\#}(x) \\ f_3^{\#}(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} f(x) \\ f(x+a) \\ f(x+b) \end{pmatrix} \tag{1}
$$

where $x \in [0, 1]$.

Then check that $\left\langle f, e^{2\pi i\lambda \cdot} \right\rangle = 0,\ \forall \lambda \in \Lambda \iff \left\langle f_j^{\#}, e^{2\pi i k \cdot} \right\rangle = 0,\ \forall k \in \mathbb{Z},\ \ j = 1, 2, 3 \iff f_j^{\#} = 0,\ j = 1, 2, 3 \iff f = 0.$

Case 2b: This time we use Newman's theorem for the set $A = \{0, 1, a, b\}$, $p = 2$, $\alpha = 2$, hence S can have at most 2 elements. Let $a = 2^j(2n + 1)$, $b = 2^k(2m + 1)$. Then the following cases arise:
1. If $j \neq 0$, $k \neq 0$, $\{0, j, k\} \subset S$. Hence $j = k$, but then $a - b = 2^j.2(n - m)$, so $j + 1 \in S$, and the set S still has cardinality ≥ 3, therefore A cannot tile $\mathbb{Z}$.
2. Suppose that $j = 0$, $k \neq 0$. Then $A = \{0, 1, a = 2n + 1,\ b = 2^k(2m + 1)\}$. If $n = 2^{l-1}r$ with $l \geq 1$ and r an odd integer, then $S = \{0, k, l\}$. Since S has at most 2 elements, we get $k = l$ in this case.
3. Let j, $k = 0$. We see then that S has at least three distinct $e_{i,j}$'s, so A cannot tile $\mathbb{Z}$.

We conclude that if Ω_4 tiles, then $a = 2^l r + 1$, $b = 2^l s$, where $r, s \in 2\mathbb{Z} + 1$ and then it is easy to see that the set $\Lambda = \mathbb{Z} \cup \mathbb{Z} + \frac{1}{2^{l+1}}$ is a spectrum. This completes the proof of the "Tiling implies Spectral" part of Fuglede's conjecture for 3 intervals.

3.2. **Spectral implies Tiling:** We now proceed to prove the converse for the two cases Ω_3 and Ω_4.

In the first case, the proof is long, but we are able to construct the spectrum as a subset of the zero set of $\widehat{\chi_\Omega}$, using only the fact $\rho(\Lambda) = |\Omega|$. In the process we get information on the end-points of the three intervals, so that Newman's theorem can be applied to conclude tiling.

Case 3: Let $\Omega_3 = [0,1] \cup [a, a+1] \cup [b, b+1]$. Without loss of generality, let $0 \in \Lambda \subset \Lambda - \Lambda \subset \mathbb{Z}_\Omega$, then

$$\begin{aligned}\widehat{\chi_\Omega}(\lambda) &= e^{2\pi i\lambda} - 1 + e^{2\pi i\lambda(a+1)} - e^{2\pi i\lambda a} + e^{2\pi i\lambda(b+1)} - e^{2\pi i\lambda b}\\ &= (e^{2\pi i\lambda} - 1)(1 + e^{2\pi i\lambda a} + e^{2\pi i\lambda b})\end{aligned}$$

Hence, if $\lambda \in \mathbb{Z}_\Omega$, then either $\lambda \in \mathbb{Z}$ or $1 + e^{2\pi i\lambda a} + e^{2\pi i\lambda b} = 0$.

Let

$$\begin{aligned}\mathbb{Z}^1_\Omega &= \{\lambda : e^{2\pi i\lambda a} = \omega,\ e^{2\pi i\lambda b} = \omega^2\},\\ \mathbb{Z}^2_\Omega &= \{\lambda : e^{2\pi i\lambda a} = \omega^2,\ e^{2\pi i\lambda b} = \omega\}.\end{aligned}$$

We collect some easy facts in the following lemma:

Lemma. Let $\lambda_1 \in \mathbb{Z}^1_\Omega,\ \lambda_2 \in \mathbb{Z}^2_\Omega$, then the following hold:

(1) $-\lambda_1,\ 2\lambda_1 \in \mathbb{Z}^2_\Omega$ and $-\lambda_2,\ 2\lambda_2 \in \mathbb{Z}^1_\Omega$.
(2) $\lambda_1 - \lambda_2 \in \mathbb{Z}^2_\Omega$ and $-\lambda_1 + \lambda_2 \in \mathbb{Z}^1_\Omega$.
(3) Let $\alpha \in \mathbb{Z}^1_\Omega \cup \mathbb{Z}^2_\Omega$, then $\lambda_1 + 3\alpha\mathbb{Z} \subset \mathbb{Z}^1_\Omega,\ \ \lambda_2 + 3\alpha\mathbb{Z} \subset \mathbb{Z}^2_\Omega$.
(4) If $\alpha_\circ$ is the smallest positive real number in $\mathbb{Z}^1_\Omega \cup \mathbb{Z}^2_\Omega$, say $\alpha_\circ \in \mathbb{Z}^1_\Omega$, then $\mathbb{Z}^1_\Omega = \alpha_\circ + 3\alpha_\circ\mathbb{Z}$ and $\mathbb{Z}^2_\Omega = 2\alpha_\circ + 3\alpha_\circ\mathbb{Z}$.

Proof. It is easy to verify (1), (2) and (3). We need to prove (4): First note that since $\widehat{\chi_\Omega}(0) = 3$, so $\widehat{\chi_\Omega} > 0$ in a neighbourhood of 0, and there exists a smallest positive real number, say $\alpha_\circ \in \mathbb{Z}^1_\Omega \cup \mathbb{Z}^2_\Omega$. We assume that $\alpha_\circ \in \mathbb{Z}^1_\Omega$. By (3), we only need to prove that $\mathbb{Z}^1_\Omega \subset \alpha_\circ + 3\alpha_\circ\mathbb{Z}$. Suppose not, then there exists $\beta \in \mathbb{Z}^1_\Omega$, such that $\beta \notin \alpha_\circ + 3\alpha_\circ\mathbb{Z}$. Now by (3), $\beta + 3\alpha_\circ \in \mathbb{Z}^1_\Omega$, so we may assume that $\beta \in (\alpha_\circ, 3\alpha_\circ]$. But $2\alpha_\circ, 3\alpha_\circ \notin \mathbb{Z}^1_\Omega$ and, hence the only possibility is $\beta \in (\alpha_\circ, 2\alpha_\circ) \cup (2\alpha_\circ, 3\alpha_\circ)$. Now in case $\beta \in (\alpha_\circ, 2\alpha_\circ)$, we use (1) and (2) to get $2\alpha_\circ - \beta \in \mathbb{Z}^1_\Omega$. But $0 < 2\alpha_\circ - \beta < \alpha_\circ$, which contradicts the minimality of $\alpha_\circ$. In the other case, that $\beta \in (2\alpha_\circ, 3\alpha_\circ)$, by (1) and (3) we get $3\alpha_\circ - \beta \in \mathbb{Z}^2_\Omega$. But then $0 < 3\alpha_\circ - \beta < \alpha_\circ$, again a contradiction. Hence $\mathbb{Z}^1_\Omega = \alpha_\circ + 3\alpha_\circ\mathbb{Z}$. By similar arguments, we get $\mathbb{Z}^2_\Omega = 2\alpha_\circ + 3\alpha_\circ\mathbb{Z}$. ∎

Therefore,

$$\mathbb{Z}_\Omega = \mathbb{Z} \cup \mathbb{Z}^1_\Omega \cup \mathbb{Z}^2_\Omega = \mathbb{Z} \cup (\alpha_\circ + 3\alpha_\circ\mathbb{Z}) \cup (2\alpha_\circ + 3\alpha_\circ\mathbb{Z}).$$

Note that if $\lambda_1, \lambda_2 \in \mathbb{Z}$ then e_{λ_1}, e_{λ_2} are mutually orthogonal. But Λ must have upper asymptotic density 3, by Landau's density theorem, hence $\rho(\Lambda \setminus \mathbb{Z}) = 2$. In other words $\rho(\Lambda \cap (\mathbb{Z}^1_\Omega \cup \mathbb{Z}^2_\Omega)) \geq 2$.

Our next step is to actually find all of Λ as a subset of $\mathbb{Z}_\Omega$.

Suppose first $\Lambda \cap \mathbb{Z}^1_\Omega \neq \emptyset$ and that λ_1 is the smallest positive element of this set. Now if $\lambda_1' \in \Lambda \cap \mathbb{Z}^1_\Omega$, then $\lambda_1 - \lambda_1' \in \Lambda - \Lambda \subset \mathbb{Z}_\Omega$, and by the definition of $\mathbb{Z}^1_\Omega$ and $\mathbb{Z}^2_\Omega$, we see that

$$e^{2\pi i(\lambda_1 - \lambda_1')a} = 1 = e^{2\pi i(\lambda_1 - \lambda_1')b}$$

This means that $\lambda_1 - \lambda_1' \in \mathbb{Z}$. Therefore $\Lambda \cap \mathbb{Z}^1_\Omega \subset \lambda_1 + \mathbb{Z}$. A similar argument shows that $\Lambda \cap \mathbb{Z}^2_\Omega \subset \lambda_2 + \mathbb{Z}$, where $\lambda_2 \in \Lambda$. Therefore,

$$\rho(\Lambda \cap \mathbb{Z}^1_\Omega) \leq 1, \quad \rho(\Lambda \cap \mathbb{Z}^2_\Omega) \leq 1$$

With this bound on the density, not only $\Lambda \cap \mathbb{Z}^1_\Omega \neq \emptyset$ and $\Lambda \cap \mathbb{Z}^2_\Omega \neq \emptyset$, but each of these sets must contribute a density 1 to Λ. Therefore, from the set $\lambda_1 + \mathbb{Z}$, two consecutive elements, say $\lambda_1 + n$ and $\lambda_1 + (n+1)$ must lie in $\Lambda \cap \mathbb{Z}^1_\Omega$, so their difference, namely 1, satisfies

$$e^{2\pi i a} = 1 = e^{2\pi i b}$$

and so a, $b \in \mathbb{Z}$, and

$$\begin{aligned} \lambda_1 + \mathbb{Z} &\subset \mathbb{Z}^1_\Omega = \alpha_\circ + 3\alpha_\circ\mathbb{Z} \\ \lambda_2 + \mathbb{Z} &\subset \mathbb{Z}^2_\Omega = 2\alpha_\circ + 3\alpha_\circ\mathbb{Z} \end{aligned}$$

Finally we see that $\alpha_\circ \in \mathbb{Q}$, since $e^{2\pi i\alpha_\circ a} = \omega$, and $a \in \mathbb{Z}$. Putting $\alpha_\circ = p/q$ and $\lambda_1 = \alpha_\circ + 3\alpha_\circ k_1$, $k_1 \in \mathbb{Z}$, we see easily that $\mathbb{Z} \subset \frac{p}{q}(3\mathbb{Z})$, which means $p = 1$ and $q = 3q_1$ for some $q_1 \in \mathbb{Z}$. But then

$$a = (3n+1)q_1, \quad b = (3m+2)q_1.$$

We conclude that Ω_3 tiles, by applying Newman's Theorem to the set $A = \{0, (3n+1)q, (3m+2)q\}$, $p = 3$, $\alpha = 1$.

Case 2b: In the case Ω_4, we use a theorem of [JP], to conclude that if Ω_4 is spectral, then the end-points of the intervals lie on $\mathbb{Z}$. For Ω_3, we deduced this without invoking this theorem.

The spectrum Λ will have density $= 4$, by Landau's Theorem. It follows from Corollary 2.4.1 of [JP] that $a, b \in \mathbb{Z}$, so that $\Omega_4 = [0,2] \cup [M, M+1] \cup [N, N+1]$ with M, $N \in \mathbb{Z}$, and so

$$\widehat{\chi_\Omega}(\lambda) = (e^{2\pi i\lambda} - 1)(1 + e^{2\pi i\lambda} + e^{2\pi i\lambda M} + e^{2\pi i\lambda N}).$$

Hence

$$\mathbb{Z}_\Omega = \mathbb{Z} \cup \mathbb{Z}^1_\Omega \cup \mathbb{Z}^2_\Omega \cup \mathbb{Z}^3_\Omega,$$

where

$$\begin{aligned} \mathbb{Z}^1_\Omega &= \{\lambda : 1 + e^{2\pi i\lambda} = 0,\ e^{2\pi i\lambda N} + e^{2\pi i\lambda M} = 0\}, \\ \mathbb{Z}^2_\Omega &= \{\lambda : 1 + e^{2\pi i\lambda N} = 0,\ e^{2\pi i\lambda M} + e^{2\pi i\lambda} = 0\}, \\ \mathbb{Z}^3_\Omega &= \{\lambda : 1 + e^{2\pi i\lambda M} = 0,\ e^{2\pi i\lambda N} + e^{2\pi i\lambda} = 0\}. \end{aligned}$$

We collect some facts, which are easy to verify:

(1) If $\lambda \in \mathbb{Z}_\Omega^i$, then $\lambda + \mathbb{Z} \subset \mathbb{Z}_\Omega^i,\ i = 1, 2, 3$. Further if $\lambda \in \Lambda$, then $\lambda + \mathbb{Z} \subset \Lambda$, i.e. Λ is 1-periodic.
(2) If $\lambda \in \mathbb{Z}_\Omega^i$, then $-\lambda \subset \mathbb{Z}_\Omega^i,\ i = 1, 2, 3$.
(3) If $\mathbb{Z}_\Omega^1 \neq \emptyset$, then $M - N$ is an odd integer, and $\mathbb{Z}_\Omega^1 = \mathbb{Z} + 1/2$. Without loss of generality, we assume that $N = 2n + 1$ and $M = 2m$. Thus, in this case $\mathbb{Z}_\Omega^1 \subset \mathbb{Z}_\Omega^2$.
(4) If $\lambda,\ \lambda' \in \mathbb{Z}_\Omega^2$ are such that $\lambda - \lambda' \in \mathbb{Z}_\Omega$, then $\lambda - \lambda' \in \mathbb{Z}$.
(5) If $\lambda,\ \lambda' \in \mathbb{Z}_\Omega^3$ are such that $\lambda - \lambda' \in \mathbb{Z}_\Omega$, then $\lambda - \lambda' \in \mathbb{Z} \cup \mathbb{Z}_\Omega^1$.
(6) If $\lambda \in \mathbb{Z}_\Omega^3$ and $\lambda' \in \mathbb{Z}_\Omega^2$ are such that $\lambda - \lambda' \in \mathbb{Z}_\Omega$, then $\lambda - \lambda' \in \mathbb{Z}_\Omega^3$. So by (5) $\lambda' \in \mathbb{Z}_\Omega^1$.

From these facts we deduce that the density contribution from each of the sets $\mathbb{Z},\ \mathbb{Z}_\Omega^2$ to Λ can be at most 1, and from $\mathbb{Z}_\Omega^3$, at most 2. But since $\rho(\Lambda) = 4$, we must have $\rho(\Lambda \cap \mathbb{Z}_\Omega^2) = 1$, and $\rho(\Lambda \cap \mathbb{Z}_\Omega^3) = 2$. From (6), it follows that $\Lambda \cap \mathbb{Z}_\Omega^2 = \Lambda \cap \mathbb{Z}_\Omega^1$, and then by (1), we get $\Lambda \cap \mathbb{Z}_\Omega^1 = \mathbb{Z}_\Omega^1 = \mathbb{Z} + 1/2$. Hence

$$\Lambda = \mathbb{Z} \cup \mathbb{Z} + 1/2 \cup (\Lambda \cap \mathbb{Z}_\Omega^3).$$

Next, as $\mathbb{Z}_\Omega^1 \neq \emptyset$, $\lambda \in \Lambda \cap \mathbb{Z}_\Omega^3$ iff $2\lambda n,\ 2\lambda m \in \mathbb{Z} + 1/2$. Let

$$2\lambda n = \frac{2k+1}{2},\ \ 2\lambda m = \frac{2l+1}{2}$$

so if j is the largest integer such that $2^j | n$, this is also true for m.

Finally we see that if Ω is spectral, then $N = 2^{j+1}r + 1,\ b = 2^{j+1}s$, with $r, s \in 2\mathbb{Z} + 1$. Newman's theorem, applied to the set $S = \{0, 1, 2^{j+1}r + 1, 2^{j+1}s\},\ p = 2,\ \alpha = 2$, ensures tiling of $\mathbb{R}$ by Ω_4. This completes the proof.

4. Structure of the Spectrum for n intervals

Let $\Omega = \cup_{j=1}^n [a_j, a_j + r_j),\ \sum_{j=1}^n r_j = 1$. In this section we assume that Ω is a spectral set with spectrum Λ. Recall that we may assume $0 \in \Lambda \subset \Lambda - \Lambda \subset \mathbb{Z}_\Omega$.

We will prove below that if, we assume that Λ is a subset of some discrete lattice $\mathcal{L}$, then Λ is rational. In one dimension, all known examples of spectra are rational and periodic, though it is not known whether it has to be so. Given Λ has positive asymptotic density, if $\Lambda \subset \mathcal{L}$, then Λ contains arbitrarily long arithmetic progressions by (Szemerèdi's theorem[S]). We now prove that in the n-interval case, as soon as Λ contains an AP of length $2n$, then the complete AP is also in Λ. We first prove this for the set $\mathbb{Z}_\Omega$:

Proposition 4.1. *If $\mathbb{Z}_\Omega$ contains an arithmetic progression of length $2n$ containing 0, say $0, d, 2d, \cdots, (2n-1)d$ then*

(1) *the complete arithmetic progression* $d\mathbb{Z} \subset \mathbb{Z}_\Omega$,
(2) $d \in \mathbb{Z}$, *and*
(3) Ω *d-tiles* $\mathbb{R}$.

Proof. Note that if $t \in \mathbb{Z}_\Omega$, then

$$\sum_{j=1}^{n} [e^{2\pi i t(a_j+r_j)} - e^{2\pi i t a_j}] = 0.$$

The hypothesis says that $\widehat{\chi_\Omega}(ld) = 0;\ l = 1, 2, ..., 2n-1$, hence

$$\sum_{j=1}^{n} [e^{2\pi i l d(a_j+r_j)} - e^{2\pi i l d a_j}] = 0;\ l = 0, 1, 2, ..., 2n-1$$

We write $\zeta_{2j} = e^{2\pi i d a_j}$, $\zeta_{2j-1} = e^{2\pi i d(a_j+r_j)}$; $j = 1, 2, ..., n$, then the above system of equations can be rewritten as

$$\zeta_1^l - \zeta_2^l \cdots + \zeta_{2n-1}^l - \zeta_{2n}^l = 0;\ l = 0, 1, \cdots, 2n-1.$$

Equivalently,

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \zeta_1 & \zeta_2 & \cdots & \zeta_{2n-1} & \zeta_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \zeta_1^{2n-2} & \zeta_2^{2n-2} & \cdots & \zeta_{2n-1}^{2n-2} & \zeta_{2n}^{2n-2} \\ \zeta_1^{2n-1} & \zeta_2^{2n-1} & \cdots & \zeta_{2n-1}^{2n-1} & \zeta_{2n}^{2n-1} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ \vdots \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{2}$$

Since $(1, -1, ..., 1, -1)^t \neq 0$, we have

$$\prod_{1=i<j}^{2n} (\zeta_i - \zeta_j) = 0 \tag{3}$$

Next, we need a lemma.

Lemma 4.2. *If (3) holds, then there exist indices i, j with $i \neq j$ and $i+j$ odd, such that $\zeta_i - \zeta_j = 0$. We call such a pair (ζ_i, ζ_j) a good pair. (If $\zeta_i - \zeta_j = 0$, with $i + j$ even, we say (ζ_i, ζ_j) is a bad pair.)*

Proof. Suppose not, then all solutions of (3) are bad. Since the solution set of (3) is non-empty, without loss of generality let $\zeta_1 = \zeta_3$. Then the system reduces to

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \zeta_2 & \zeta_3 & \cdots & \zeta_{2n-1} & \zeta_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \zeta_2^{2n-3} & \zeta_3^{2n-3} & \cdots & \zeta_{2n-1}^{2n-3} & \zeta_{2n}^{2n-3} \\ \zeta_2^{2n-2} & \zeta_3^{2n-2} & \cdots & \zeta_{2n-1}^{2n-2} & \zeta_{2n}^{2n-2} \end{pmatrix} \begin{pmatrix} -\zeta_2 \\ 2\zeta_3 \\ \vdots \\ \zeta_{2n-1} \\ -\zeta_{2n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{4}$$

Hence

$$\prod_{2=i<j}^{2n} (\zeta_i - \zeta_j) = 0 \tag{5}$$

Note that any solution(good) of (5) is a solution(good) of (3). Thus if (5) has a good solution that would lead to a contradiction. So (5) does not have any good solution. Repeating this $2n-2$ times we would be left with a 2×2 Vandermonde matrix which is singular as

$$\begin{pmatrix} 1 & 1 \\ \zeta_i & \zeta_j \end{pmatrix} \begin{pmatrix} n\zeta_i \\ -n\zeta_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6}$$

where i is odd and j is even. This implies that $\zeta_i = \zeta_j$ with $i+j$ odd. This is a contradiction to our assumption. □

Now we can complete the Proof of Proposition 4.1. Observe that whenever we get a good solution of (3), the system of equations in (2) reduces to one involving a $(2n-2) \times (2n-2)$ Vandermonde matrix. Now by the same arguments the reduced matrix would again have a good solution. Repeating this process $n-1$ times, we get a partition of $\{\zeta_i\}$ into n distinct pairs (ζ_i, ζ_j) such that each (ζ_i, ζ_j) is a good solution of (3). But then $\zeta_i^k = \zeta_j^k$, $\forall k \in \mathbb{Z}$. We can relabel the ζ_{2j}'s, $j = 1, 2, \cdots, n$ so that $\zeta_{2j-1} = \zeta_{2j}$. Then we get,

$$\widehat{\chi_\Omega}(kd) = \sum_{j=1}^{n} (\zeta_{2j-1}^k - \zeta_{2j}^k) = 0; \ \forall\, k \in \mathbb{Z} \setminus \{0\} \tag{7}$$

Thus $d\mathbb{Z} \subset \mathbb{Z}_\Omega$.

Now consider

$$F(x) = \sum_{k\in\mathbb{Z}} \chi_\Omega\left(x + k/d\right),\, x \in [0, 1/d) \tag{8}$$

Thus F is $\frac{1}{d}$ periodic and integer valued thus

$$\begin{aligned} \widehat{F}(ld) &= d \sum_{k\in\mathbb{Z}} \int_0^{\frac{1}{d}} \chi_\Omega(x + k/d) e^{-2\pi i l d x} dx \\ &= d\widehat{\chi_\Omega}(ld) = d\, \delta_{l,0} \end{aligned} \tag{9}$$

Thus $F(t) = d$ a.e. So $d \in \mathbb{Z}$ and Ω d-tiles the real line. □

Using Proposition 4.1, we now prove the corresponding result for the spectrum.

Theorem 4.3. *Suppose Ω is spectral and Λ a spectrum with $0 \in \Lambda$. If for some $a, d \in \mathbb{R}$, $a, a+d, ..., a+(2n-1)d \in \Lambda$, then $a + d\mathbb{Z} \subseteq \Lambda$. Further $d \in \mathbb{Z}$ and Ω d-tiles $\mathbb{R}$.*

Proof. Since

$$a, a+d, ..., a+(2n-1)d \in \Lambda$$

Shifting Λ by a we get $\Lambda_1 = \Lambda - a$ is a spectrum for Ω and

$$0, d, ..., (2n-1)d \in \Lambda_1 \subset \Lambda_1 - \Lambda_1 \subset \mathbb{Z}_\Omega.$$

Thus surely $d\mathbb{Z} \subset \mathbb{Z}_\Omega$ by Proposition 4.1. Now let $\lambda \in \Lambda_1$. Then by orthogonality,

$$-\lambda, d-\lambda, 2d-\lambda, ..., (2n-1)d-\lambda \in \mathbb{Z}_\Omega.$$

Let

$$\xi_{2j} = e^{-2\pi i\lambda a_j}, \xi_{2j-1} = e^{-2\pi i\lambda(a_j+r_j)}; j = 1, ..., n$$
$$\zeta_{2j} = e^{2\pi i d a_j}, \zeta_{2j-1} = e^{2\pi i d(a_j+r_j)}; j = 1, ..., n$$

Since $\widehat{\chi_\Omega}(kd-\lambda) = 0$, $k = 0, ..., 2n-1$, we have

$$\xi_1\zeta_1^k - \xi_2\zeta_2^k + ... + \xi_{2n-1}\zeta_{2n-1}^k - \xi_{2n}\zeta_{2n}^k = 0 \text{ for } k = 0, ..., 2n-1. \tag{10}$$

Since the ζ_i's can be partitioned into n disjoint pairs (ζ_i, ζ_j) such that $\zeta_i = \zeta_j$ and $i+j$ odd, without loss of generality, we relabel the ζ_{2j}'s and the corresponding ξ_{2j}'s so that $\zeta_{2j-1} = \zeta_{2j}$, $j = 1, 2, ..., n$. Thus (10) can be written as

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \zeta_1 & \zeta_3 & \cdots & \zeta_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_1^{n-1} & \zeta_3^{n-1} & \cdots & \zeta_{2n-1}^{n-1} \end{pmatrix} \begin{pmatrix} \xi_1 - \xi_2 \\ \xi_3 - \xi_4 \\ \vdots \\ \xi_{2n-1} - \xi_{2n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{11}$$

Now if $[\xi_1 - \xi_2, \xi_3 - \xi_4, ..., \xi_{2n-1} - \xi_{2n}]^t$ is the trivial solution i.e. $\xi_{2j-1} - \xi_{2j} = 0$, $\forall\, j = 1, ..., n$, then $\forall\, k \in \mathbb{Z}$,

$$\begin{aligned} \widehat{\chi_\Omega}(kd-\lambda) &= \xi_1\zeta_1^k - \xi_2\zeta_2^k + \cdots + \xi_{2n-1}\zeta_{2n-1}^k - \xi_{2n}\zeta_{2n}^k \\ &= \zeta_1^k(\xi_1 - \xi_2) + \cdots + \zeta_{2n-1}^k(\xi_{2n-1} - \xi_{2n}) = 0 \end{aligned}$$

Thus $d\mathbb{Z} - \lambda \in \mathbb{Z}_\Omega$.

If however $[\xi_1 - \xi_2, \xi_3 - \xi_4, ..., \xi_{2n-1} - \xi_{2n}]^t$ is not the trivial solution, then $\zeta_{2l-1} = \zeta_{2k-1}$ for some $l, k \in 1, ..., n; l \neq k$.

Removing all the redundant variables and writing the remaining variables as η_{2j+1}^l, $j, l = 0, 1 \cdots k-1$, we get a non-singular Vandermonde matrix satisfying

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \eta_1 & \eta_3 & \cdots & \eta_{2k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_1^{k-1} & \eta_3^{k-1} & \cdots & \eta_{2k-1}^{k-1} \end{pmatrix} \begin{pmatrix} \sum_1 \\ \sum_3 \\ \vdots \\ \sum_{2k-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{12}$$

where

$$\sum\nolimits_k = \sum_{j:\zeta_{2j-1}=\eta_k} \xi_{2j-1} - \xi_{2j}.$$

Then each of the $\sum_i = 0, i = 1, \cdots, k$. But then once again

$$\widehat{\chi_\Omega}(pd-\lambda) = \eta_1^p \sum\nolimits_1 + \eta_3^p \sum\nolimits_3 + \cdots + \eta_{2k-1}^p \sum\nolimits_{2k-1} = 0 \;\; \forall p \in \mathbb{Z}$$

Thus $d\mathbb{Z} - \lambda \in \mathbb{Z}_\Omega$. We already have $d\mathbb{Z} \subseteq \mathbb{Z}_\Omega$ and now we have seen if $\lambda \in \Lambda_1$, then $d\mathbb{Z} - \lambda \in \mathbb{Z}_\Omega$. Thus $d\mathbb{Z} \subseteq \Lambda_1$, hence $a + d\mathbb{Z} \subset \Lambda$. ☐

4.1. **Existence of APs.** With the above Proposition it is clear that we need to explore conditions under which the spectrum, when it exists, contains arithmetic progressions. In the following lemma, we are able to relax the condition that Λ be contaned in a lattice condition to a local condition. We say that a discrete set Δ is $\delta-$ separated, if $\inf\{|\lambda - \lambda'| : \lambda, \lambda' \in \Delta, \lambda \neq \lambda'\} = \delta > 0$.

Lemma 4.4. . *If Λ is a spectrum, such that $\Gamma = \Lambda - \Lambda$ is δ-separated, then Λ contains a complete arithmetic progression. Further, Λ is contained in a lattice with a base.*

Proof. : The δ-separability of Γ means that for some $\delta > 0$

$$inf\{|\gamma_j - \gamma_k| : \gamma_j, \gamma_k \in \Gamma, \;\gamma_j \neq \gamma_k\} = \delta > 0$$

Let $\delta_1 = \delta/2$ and consider the map from Λ to the lattice $\delta_1\mathbb{Z}$ given by

$$\psi(\lambda) = [\lambda/\delta_1]\delta_1,$$

where $[x]$ denotes the largest integer less than x.

Then the subset $\psi(\Lambda)$ has positive density, hence by [S], given any N, there exists an arithmentic progression of length N, say

$$\psi(\lambda_1), \psi(\lambda_2), ..., \psi(\lambda_N),$$

with $\psi(\lambda_{j+1}) = \psi(\lambda_j) + d\delta_1,\; d \in \mathbb{Z},\; j = 1, 2, ...N-1$. But then $\lambda_{j+1} - \lambda_j$ must lie in the interval $(d\delta_1 - \delta_1, d\delta_1 + \delta_1)$ for each $j = 1, 2, ..., N-1$. Since this interval has length δ, all the above elements must be the same (in any interval of length δ, there can be at most one element of $\Lambda - \Lambda$). But this means that the λ_j's are in arithmetic progression. Now using Proposition 4.3, we get that Λ contains the complete AP $M_d\mathbb{Z}$, where $M_d = d\delta_1$, and $M_d \in \mathbb{Z}$.

Let $\Lambda_s := \{\lambda_{n+1} - \lambda_n | \lambda_n \in \Lambda\}$ be the set of successive differences of spectral sequences, the spectral gaps. We will show that Λ_s is finite. $\Lambda_s \subseteq \Lambda - \Lambda \subseteq \mathbb{Z}_\Omega$. As Ω is measurable and of finite measure there exists a neighbourhood around 0 which does not intersect $\mathbb{Z}_\Omega$. Thus

Λ_s is bounded below . But Λ_s is bounded above [see IP]. So by the compactness of Ω we get that Λ_s is finite as $\widehat{\chi_\Omega}$ can be extended analytically to the entire complex plane and zeros of an entire function are isolated.

Let $\Lambda_s = \{r_1, r_2, ..., r_k\}$. The set of solutions for $\sum_{i=1}^{k} a_i r_i = M_d$ with $a_i \in \mathbb{N} \cup \{0\}$ is finite. Thus $\Lambda \subseteq \{0, b_1, ..., b_l\} + M_d\mathbb{Z}$ for some l. In other words, Λ is contained in a lattice with a base. So we have $M_d\mathbb{Z} \subseteq \Lambda \subseteq \{0, b_1, ..., b_l\} + M_d\mathbb{Z}$ □

We end this section by showing that the hypothesis $\Lambda - \Lambda \subseteq \mathcal{L}$ gives more information on the spectrum.

Theorem 4.5. *If Ω as above is spectral, with spectrum Λ such that $0 \in \Lambda \subset \Lambda - \Lambda \subset \mathcal{L}$, $\mathcal{L}$ a lattice, then Λ is rational.*

Proof. : Let $\mathcal{L} = \theta\mathbb{Z}$, and suppose $\Lambda \subseteq \theta\mathbb{Z}$. Since Λ is a spectrum it has asymptotic density 1. By Szemerèdi's theorem Λ contains arbitrarily long arithmetic progressions, and so in particular, of length 2n. Without loss of generality, suppose

$$0, K\theta, ..., (2n-1)K\theta \in \Lambda \subseteq \mathbb{Z}_\Omega$$

where $K \in \mathbb{Z}$. We already know that if Λ contains an arithmetic progression of length $2n$ then the complete arithmetic progression is in Λ and the common difference $d \in \mathbb{N}$. Hence $K\theta \in \mathbb{Z}$, which means that $\theta \in \mathbb{Q}$. So $\Lambda \subseteq \mathbb{Q}$. □

5. Three Intervals: Spectral implies Tiling

Consider now the particular case of three intervals. Let $\Omega = [0, r_1) \cup [a_2, a_2 + r_2) \cup [a_3 + r_3)$, $r_1 + r_2 + r_3 = 1$. Suppose that Ω is spectral and that its spectrum Λ contains an arithmetic progression of length 6 and common difference d. By translating if necessary we may assume that $d\mathbb{Z} \subset \Lambda$, $d \in \mathbb{Z}$, and so by Proposition 4.1, Ω $d-$tiles $\mathbb{R}$. Thus, if $d = 1$, Ω tiles $\mathbb{R}$ by $\mathbb{Z}$. If $d \neq 1$, let $\lambda \in \Lambda \setminus d\mathbb{Z}$. Put

$$\zeta_{2j-1} = e^{2\pi i d(a_j + r_j)}, \zeta_{2j} = e^{2\pi i d a_j}$$

$$\xi_{2j-1} = e^{2\pi i \lambda(a_j + r_j)}, \xi_{2j} = e^{2\pi i \lambda a_j},$$

for $j = 1, 2, 3$. Now since λ, $0, d, \cdots, 5d \in \Lambda$, by orthogonality we get,

$$\xi_1\zeta_1^k - \xi_2\zeta_2^k + \cdots + \xi_{2n-1}\zeta_{2n-1}^k - \xi_{2n}\zeta_{2n}^k = 0$$

for $k = 0, 1, \cdots, 5$.

We know that among the ζ_j's there are good pairs. We can reindex the ζ_{2j}'s and simultaneously the corresponding ξ_{2j}'s such that $\zeta_{2j-1} = \zeta_{2j}$. Then we have

$$\begin{pmatrix} 1 & 1 & 1 \\ \zeta_1 & \zeta_3 & \zeta_5 \\ \zeta_1^2 & \zeta_3^2 & \zeta_5^2 \end{pmatrix} \begin{pmatrix} \xi_1 - \xi_2 \\ \xi_3 - \xi_4 \\ \xi_5 - \xi_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{13}$$

Let A denote the linear transformation represented by the above matrix. We consider three cases:

Case 1. Rank A=3. In this case, $\xi_1 = \xi_2$, $\xi_3 = \xi_4$ and $\xi_5 = \xi_6$. But then it follows that Λ is a group. Further, $\lambda \in \Lambda$ implies $n\lambda \in \Lambda\ \forall\ n \in \mathbb{Z}$. This means that Ω $d-$tiles $\mathbb{Z}$, so infact, $\lambda \in \mathbb{Z}$, i.e. $\Lambda = k\mathbb{Z}$ for some $k \in \mathbb{N}$. But Λ has density 1, we must have $\Lambda = \mathbb{Z}$.

Case 2. Rank A=2. In this case, without loss of generality, assume that $\zeta_1 = \zeta_5$, then

$$\begin{pmatrix} 1 & 1 \\ \zeta_1 & \zeta_3 \end{pmatrix} \begin{pmatrix} \xi_1 - \xi_2 + \xi_5 - \xi_6 \\ \xi_3 - \xi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{14}$$

Hence

$$\xi_1 - \xi_2 + \xi_5 - \xi_6 = 0 \text{ and } \xi_3 - \xi_4 = 0$$

It follows that in this case

$$\mathbb{Z}_\Omega = \mathbb{Z}_\Omega(1) \cup \mathbb{Z}_\Omega(2) \cup \mathbb{Z}_\Omega(3)$$

where

$$\mathbb{Z}_\Omega(1) = \{\lambda : \xi_3 = \xi_4, \xi_1 = \xi_2, \xi_5 = \xi_6\}$$
$$\mathbb{Z}_\Omega(2) = \{\lambda : \xi_3 = \xi_4, \xi_1 = \xi_6, \xi_2 = \xi_5\}$$
$$\mathbb{Z}_\Omega(3) = \{\lambda : \xi_3 = \xi_4, \xi_1 = -\xi_5, \xi_2 = -\xi_6\}.$$

We proceed as in ([L1]) to conclude that

(1) Either $\Lambda \subset \mathbb{Z}_\Omega(1) \cup \mathbb{Z}_\Omega(3)$ or $\Lambda \subset \mathbb{Z}_\Omega(2) \cup \mathbb{Z}_\Omega(3)$,
(2) $\Lambda \subset \Lambda - \Lambda \subset \mathbb{Z}_\Omega(3) \cup (\mathbb{Z}_\Omega(1) \cap \mathbb{Z}_\Omega(2))$
(3) $\mathbb{Z}_\Omega(1) \cap \mathbb{Z}_\Omega(2) = k\mathbb{Z},\ k \in \mathbb{Z}$, since this set is a subgroup of $\mathbb{Z}$, and
(4) Let $\lambda_1,\ \lambda_2 \in \Lambda \cap \mathbb{Z}_\Omega(3)$, then $\lambda_1 - \lambda_2 \in (\mathbb{Z}_\Omega(1) \cap \mathbb{Z}_\Omega(2)) \setminus \mathbb{Z}_\Omega(3)$.

It follows that $\Lambda \cap \mathbb{Z}_\Omega(3) \subseteq \beta + k\mathbb{Z}$. Then by density considerations, and Theorem 4.3, we get that $\Lambda = k\mathbb{Z} \cup (k\mathbb{Z} + \beta)$, with $k = 2$. Then *Spectral implies Tiling* follows from [P2], where it is proved that if the spectrum is a union of two lattices, then Ω tiles $\mathbb{R}$.

Case 3. Rank A=1
In this case, $\zeta_1 = \zeta_3 = \zeta_5 = \zeta_2 = \zeta_4 = \zeta_6$. Then

$$e^{2\pi i d a_1} = e^{2\pi i d(a_1 + r_1)} = e^{2\pi i d a_2} = e^{2\pi i d(a_2 + r_2)} = e^{2\pi i d a_3} = e^{2\pi i d(a_3 + r_3)}.$$

Taking $a_1 = 0$ we get

$$a_1 = 0, a_2 = \frac{l_2}{d}, a_3 = \frac{l_3}{d}; r_1 = \frac{k_1}{d}, r_2 = \frac{k_2}{d}, r_3 = \frac{k_3}{d}$$

where $l_2, l_3, k_1, k_2, k_3 \in \mathbb{Z}$ and $k_1 + k_2 + k_3 = d$.

It follows from this that we are in the case of d equal intervals, in three groups. But then the spectrum is periodic and of the form

$$\Lambda = L + d\mathbb{Z}$$

If $d = 1$ then $\Lambda = \mathbb{Z}$, so there is nothing to prove. If $d = 2$, then $r_1, r_2, r_3 \geq 1/2$, which is not possible. Hence, we may assume that $d \geq 3$. We beleive that $d = 3$ in this case, but we are unable to prove this. However, if we make a further assumption such as $\Lambda \subset \mathcal{L}$, a lattice, or that $\Lambda \subset \mathbb{Q}$, then we are led to questions of vanishing sums of roots of unity given below. Let $\Lambda \subset \mathbb{Q}$

$$e^{2\pi i\lambda a_1} - 1 + e^{2\pi i\lambda(a_2+r_2)} - e^{2\pi i\lambda a_2} + e^{2\pi i\lambda(a_3+r_3)} - e^{2\pi i\lambda a_3} = 0$$

and this is a case of six roots of unity summing to 0, say $\alpha_1 + \cdots + \alpha_6 = 0$. Poonen and Rubinstein [PR] (have classified all minimal vanishing sums of roots of unity $\alpha_1 + \cdots + \alpha_n = 0$ of weight $n \leq 12$ (see also[LL]). There are three possible ways in which six roots of unity $(\alpha_1, \alpha_2, ..., \alpha_6)$ can sum up to zero. If σ denotes an element of S_6, the group of permutations of 6 objects, the possible cases are:

(T1) (a 2 sub-sum = 0). $\alpha_{\sigma(1)} + \alpha_{\sigma(2)} = \alpha_{\sigma(3)} + \alpha_{\sigma(4)} = \alpha_{\sigma(5)} + \alpha_{\sigma(6)} = 0$.

(T2) (a 3 sub-sum = 0). $\alpha_{\sigma(1)} + \alpha_{\sigma(2)} + \alpha_{\sigma(3)} = \alpha_{\sigma(4)} + \alpha_{\sigma(5)} + \alpha_{\sigma(6)} = 0$.

(T3) (no sub-sum = 0). $\alpha_{\sigma(n)} = \rho^n;\ n = 1, \cdots, 4;\ \alpha_{\sigma(5)} = -\omega, \alpha_{\sigma(6)} = -\omega^2$ (after normalizing) where ρ is a fifth root of unity and ω is a cube root of unity.

It turns out that there are now many possibilities and we use a symbolic computation explained in the next section.

6. A symbolic computation using Mathematica

We begin with the setting of Case 3 in the previous section, where Rank$A = 1$. Let d be the smallest positive integer in $\mathbb{Z}_\Omega$ such that $d\mathbb{Z} \subseteq \Lambda$, and let $\Lambda = L + d\mathbb{Z} = \cup_{j=1}^{d}\{\lambda_j + d\mathbb{Z}\}$, with $\lambda_1 = 0$. We also have $d \geq 3$. Observe that if $d' \in \mathbb{Z}_\Omega$ is such that $e^{2\pi i d' a_j} = e^{2\pi i d'(a_j+r_j)} = 1$ for all $j \in \{1, 2, 3\}$, then $d' \in d\mathbb{Z}$.

We now introduce some notation and explain the analysis behind the computation carried out. First we map $\mathbb{Z}_\Omega$ into $\mathbb{C}^6$; $\lambda \to v_\lambda = (e^{2\pi i\lambda(a_1+r_1)}, -e^{2\pi i\lambda a_1}, e^{2\pi i\lambda(a_2+r_2)}, -e^{2\pi i\lambda a_2}, e^{2\pi i\lambda(a_3+r_3)}, -e^{2\pi i\lambda a_3})$. In particular $v_0 = (1, -1, 1, -1, 1, -1)$

Define a conjugate bilinear form on $\mathbb{C}^6$ as follows. For $v = (x_1, x_2, ..., x_6)$, $w = (y_1, y_2, ..., y_6)$, (skew dot product)

$$SDP(v, w) = x_1\bar{y}_1 - x_2\bar{y}_2 + x_3\bar{y}_3 - x_4\bar{y}_4 + x_5\bar{y}_5 - x_6\bar{y}_6$$

Note that $SDP(v_\lambda, v_0) = 0\ \forall\lambda \in \mathbb{Z}_\Omega$. Let

$$G(v, w) = (x_1\bar{y}_1, -x_2\bar{y}_2, x_3\bar{y}_3, -x_4\bar{y}_4, x_5\bar{y}_5, -x_6\bar{y}_6)$$

We will say that a vector v_λ is of Type1, Type2 or Type3 if it satisfies (T1), (T2) or (T3) respectively, listed at the end of the last section.

We make some observations

(1) $SDP(v_\lambda, v_0) = 0,\ \forall\lambda \in \mathbb{Z}_\Omega$.
(2) $G(v_{\lambda_1}, v_{\lambda_2}) = v_{\lambda_1 - \lambda_2}$.
(3) If $\lambda \subset \mathbb{Q}$, then all components of $v_\lambda,\ \lambda \in \lambda$ are roots of unity.
(4) Since $a_1 = 0$, the second coordinate in the image of $\mathbb{Z}_\Omega$ in $\mathbb{C}^6$ is always -1.
(5) The image of Λ in $\mathbb{C}^6$ consists of precisely d elements corresponding to the different cosets of $d\mathbb{Z}$ (for, if $v_{\lambda_1} = v_{\lambda_2}$, then $G(v_{\lambda_1}, v_{\lambda_2}) = v_0$, so $\lambda_1 - \lambda_2 \in d\mathbb{Z}$.

The computation is done under the following assumption:

Assumption: For $\lambda_1, \lambda_2 \in \Lambda,\ v_{\lambda_1 - \lambda_2}$ is of Type 1 if and only if $\lambda_1 - \lambda_2 \in d\mathbb{Z}$

Through the symbolic computation we shall investigate the maximum possible cardinality of a set $\{\lambda_1, \lambda_2, ..., \lambda_d\}$ such that $\lambda_i - \lambda_j \in \mathbb{Z}_\Omega$ and such that $v_{\lambda_i - \lambda_j}$ is a vector of either Type 2 or Type 3.

The first case of this investigation is analyzed below.

Case 1. $v_{\lambda_1}, v_{\lambda_2}$ are both Type 2 vectors. First we make some observations.

1. if $(\alpha_1, \alpha_2, ..., \alpha_6)$ are 6 roots of unity such that their sum is zero and if $\alpha_{\sigma(1)} + \alpha_{\sigma(2)} + \alpha_{\sigma(3)} = 0$ then $\frac{\alpha_{\sigma(1)}}{\alpha_{\sigma(2)}}, \frac{\alpha_{\sigma(2)}}{\alpha_{\sigma(3)}}, \frac{\alpha_{\sigma(3)}}{\alpha_{\sigma(1)}}$ are powers of ω where $\omega^3 = 1$.

2. if $(\alpha_1, \alpha_2, ..., \alpha_6)$ is as above and has no subsum zero i.e. it is a Type3 vector, then it has to be a permutation of $(x\rho, x\rho^2, x\rho^3, x\rho^4, -x\omega, -x\omega^2)$ where x is some root of unity. So there cannot be two pairs among these six elements whose ratios are powers of $-\omega$.

3. If the vector $(\pm\omega^*, \pm\omega^*, \pm\omega^*, \pm\omega^*, \pm\omega^*, \pm\omega^*)$ where ω^* is some power of ω, with exactly 3 positive signs and 3 negative signs has sum zero then it has to be of Type1, because Type2 would imply that it is

a permutation of $(1, \omega, \omega^2, -1, -\omega, -\omega^2)$ which is a Type1 vector.

4. For a Type2 vector all elements in a 3 subsum adding up to zero have the same sign.

Let $u[x] = (1, \omega, \omega^2, x, x\omega, x\omega^2)$ and $u[y] = (1, \omega, \omega^2, y, y\omega, y\omega^2)$ be two vectors of Type2 where x and y are roots of unity. Take the conjugate SDP of these two vectors by permuting and conjugating the second vector $u[y]$. The terms in the conjugate SDP, ignoring the signs, will fall into one of the following four categories upto a permutation of the first 3 elements and last 3 elements.

1. $(\omega^*, \omega^*, \omega^*, x\bar{y}\omega^*, x\bar{y}\omega^*, x\bar{y}\omega^*)$
2. $(\bar{y}\omega^*, \bar{y}\omega^*, \bar{y}\omega^*, x\omega^*, x\omega^*, x\omega^*)$
3. $(\omega^*, \omega^*, \bar{y}\omega^*, x\omega^*, x\bar{y}\omega^*, x\bar{y}\omega^*)$
4. $(\omega^*, \bar{y}\omega^*, \bar{y}\omega^*, x\omega^*, x\omega^*, x\bar{y}\omega^*)$

Here ω^* represents some power of ω. The first case arises when we take conjugate SDP of u[x] and permuted u[y] which is of the type $(\omega^*, \omega^*, \omega^*, y\omega^*, y\omega^*, y\omega^*)$. The second case is similar. The third case arises when we take the conjugate SDP of $u[x]$ and permuted $u[y]$ which is of the type $(\omega^*, \omega^*, y\omega^*, \omega^*, y\omega^*, y\omega^*)$ upto a permutation of the first 3 elements and last 3 elements. The fourth case is again similar.

In all cases, after putting in the signs, they are not of type 3, as there are atleast two pairs whose ratios are powers of $-\omega$.

Now we need to consider only two cases.

Case 1. $(\omega^*, \omega^*, \omega^*, x\bar{y}\omega^*, x\bar{y}\omega^*, x\bar{y}\omega^*)$
If there is a 3 subsum being zero, after the signs are put in appropriately, which involves ω^* and $x\bar{y}\omega^*$ then the ratio $(x\bar{y}\omega^*/\omega^*) = x\bar{y}$ is a power of $-\omega$. Hence the set $(\omega^*, \omega^*, \omega^*, x\bar{y}\omega^*, x\bar{y}\omega^*, x\bar{y}\omega^*)$ becomes $(\pm\omega^*, \pm\omega^*, \pm\omega^*, \pm\omega^*, \pm\omega^*, \pm\omega^*)$ with exactly 3 positive signs and 3 negative signs. So the terms in this SDP form a Type1 vector. Hence this possibility is not considered. So if there is a 3 subsum being zero then it should be $\omega^* + \omega^* + \omega^* = 0.$ and $x\bar{y}\omega^* + x\bar{y}\omega^* + x\bar{y}\omega^* = 0.$ The 3 pluses occur with first 3 elements and 3 minuses occur with the last 3 elements or vice versa. Once u[x] is fixed with these signs $(1, \omega, \omega^2, y, y\omega, y\omega^2)$ can be permuted in the first 3 and last 3 elements. In this case x and y can be any root of unity other than $-1, -\omega, -\omega^2$.

Case 3. $(\omega^*, \omega^*, \bar{y}\omega^*, x\omega^*, x\bar{y}\omega^*, x\bar{y}\omega^*)$
If there is a 3 subsum being zero after the signs are put in appropriately, then the 3 subsum which involves ω^* has to involve one of the

terms $\bar{y}\omega^*, x\omega^*, x\bar{y}\omega^*$. So either x is a power of $-\omega$ or y is a power of $-\omega$ or $x\bar{y}$ is a power of $-\omega$. If $x\bar{y}$ is a power of $-\omega$ then both $\bar{y}\omega^*, x\omega^*$ will also be involved in a 3 subsum which has ω^*. So both x and y are powers of $-\omega$. Hence the set $(\omega^*, \omega^*, \bar{y}\omega^*, x\omega^*, x\bar{y}\omega^*, x\bar{y}\omega^*)$ becomes $(\pm\omega^*, \pm\omega^*, \pm\omega^*, \pm\omega^*, \pm\omega^*, \pm\omega^*)$ with exactly 3 positive signs and 3 negative signs. So the terms in this SDP form a Type1 vector. Hence this possibility is not considered. Hence either x is a power of $-\omega$ or y is a power of $-\omega$ but not both.

If x is a power of $-\omega$ then x is a power of ω as $u[x]$ is a Type2 vector and the 3 subsum would be $\omega^* + \omega^* + \omega^* = 0$. and $\bar{y}\omega^* + \bar{y}\omega^* + \bar{y}\omega^* = 0$. The 3 pluses occur with elements which involve $\bar{y}$ and 3 minuses occur with the other 3 elements or viceversa. So, keeping $u[y] = (1, \omega, \omega^2, y, y\omega, y\omega^2)$ fixed, the 3 pluses occur with first 3 elements and 3 minuses occur with the last three elements or viceversa. The vector $u[x] = (1, \omega, \omega^2, x, x\omega, x\omega^2)$ with $x = \omega^*$ is itself a permutation of $(1, \omega, \omega^2, 1, \omega, \omega^2)$. Now $u[x]$ has to be permuted in such a way that the terms in the SDP which involve $\bar{y}$ have to be a permutation of $\pm(\bar{y}, \bar{y}\omega, \bar{y}\omega^2)$. Hence permuted $u[x]$ has to be of the form $(\sigma(1), \sigma(omega), \sigma(\omega^2), \mu(1), \mu(\omega), \mu(\omega^2))$ where σ and μ are permutations of $(1, \omega, \omega^2)$. In this case y can be any root of unity other than $-1, -\omega, -\omega^2$.

The analysis when y is a power of $-\omega$ is identical.

So all the cases where the two Type2 vectors $u[x]$ and $u[y]$ can have conjugate SDP zero and the terms in the SDP forms a Type2 vector, reduce to the case where the 3 pluses occur at the first 3 positions and 3 minuses occur at the last 3 positions or viceversa, with $u[x]$ fixed and $u[y]$ permuted among the first 3 positions and last 3 positions or where all the permuted y terms in $u[y]$ occur in the first 3 positions and other permuted 3 terms occur at the last 3 positions, as given below

$$\begin{pmatrix} +1 & +1 & +1 & -1 & -1 & -1 \\ 1 & \omega & \omega^2 & x & x\omega & x\omega^2 \\ \sigma(1) & \sigma(\omega) & \sigma(\omega^2) & y\mu(1) & y\mu(\omega) & y\mu(\omega^2) \end{pmatrix} \tag{15}$$

It follows that if the spectrum contains only vectors of Type1 and Type2, then $d \leq 3$. This follows from the computation given at the end of
http://www.imsc.res.in/ $\sim$ rkrishnan/FugledeComputation.html - Type2withType2.nb

Now in the case when Λ contains one vector of Type3, Two cases can arise:

1. Λ contains only vectors of Type1 and Type3, then from the Mathematica symbolic computation, available in the file http://www.imsc.res.in/∼rkrishnan/FugledeComputation.html - Type3withType3.nb we conclude again the $d = 3$.

2. Lastly if Λ contains vectors of Type 3 as well as of Type 2, then there can be at most one coset coming from each type. The details of this computation are available at http://www.imsc.res.in/∼rkrishnan/FugledeComputation.html - Type3withType2.nb and vwithuandv1.nb

We conclude that $d = 3$ in all the above cases, which reduces to the case of three equal intervals, for which we have already proved the conjecture.

We have thus proved the spectral implies tiling part of Fuglede's conjecture for three intervals under two assumptions, namely (a) Λ is contained in a lattice, and (b) $(\Lambda - \Lambda)\cap$ Type1 $= d\mathbb{Z}$.

Final Remark. Note that the additional assumptions made on the spectrum are used to reduce the Rank $A = 1$ case to the case of three equal intervals. However, without any additional assumptions on Λ, this case still corresponds to an equal interval case grouped together in three bunches.

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