Some Properties of a New Class of Generalized Cauchy Numbers

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Abstract. In this paper, we investigate properties of a new class of generalized Cauchy numbers. By using the method of coefficient, we establish a series of identities involving generalized Cauchy numbers, which generalize some results for the Cauchy numbers. Furthermore, we give some asymptotic approximations of certain sums related to the generalized Cauchy numbers.

Key words. Cauchy numbers, harmonic numbers, Stirling numbers, asymptotic approximation.

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1 Introduction

As known to us, Cauchy numbers play important roles in many areas such as approximate integrals and difference-differential equations. One of the important applications is the so-called Laplace summation formula. The Laplace summation formula is analogous to the Euler-Maclaurin formula, but it uses Cauchy numbers and the difference operator instead of the Bernoulli numbers and the differentiation. Cauchy numbers are related to Stirling numbers of the first kind (see [4]), and this relates them to combinatorics. For more details about Cauchy numbers, see [3, 7, 8, 11].

Properties of Cauchy numbers of the first kind a_n and Cauchy numbers of the second kind b_n have been investigated in [4, 9, 12], where a_n and b_n are defined by

$$\sum_{n=0}^{\infty} a_n \frac{t^n}{n!} = \frac{t}{\ln(1+t)}, \quad \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} = \frac{-t}{(1-t)\ln(1-t)}.$$

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More recently, the generalized Cauchy numbers $c_n^{[r]}$, given by

$$\sum_{n=0}^{\infty} c_n^{[r]} \frac{t^n}{n!} = \frac{t(1+t)^{1-r}}{\ln(1+t)},$$

where r is an integer, are studied in [9,13]. In this paper, we consider a new class of generalized Cauchy numbers $c_n^{[r]}(\alpha,\beta)$ defined by

$$\sum_{n=0}^{\infty} c_n^{[r]}(\alpha,\beta) \frac{t^n}{n!} = \frac{t}{(1-\beta t)^r \ln(1-\alpha t)}, \tag{1.1}$$

where α and β are real numbers with $\alpha \neq 0$, and r is an integer. Since $c_n^{[r]}(-1,-1) = c_n^{[r+1]}$, $c_n^{[0]}(-1,0) = a_n$ and $c_n^{[1]}(1,1) = -b_n$, $c_n^{[r]}(\alpha,\beta)$ is therefore a generalization of the Cauchy numbers a_n and b_n, and the generalized Cauchy numbers $c_n^{[r]}$.

The paper is organized as follows. In Section 2, by means of the method of coefficient, we establish a series of identities involving $c_n^{[r]}(\alpha,\beta)$. In particular, we obtain identities for $c_n^{[r]}(\alpha,\beta)$ and generalized harmonic numbers, and derive some recurrence relations for $c_n^{[r]}(\alpha,\beta)$. In Section 3, we compute some asymptotic approximations of certain sums related to $c_n^{[r]}(\alpha,\beta)$.

Now we give some notations and definitions involved in this paper. For $n \geq 0$, we denote $(x)_n$ by

$$(x)_n = \begin{cases} x(x-1)\cdots(x-n+1), & n \geq 1, \\ 1, & n = 0. \end{cases}$$

The binomial coefficient $\binom{n}{m}$ is defined by

$$\binom{n}{m} = \begin{cases} \dfrac{(n)_m}{m!}, & n \geq m, \\ 0, & n < m, \end{cases}$$

where n and m are nonnegative integers. We denote by $s(n,k)$ and $S(n,k)$ Stirling numbers of the first kind and Stirling numbers of the second kind, respectively, and use $H_n^{[m]}(\lambda,\mu)$ and $P_n^{[q]}(g_1, g_2, \cdots, g_n)$ to stand for the generalized hyperharmonic numbers and the potential polynomials, respectively. That is,

$$\begin{aligned} \sum_{n=k}^{\infty} s(n,k) \frac{t^n}{n!} &= \frac{\ln^k(1+t)}{k!}, \\ \sum_{n=k}^{\infty} S(n,k) \frac{t^n}{n!} &= \frac{(e^t-1)^k}{k!}, \end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty} H_n^{[m]}(\lambda,\mu)t^n &= -\frac{\ln(1-\lambda t)}{(1-\mu t)^m},\\ \left(\sum_{n=0}^{\infty} g_n\frac{t^n}{n!}\right)^q &= \left(1+g_1t+g_2\frac{t^2}{2!}+\cdots\right)^q\\ &= 1+\sum_{n=1}^{\infty} P_n^{[q]}(g_1,g_2,\cdots,g_n)\frac{t^n}{n!},\end{aligned}$$

where $k \geq 1$ and $m \geq 0$ are integers, λ and μ are real numbers with $\lambda \neq 0$, q is a complex number. Especially, $H_n^{[1]}(1,1)$ is the ordinary harmonic number, $H_n^{[m]}(1,1)(m \geq 1)$ is the hyperharmonic number, and $H_n^{[0]}(1,\mu) = 1/n$. See [2] for various properties of the hyperharmonic numbers. Moreover, $[t^n]f(t)$ denotes the coefficient of t^n in $f(t)$, where

$$f(t) = \sum_{n=0}^{\infty} f_n t^n.$$

If $f(t)$ and $g(t)$ are formal power series, the following relations hold [10]:

$$[t^n](af(t)+bg(t)) = a[t^n]f(t)+b[t^n]g(t), \tag{1.2}$$

$$[t^n]tf(t) = [t^{n-1}]f(t), \tag{1.3}$$

$$[t^n]f(t)g(t) = \sum_{j=0}^{n}[t^j]f(t)[t^{n-j}]g(t), \tag{1.4}$$

$$[t^n]f(g(t)) = \sum_{k=0}^{\infty}[y^k]f(y)[t^n]g^k(t). \tag{1.5}$$

2 Some Identities

In this section, we give some identities related to $c_n^{[r]}(\alpha,\beta)$ by using (1.2)-(1.5). From [4], we know that Cauchy numbers of the first kind a_n satisfy that

$$a_n = \sum_{k=1}^{n}\frac{(-1)^{k-1}(n)_k a_{n-k}}{k+1}. \tag{2.1}$$

For $c_n^{[r]}(\alpha,\beta)$, we have

Theorem 2.1 *Let n be a positive integer. For $c_n^{[r]}(\alpha,\beta)$, we have*

$$c_n^{[r]}(\alpha,\beta) - n\beta c_{n-1}^{[r]}(\alpha,\beta) = c_n^{[r-1]}(\alpha,\beta), \tag{2.2}$$

$$(m-1)!c_n^{[r+m]}(\alpha,\beta) = \sum_{j=0}^{n}\binom{n}{j}c_j^{[r]}(\alpha,\beta)(n-j+m-1)_{n-j}\beta^{n-j}, \quad m \geq 1, \tag{2.3}$$

$$c_n^{[r]}(\alpha,\beta) = \sum_{j=0}^{n}\binom{j+r-1}{j}(-1)^{n-j-1}(n)_j\alpha^{n-j-1}\beta^j a_{n-j}, \quad r \geq 1, \tag{2.4}$$

$$a_n = \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^{j+1} c_j^{[r]}(\alpha,\beta)(r)_{n-j}\beta^{n-j}}{\alpha^{n-1}}, \quad r<0, \tag{2.5}$$

$$c_n^{[r]}(\alpha,\beta) = -\binom{n+r-1}{n}\frac{\beta^n n!}{\alpha} - \sum_{j=1}^{n} \frac{(n)_j c_{n-j}^{[r]}(\alpha,\beta)\alpha^j}{j+1}, \quad r\geq 1. \tag{2.6}$$

Proof. From (1.1), one can obtain

$$\sum_{n=0}^{\infty} c_n^{[r]}(\alpha,\beta)\frac{t^n}{n!} - \beta\sum_{n=0}^{\infty} c_n^{[r]}(\alpha,\beta)\frac{t^{n+1}}{n!} = \frac{t}{(1-\beta t)^{r-1}\ln(1-\alpha t)}.$$

Then we have

$$\begin{aligned} \frac{c_n^{[r]}(\alpha,\beta)}{n!} - \beta\frac{c_{n-1}^{[r]}(\alpha,\beta)}{(n-1)!} &= [t^n]\frac{t}{(1-\beta t)^{r-1}\ln(1-\alpha t)} \\ &= \frac{c_n^{[r-1]}(\alpha,\beta)}{n!}. \end{aligned}$$

Naturally, (2.2) holds. Moreover, we have

$$\begin{aligned} \sum_{j=0}^{n}\binom{n}{j}\frac{c_j^{[r]}(\alpha,\beta)(n-j+m-1)_{n-j}\beta^{n-j}}{(m-1)!n!} &= \sum_{j=0}^{n}\frac{c_j^{[r]}(\alpha,\beta)}{j!}\binom{n-j+m-1}{m-1}\beta^{n-j} \\ &= \sum_{j=0}^{n}[t^j]\frac{t}{(1-\beta t)^r\ln(1-\alpha t)}[t^{n-j}]\frac{1}{(1-\beta t)^m} \\ &= \frac{c_n^{[r+m]}(\alpha,\beta)}{n!}, \quad m\geq 1, \end{aligned}$$

$$\begin{aligned} \sum_{j=0}^{n}\binom{j+r-1}{j}\frac{(-1)^{n-j-1}\alpha^{n-j-1}\beta^j a_{n-j}}{(n-j)!} &= -\frac{1}{\alpha}\sum_{j=0}^{n}[t^j]\frac{1}{(1-\beta t)^r}[t^{n-j}]\frac{-\alpha t}{\ln(1-\alpha t)} \\ &= [t^n]\frac{t}{(1-\beta t)^r\ln(1-\alpha t)} \\ &= \frac{c_n^{[r]}(\alpha,\beta)}{n!}, \quad r\geq 1, \end{aligned}$$

$$\begin{aligned} \sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^{j+1}c_j^{[r]}(\alpha,\beta)(r)_{n-j}\beta^{n-j}}{\alpha^{n-1}} &= -\alpha n!\sum_{j=0}^{n}\frac{(-1)^j c_j^{[r]}(\alpha,\beta)}{j!\alpha^j}\times\frac{(r)_{n-j}}{(n-j)!}\left(\frac{\beta}{\alpha}\right)^{n-j} \\ &= -\alpha n!\sum_{j=0}^{n}[t^j]\frac{-t/\alpha}{(1+\beta t/\alpha)^r\ln(1+t)}[t^{n-j}]\left(1+\frac{\beta t}{\alpha}\right)^r \\ &= -n![t^n]\frac{t}{\ln(1+t)}, \quad r<0, \end{aligned}$$

$$\begin{aligned}
\sum_{j=0}^{n}\frac{(n)_j c_{n-j}^{[r]}(\alpha,\beta)\alpha^{j+1}}{j+1} &= n!\sum_{j=0}^{n}\frac{c_{n-j}^{[r]}(\alpha,\beta)\alpha^{j+1}}{(n-j)!(j+1)}\\
&= n!\sum_{j=0}^{n}[t^{n-j}]\frac{t}{(1-\beta t)^r\ln(1-\alpha t)}[t^j]\frac{-\ln(1-\alpha t)}{t}\\
&= -n![t^n]\frac{1}{(1-\beta t)^r}\\
&= -n!\binom{n+r-1}{n}\beta^n, \quad r\geq 1.
\end{aligned}$$

That indicates that (2.3)-(2.6) hold. ■

Obviously, (2.4) correlates $c_n^{[r]}(\alpha,\beta)$ and a_n, and (2.6) is a generalization of (2.1). On the other hand, there holds some relations between Stirling numbers and Cauchy numbers [4, 9] such as

$$\begin{aligned}
a_n &= \sum_{j=0}^{n}\frac{s(n,j)}{j+1},\\
\frac{1}{n+1} &= \sum_{j=0}^{n}S(n,j)a_j. \qquad (2.7)
\end{aligned}$$

Now we give some relations for $c_n^{[r]}(\alpha,\beta)$ and Stirling numbers.

Theorem 2.2 *Let $n\geq 0$ and r be integers. We have the following identities:*

$$\begin{aligned}
&\sum_{j=0}^{n}\binom{n+r+1}{j}c_j^{[r]}(\alpha,\beta)(-\alpha)^{n-j}s(n-j+r+1,r+1)\\
&= -\frac{1}{\alpha}\binom{n+r+1}{r+1}P_n^{[r]}(H_2^{[1]}(\alpha,\beta)/\alpha, H_3^{[1]}(\alpha,\beta)/\alpha,\cdots,H_{n+1}^{[1]}(\alpha,\beta)/\alpha), \quad r\geq 0, \qquad (2.8)
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=0}^{n}S(n,j)c_j^{[r]}(-1,\beta)\\
&= \frac{1}{n+1}\sum_{j=0}^{-r}S(n+1,j+1)(-1)^j(j+1)(-r)_j\beta^j, \qquad r\leq 0, \quad n\geq -r. \qquad (2.9)
\end{aligned}$$

Proof. From the generating functions of $c_n^{[r]}(\alpha,\beta)$, $s(n,k)$ and $S(n,k)$, we have

$$\begin{aligned}
&\sum_{j=0}^{n}\binom{n+r+1}{j}c_j^{[r]}(\alpha,\beta)(-\alpha)^{n-j+r+1}s(n-j+r+1,r+1)\\
&=\frac{(n+r+1)!}{(r+1)!}\sum_{j=0}^{n}[t^j]\frac{t}{(1-\beta t)^r\ln(1-\alpha t)}[t^{n-j+r+1}]\ln^{r+1}(1-\alpha t)\\
&=\frac{(n+r+1)!}{(r+1)!}[t^{n+r+1}]\frac{t\ln^r(1-\alpha t)}{(1-\beta t)^r}\\
&=\frac{(n+r+1)!}{(r+1)!}[t^{n+r}]\frac{\ln^r(1-\alpha t)}{(1-\beta t)^r}\\
&=\frac{(n+r+1)!}{(r+1)!}[t^{n+r}]t^r\left(-\sum_{n=0}^{\infty}H_{n+1}^{[1]}(\alpha,\beta)t^n\right)^r\\
&=\frac{\alpha^r(n+r+1)!}{(r+1)!}[t^n]\left(-\sum_{n=0}^{\infty}\frac{H_{n+1}^{[1]}(\alpha,\beta)}{\alpha}t^n\right)^r\\
&=\frac{(-\alpha)^r(n+r+1)!}{n!(r+1)!}P_n^{[r]}(H_2^{[1]}(\alpha,\beta)/\alpha,H_3^{[1]}(\alpha,\beta)/\alpha,\cdots,H_{n+1}^{[1]}(\alpha,\beta)/\alpha)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=0}^{n}\frac{S(n,j)c_j^{[r]}(-1,\beta)}{n!} &= [t^n]\frac{y}{(1-\beta y)^r\ln(1+y)}\bigg|_{y=e^t-1}\\
&= [t^n]\frac{e^t-1}{t(1-\beta e^t+\beta)^r}\\
&= [t^n]\sum_{j=0}^{-r}(-1)^j(j+1)(-r)_j\beta^j\sum_{p=0}^{\infty}\frac{S(p+j+1,j+1)t^{p+j}}{(p+j+1)!},\quad r\le 0.
\end{aligned}$$

Hence (2.8)-(2.9) hold. ■

We note that (2.9) becomes 2.7) when $r=0$.

In a similar way, we can prove the following result.

Theorem 2.3 *Let $m\ge 1$, $n\ge 1$ and $r\ge 0$ be integers. For $c_n^{[r]}(\alpha,\beta)$ and $H_n^{[m]}(\alpha,\beta)$, we have*

$$\sum_{j=0}^{n}\frac{c_j^{[r]}(\alpha,\beta)H_{n-j+1}^{[m]}(\alpha,\beta)}{j!} = -\binom{n+m+r-1}{n}\beta^n.$$

3 Some Asymptotic Approximations

In general, it is difficult to compute the exact values of the sums involving $c_n^{[r]}(\alpha,\beta)$; therefore, it is very important to give their asymptotic approximations. We first recall two lemmas [1,5,6].

Lemma 3.1 *(Bender's Theorem) Suppose that* $A(z) = \sum A_n z^n$ *and* $B(z) = \sum B_n z^n$ *are power series with radii of convergence* $a > b \geq 0$*, respectively, and* B_{n-1}/B_n *approaches* b *as* $n \to \infty$*. If* $A(b) \neq 0$*, then* $C_n \sim A(b)B_n$*, where* $\sum C_n z^n = A(z)B(z)$*.*

Lemma 3.2 *Let* k *be a positive integer,* m *be a nonnegative integer,* λ *and* ν *be complex numbers with* $\lambda, \nu \notin \mathbf{Z}_{\geq 0}$ *and*

$$L(t) = \ln \frac{1}{1-t}.$$

Then

$$[t^n](1-t)^\lambda \left(\frac{1}{t}L(t)\right)^\nu \quad \sim \quad \frac{\ln^\nu n}{n^{\lambda+1}\Gamma(-\lambda)}, \quad n \to \infty, \tag{3.1}$$

$$[t^n](1-t)^\lambda (L(t))^k \quad \sim \quad \frac{\ln^k n}{n^{\lambda+1}\Gamma(-\lambda)}, \quad n \to \infty, \tag{3.2}$$

$$[t^n](1-t)^m (L(t))^k \sim \frac{(-1)^m k m! \ln^{k-1} n}{n^{m+1}}, \quad n \to \infty. \tag{3.3}$$

As a consequence of Lemmas 3.1 and 3.2, we obtain the asymptotic approximation for the Cauchy numbers of both kinds [4]:

$$\frac{a_n}{n!} \quad \sim \quad \frac{(-1)^{n+1}}{n \ln^2 n}, \quad n \to \infty, \tag{3.4}$$

$$\frac{b_n}{n!} \quad \sim \quad \frac{1}{\ln n}, \quad n \to \infty. \tag{3.5}$$

We next derive some asymptotic approximations related to $c_n^{[r]}(\alpha, \beta)$.

Theorem 3.1 *Let* r *be a positive integer. When* $n \to \infty$*, we have*

$$\frac{c_n^{[r]}(\alpha, \beta)}{n!} \quad \sim \quad \frac{\alpha^{n+r-1}}{(\alpha-\beta)^r n \ln^2 n}, \quad |\beta| < |\alpha|, \tag{3.6}$$

$$\frac{c_n^{[r]}(\alpha, \alpha)}{n!} \quad \sim \quad -\frac{\alpha^{n-1} n^{r-1}}{(r-1)! \ln n}. \tag{3.7}$$

Proof. It is obvious that

$$\sum_{n=0}^{\infty} c_n^{[r]}(\alpha, \beta) \frac{(-1)^n t^n}{\alpha^n n!} \quad = \quad -\frac{t}{\alpha(1+\beta t/\alpha)^r \ln(1+t)},$$

$$\sum_{n=0}^{\infty} \frac{c_n^{[r]}(\alpha, \alpha)}{\alpha^n n!} t^n \quad = \quad \frac{t}{\alpha(1-t)^r \ln(1-t)}.$$

One observes that the radius of $\frac{t}{\ln(1+t)}$ is 1. By means of Lemma 3.1, we get

$$\frac{(-1)^n c_n^{[r]}(\alpha,\beta)}{\alpha^n n!} \sim \frac{(-1)^n \alpha^{r-1}}{(\alpha-\beta)^r n \ln^2 n}, \quad n \to \infty.$$

Then (3.6) holds.

By (3.1), we have

$$\begin{aligned} \frac{c_n^{[r]}(\alpha,\alpha)}{\alpha^n n!} &= -\frac{1}{\alpha}[t^n](1-t)^{-r}\left(\frac{1}{t}L(t)\right)^{-1} \\ &\sim -\frac{1}{\alpha n^{-r+1}\Gamma(r)\ln n}. \end{aligned}$$

Noting that $\Gamma(r) = (r-1)!$, we have (3.7). ■

Note that (3.4) and (3.5) are special cases of Theorem 3.1.

Theorem 3.2 *Suppose that n and $k \geq 2$ are positive integers. When k is fixed and $n \to \infty$, we have*

$$\sum_{j=0}^{n} \frac{(-1)^j c_j^{[r]}(\alpha,\beta) s(n-j+k,k)}{\alpha^j j!(n-j+k)!} \sim \frac{(-1)^{n+k+1}\alpha^{r-1}\ln^{k-2}(n+k-1)}{k(k-2)!(\alpha-\beta)^r(n+k-1)}, \quad r \geq 1, \quad |\beta| < |\alpha|, \tag{3.8}$$

$$\sum_{j=0}^{n} \frac{(-1)^j c_j^{[r]}(\alpha,\alpha) s(n-j+k,k)}{\alpha^j j!(n-j+k)!} \sim \begin{cases} \dfrac{(-1)^{n+k+1}\ln^{k-1}(n+k-1)}{k!\alpha(r-1)!(n+k-1)^{-r+1}}, & r \geq 1 \\ \dfrac{(-1)^{n+k+r+1}(-r)!\ln^{k-2}(n+k-1)}{k(k-2)!\alpha(n+k-1)^{-r+1}}, & r \leq 0, \end{cases} \tag{3.9}$$

$$\sum_{j=0}^{n} \frac{c_j^{[r]}(\alpha,\alpha)}{\alpha^j j!}\binom{n-j+k-1}{k-1} \sim -\frac{n^{k+r-1}}{\alpha(k+r-1)!\ln n}, \quad k+r > 0. \tag{3.10}$$

Proof. One can verify that

$$\begin{aligned} \sum_{j=0}^{n} \frac{(-1)^{n-j} c_j^{[r]}(\alpha,\beta) s(n-j+k,k)}{\alpha^j j!(n-j+k)!} &= \sum_{j=0}^{n}[t^j]\frac{t}{\alpha(1-\beta t/\alpha)^r \ln(1-t)}[t^{n-j}]\frac{\ln^k(1-t)}{k!t^k} \\ &= [t^n]\frac{\ln^{k-1}(1-t)}{\alpha(1-\beta t/\alpha)^r k! t^{k-1}} \\ &= (-1)^{k-1}[t^n]\frac{(L(t))^{k-1}}{\alpha(1-\beta t/\alpha)^r k! t^{k-1}} \\ &= (-1)^{k-1}[t^{n+k-1}]\frac{(L(t))^{k-1}}{\alpha(1-\beta t/\alpha)^r k!}, \end{aligned}$$

$$\sum_{j=0}^{n} \frac{(-1)^{n-j} c_j^{[r]}(\alpha,\alpha) s(n-j+k,k)}{\alpha^j j!(n-j+k)!} = -[t^{n+k-1}]\frac{(L(t))^{k-1}}{\alpha k!(1-t)^r},$$

and

$$\sum_{j=0}^{n} \frac{c_j^{[r]}(\alpha,\alpha)}{\alpha^j j!}\binom{n-j+k-1}{k-1} = [t^n]\frac{t}{\alpha(1-t)^{k+r}\ln(1-t)}$$

$$= (-1)^{k-1}\frac{1}{\alpha}[t^n](1-t)^{-k-r}\left(\frac{1}{t}L(t)\right)^{-1}.$$

When k is fixed, the radius of $\ln^k(1+t)$ is 1. Owing to (3.3), we get

$$[t^{n+k-1}](L(t))^{k-1} \sim \frac{(k-1)!\ln^{k-2}(n+k-1)}{n+k-1}, \quad n\to\infty.$$

By Lemma 3.1, we have (3.8). Furthermore, (3.9) follows from (3.2)-(3.3) and (3.10) follows from (3.1). This completes the proof. ∎

In the rest of this section, we give some examples to compare the exact values with the asymptotic values. In (3.8) of Theorem 3.2, let $\alpha=1,\quad \beta=1/2,\quad k=2,$

$$X_{n,r} = \sum_{j=0}^{n} \frac{(-1)^j c_j^{[r]}(1,1/2)s(n-j+2,2)}{j!(n-j+2)!}, \quad Y_{n,r} = \frac{2^{r-1}(-1)^{n+1}}{n+1}.$$

We know that

$$X_{n,1} = \frac{(-1)^{n+1}}{2}\sum_{j=0}^{n}\frac{1}{2^j(n-j+1)}, \quad X_{n,2} = \frac{(-1)^{n+1}}{2}\sum_{j=0}^{n}\frac{j+1}{2^j(n-j+1)}.$$

Table 1 Values of $X_{n,1}$ and $Y_{n,1}$

n	$X_{n,1}$	$Y_{n,1}$	$\lvert 1-Y_{n,1}/X_{n,1}\rvert$
50	-0.0200171	-0.0196078	2.044549×10^{-2}
100	-0.0100021	-0.0099099	1.010531×10^{-2}
150	-0.00666727	-0.00662252	6.712653×10^{-3}
200	-0.00500025	-0.00497512	5.025644×10^{-3}

Table 2 Values of $X_{n,2}$ and $Y_{n,2}$

n	$X_{n,2}$	$Y_{n,2}$	$\lvert 1-Y_{n,2}/X_{n,2}\rvert$
50	-0.0408893	-0.0392157	4.093128×10^{-2}
100	-0.0202105	-0.019802	2.021508×10^{-2}
150	-0.0134253	-0.013245	1.342658×10^{-2}
200	-0.00100513	-0.00995025	1.005181×10^{-2}

The results are reported in Tables 1 and 2. From the tables, we find that, at least for these examples, the values of $|1-Y_{n,1}/X_{n,1}|$ and $|1-Y_{n,2}/X_{n,2}|$ get smaller and smaller with the increasing of n.

4 Conclusions

We have presented a new class of generalized Cauchy numbers. Some identities related to the new generalized Cauchy numbers have been established. We have also discussed the asymptotic approximations of $c_n^{[r]}(\alpha, \beta)$ when r is fixed and $n \to \infty$. Perhaps, we may consider the asymptotic approximations of $c_n^{[r]}(\alpha, \beta)$ when $r \to \infty$ and $n \to \infty$. We leave it as a future work.

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