SOME PROPERTIES OF CYCLIC FLATS OF AN INFINITE MATROID

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ABSTRACT. We prove that when a pre-independence space satisfies some natural properties, then its cyclic flats form a bounded lattice under set inclusion. Additionally, we show that a bounded lattice is isomorphic to the lattice of cyclic flats of a pre-independence space. We also prove that the notion of cyclic width gives rise to dual-closed and minor-closed classes of B-matroids. Finally, we find a difference between finite matroids and B-matroids by using the notion of well-quasi-ordering.

1. INTRODUCTION

For finite matroids, J. Bonin et.al. said in [1]: cyclic flats of a matroid have played several important roles in matroid theory. This paper will discuss some properties relating to cyclic flats of an infinite matroid. Though J. Oxley pointed out in [2]: there is no single class of structures that one calls infinite matroids, one does notice that most of infinite matroids presented in [2, 3, 9], etc. are pre-independence spaces. Thus we find that pre-independence spaces are widespread and have a profound influence among infinite matroids. Therefore, this paper adopts pre-independence spaces to study and establish the relationship between bounded lattices and cyclic flats of pre-independence spaces. Simultaneously, it explores the class of B-matroids– a special class of pre-independence spaces and shows that the class of cyclic width k or less among B-matroids is closed under duals and minors.

We will narrate our working investigations in this paper as follows.

In light of the definition of pre-independence space (cf. Definition 1) and the definition of independence space (cf. [2], p. 387& 3, p. 74), we know that an independence space is a pre-independence space, but [2], Example 3.1.1, and [3], p.387, Theorem 3 and Theorem 4, together hint that a pre-independence space is perhaps not an independence space.

In addition, we notice that Sims [10] presents a way to show that any lattice of finite length is isomorphic to the lattice of fully dependent flats of a rank-finite independence space (*a fully dependent flat is called a cyclic flat in the present paper*).

On the other hand, according to [4, 5], we may observe that a finite length lattice is a bounded one, but not vice versa.

Hence, we need to build up the relationship between a bounded lattice and an infinite matroid. We may solve this relationship with the help of cyclic flats of an infinite matroid. Analyzing the results of Sims, we will not choose the set of rank-finite independence spaces. Recalling back that the viewpoint of J. Oxley in [2]: *there is no single class of structures that one calls infinite matroids, ..., it is natural to ask which family of infinite matroids are suitable for playing the role in the above relationship?*

Date: 2012-05-06.

2010 *Mathematics Subject Classification.* 05B35.

Key words and phrases. cyclic flat; lattice; pre-independence space; B-matroid.

This research is supported by NSF of China(11101115,61202178,61073121) and NSF of Hebei Province (F2012402037, A2013201119).

We exactly choose the family of pre-independence space as the discussing object. The results from Section 2 to Section 4 will demonstrate the correctness of this choice. Of course, in view of Section 2 and Section 3, we can say that the results here are a generalization of [1] with [10].

First of all, we may begin by reviewing some knowledge what will be needed later on. In what follows, we assume that S is some arbitrary–possibly infinite–set.

Definition 1.1. From [2], p. 74 & 3, pp. 385–387, A *pre-independence space* $M_p(S)$ is a set S together with a collection $\mathscr{I}$ of subsets of S(called *independent sets*) such that
(i1) $\mathscr{I} \neq \emptyset$;
(i2) A subset of an independent set is independent;
(i3) If $I_1, I_2 \in \mathscr{I}$ with $|I_1| < |I_2| < \infty$, then there exists $x \in I_2 - I_1$ such that $I_1 \cup x \in \mathscr{I}$.
$X \subseteq S$ is called *dependent* if $X \notin \mathscr{I}$. A *circuit* of $M_p(S)$ is a minimal dependent set, and a *basis* of $M_p(S)$ is a maximal independent set.

Definition 1.2. From [2], p. 80, A *B-matroid* $M_B(S)$ is a set S together with a collection $\mathscr{I}$ of subsets of S(called *independent sets*) such that $\mathscr{I}$ satisfies (i1) and (i2) together with
(I_B1) If $T \subseteq X \subseteq S$ and $T \in \mathscr{I}$, then there is a maximal $\mathscr{I}$-subset of X containing T.
(I_B2) For all $X \subseteq S$, if B_1 and B_2 are maximal $\mathscr{I}$-subsets of X and $x \in B_1 - B_2$, then there is an element y of $B_2 - B_1$ such that $(B_1 - x) \cup y$ is a maximal $\mathscr{I}$-subset of X.
$X \notin \mathscr{I}$ is called *dependent*. A maximal member of $\mathscr{I}$ is called a *basis*.
From [2], pp. 83–84, if f is defined as $f(X) = X \cup \{x : I \cup x \notin \mathscr{I}$ for some $I \subseteq X$ such that $I \in \mathscr{I}\}$ for all $X \subseteq S$, then we call f the *closure operator* of $M_B(S)$.

Remark 1.3. (1) A minimal dependent set of $M_B(S)$ is a *circuit*. Let f be the closure operator of $M_B(S)$. If $f(X) = X$, then X is called a *flat* of $M_B(S)$.
(2) Our lattice theory follows [4, 5]. A *bounded poset* is referred to [5], p. 62. We assume that the reader is familiar with vector space theory (cf.[6]).
We use 0_L and 1_L for the least and greatest elements of a bounded lattice L. For $x, y \in L$, if x and y are incomparable, then it will be denoted by $x||y$.
(3) In this paper, the family of all the circuits of $M_p(S)$ is denoted by $\mathscr{C}(M_p(S))$.
(4) Since all the discussion in [1] are of finite cases, it follows that all the lattices appearing in [1] are bounded, though this is not pointed out in [1]. Thus some of results here are the generalizations of that found in [1].

2. Some properties of pre-independence spaces

This section discusses some properties about pre-independence spaces and postulates that if a pre-independence space satisfies some special properties, its cyclic flats form a lattice.

Property 2.1. Let $M_p(S) = (S, \mathscr{I}_p)$ be a pre-independence space. The *closure operator* of $M_p(S)$ is a function $\sigma_p : 2^S \to 2^S$ as $X \mapsto X \cup \{x : I \cup x \notin \mathscr{I}$ for some $I \subseteq X$ and $I \in \mathscr{I}\}$. A subset $X \subseteq S$ is a *flat* of $M_p(S)$ if $\sigma_p(X) = X$.
(1) σ_p satisfies the following properties
(cl1) $X \subseteq \sigma_p(X)$ for all $X \subseteq S$.
(cl2) $X \subseteq Y \subseteq S$ implies $\sigma_p(X) \subseteq \sigma_p(Y)$.
(2) If σ_p satisfies (cl3): $\sigma_p(\sigma_p(X)) = \sigma_p(X)$ for $\forall X \subseteq S$. Then
(2I) $\sigma_p(X)$ is the smallest flat containing X.

(2II) That A, B are flats of $M_p(S)$ asserts that $A \cap B$ is a flat of $M_p(S)$.
(2III) $\mathscr{I}_p|T = \{Y \subseteq T : Y \in \mathscr{I}_p\}$ is the collection of independent sets of a pre-independence space $M_p|T$ on $T \subseteq S$. One calls it the *restriction* of $M_p(S)$ to T. The closure operator σ_{pT} of $M_p|T$ states that for any $Y \subseteq T$, there is a flat X of $M_p(S)$ satisfying $\sigma_{pT}(Y) = X \cap T$.
(2IV) The set of circuits of $M_p|T$ is $\mathscr{C}(M_p|T) = \{C \subseteq T : C \in \mathscr{C}(M_p(S))\}$.

Proof. (1) Routine verification.

(2) (2I) $\sigma_p(X)$ is a flat since (cl3). $X \subseteq \sigma_p(X)$ as (cl1). Let Y be a flat and $X \subseteq Y \subseteq \sigma_p(X)$. By (cl2) and (cl3), $\sigma_p(X) \subseteq \sigma_p(Y) \subseteq \sigma_p(\sigma_p(X)) = \sigma_p(X)$, and hence, $Y = \sigma_p(X)$.

(2II) $x \in \sigma_p(A \cap B) - (A \cap B)$ means $I \cup x \notin \mathscr{I}_p$ for some $I \in \mathscr{I}_p$ with $I \subseteq A \cap B$, further, $I \subseteq A, B$, and so, $x \in \sigma_p(A) = A, \sigma_p(B) = B$. Thus, $x \in A \cap B$. Say $\sigma_p(A \cap B) = A \cap B$.

(2III) It is a straightforward to assurance that $\mathscr{I}_p|T$ satisfies (i1)-(i3).

Let $Y \subseteq T$ and $X = \sigma_p(Y)$. Then $\sigma_{pT}(Y) = Y \cup \{y : I_T \cup y \notin \mathscr{I}_p|T$ for some $I_T \subseteq Y$ and $I_T \in \mathscr{I}_p|T\} = Y \cup \{y \in T : I_T \cup y \notin \mathscr{I}$ for some $I_T \subseteq Y$ and $I_T \notin \mathscr{I}_p\}$. On the other hand, $X \cap T = (Y \cap T) \cup (\{x : I \cup x \notin \mathscr{I}_p$ for some $I \subseteq Y$ and $I \in \mathscr{I}_p\} \cap T)$. Hence, $\sigma_{pT}(Y) = X \cap T$.

(2IV) It is straightforward.

□

Remark 2.2. In [2], p. 74, Example 3.1.1, shows that some of pre-independence spaces have no circuits or no bases, and so $\mathscr{I}_p^* = \{X : S - X$ contains a basis of $M_p(S)\}$ and $\mathscr{I}_p.T = \{Y \subseteq T : Y \cup B \in \mathscr{I}_p$ for some basis B of $M_p|T\}$ is not guaranteed to be the set of independent sets of a pre-independence space on S and on $T \subseteq S$ respectively. Besides, this example implies that $\mathscr{I}_p$ does not satisfy (I_B1). However, for some class of pre-independence spaces such as B-matroids, both $\mathscr{I}_p^*$ and $\mathscr{I}_p.T$ exist as B-matroids $M_B^*(S) = (S, \mathscr{I}_p^*)$(the *dual* of $M_B(S)$) and $M_B.T = (T, \mathscr{I}_p.T)$ (the *contraction* of $M_B(S)$ to T) (cf.[2]).

As in finite matroids, a flat of $M_p(S)$ is *cyclic* if it is a (possibly empty) union of circuits. The family of cyclic flats of $M_p(S)$ is denoted by $L(M_p)$. Then

Theorem 2.3. *Let $M_p(S) = (S, \mathscr{I})$ be a pre-independence space such that $\mathscr{C}(M_p(S))$ exists and for $X \notin \mathscr{I}$, there is a circuit $C \in \mathscr{C}(M_p(S))$ satisfying $C \subseteq X$, and additionally, σ_p satisfies (cl3). Then $(L(M_p), \subseteq)$ forms a bounded lattice and for any $A, B \in L(M_p)$, $A \vee B = \sigma_p(A \cup B)$ and $A \wedge B$ is the union of all circuits contained in $A \cap B$.*

Proof. By Property 2.1, $\sigma_p(A \cup B)$ is the smallest flat of $M_p(S)$ containing A and B, and so, one does need only to prove that $\sigma_p(A \cup B)$ is the union of circuits of $M_p(S)$.

Let $x \in \sigma_p(A \cup B) - (A \cup B)$. One has $I_{A\cup B} \cup x \notin \mathscr{I}$ for some $I_{A\cup B} \in \mathscr{I}$ and $I_{A\cup B} \subseteq A \cup B$, and so, there is a circuit C_x satisfying $C_x \subseteq I_{A\cup B} \cup x$. Further, by (i2), $x \in C_x$ is true. Thus $\sigma_p(A \cup B)$ is a cyclic flat. Considered with the definition of $L(M_p)$, then $A \vee B = \sigma_p(A \cup B)$ is followed.

Let $X = \bigcup_{A\cap B \supseteq C \in \mathscr{C}(M_p(S))} C$. We seek to prove that X is a flat of $M_p(S)$. Put $x \in \sigma_p(X) - X$. Then $I \cup x \notin \mathscr{I}$ for some $I \subseteq X$ and $I \in \mathscr{I}$. Considering (i2) with the given, it follows that there is a circuit C_x of $M_p(S)$ satisfying $x \in C_x \subseteq I \cup x$. Because $A \cap B$ is a flat by Property 2.1, one has $\sigma_p(X) \subseteq A \cap B$, and so $x \in A \cap B$, further, $C_x \subseteq A \cap B$. Hence $C_x \subseteq X$. Herein $\sigma_p(X) = X$. Obviously, $A \wedge B = X$.

Similarly to the above, $\bigcup_{S \supseteq C \in \mathscr{C}(M_p(S))} C$ is the greatest element in $(L(M_p), \subseteq)$. Evidently, $\sigma_p(\emptyset)$ is the least element in $(L(M_p), \subseteq)$.

□

In what follows, the lattice $(L(M_p), \subseteq)$ is said to be $L(M_p)$, if there is no confusion from the context.

3. The lattice of cyclic flats

In [7], H. Mao presents that every finite height geometric lattice is isomorphic to the lattice of flats of some matroid of arbitrary cardinality. Here, we only discuss more general cases and obtains that for any bounded lattice L, there is a pre-independence space $M_p(S)$ for which the lattice $L(M_p)$ is isomorphic to L.

Theorem 3.1. *Every bounded lattice L is isomorphic to the lattice $L(M_p)$ of cyclic flats of a pre-independence space M_p in which $\mathscr{C}(M_p)$ exists and if X is not an independent set of M_p, then there is $C \in \mathscr{C}(M_p)$ satisfying $C \subseteq X$, and additionally, σ_p satisfies (cl3).*

Proof. We construct a pre-independence space M_p for which $L(M_p)$ is isomorphic to L. We will finish the proof by five steps.

Step 1. (I) Let $\mathbb{B}$ be the set of elements in L other than 1_L. For $z \in L$, let $V_z = \{y : z \nleq y\}$. Evidently, $V_z \subseteq \mathbb{B}$ and $z \notin V_z$ for $z \in L$.

Let $f : z \mapsto V_z$. Then $(\{V_z : z \in L\}, \subseteq)$ is a lattice, where $V_{x\wedge z} = V_x \wedge V_z, V_{x\vee z} = V_x \vee V_z$ for $x, z \in L$. The closely following proofs (III) and (V) inform us that these definitions of lattice operators for $(\{V_z : z \in L\}, \subseteq)$ are effective.

(II) We will prove that f is an isomorphism between L and the lattice $(\{V_z : z \in L\}, \subseteq)$.

Let $x, z \in L$ and $x < z$. As $V_z = \{y \in \mathbb{B} : y < z\} \cup \{y \in \mathbb{B} : y||z\}$ and $V_x = \{w \in \mathbb{B} : w < x\} \cup \{w \in \mathbb{B} : w||x\}$, $w \in V_x$ assures $w < x$ or $w||x$. But if $w < x$, then $w < x < z$; if $w||x$, then $w||z$ or $w < z$. No matter which status happens, it has $w \in V_z$. Namely, $V_x \subset V_z$.

Conversely, let $V_x \subset V_z$.

Assume that $z < x$. By the above proof, one has $V_z \subset V_x$, a contradiction with $V_x \subset V_z$.

Assume that $x||z$. It has $z \in V_x$, and so $z \in V_z$ since $V_x \subset V_z$, a contradiction to $z \notin V_z$ according to the definition of V_z.

Therefore, $x < z$ is real.

It is not difficult to see that f is a bijection. By the definition of isomorphism between two lattices in [4], p. 19, it follows that f is an isomorphism. Besides, $f(0_L) = V_{0_L} = \{y : 0_L \nleq y\} = \emptyset$, $f(1_L) = V_{1_L} = \mathbb{B}$ and $f(z) \neq \emptyset$ for $z \in \mathbb{B} - 0_L$.

(III) The following sets to prove that $V_{x\wedge z} = V_x \wedge V_z$ is effective for $x, z \in L$.

The above (II) and $x \wedge z \leq x, z$ causes $V_{x\wedge z} \subseteq V_x, V_z$, and so $V_{x\wedge z} \subseteq V_x \cap V_z$. Let $V_t \subseteq V_x \cap V_z$. Then $t \leq x, z$ by (II), and so $t \leq x \wedge z$, further $V_t \subseteq V_{x\wedge z}$. Thus, $V_{x\wedge z} = V_x \wedge V_z$.

(IV) Next to prove $V_{x\vee z} = V_x \cup V_z$ for $x, z \in L$.

$x, z \leq x \vee z$ and the above (II) taken together leads to $V_x \cup V_z \subseteq V_{x\vee z}$. Let $y \in V_{x\vee z}$.

Assume that $y < x \vee z$. Under this assumption, there are three cases to be considered as follows.

If $x < y$. Then $z \nleq y$, otherwise, $x \vee z \leq y$, a contradiction. It implies $y \in V_z \subseteq V_x \cup V_z$.
If $x||y$. Then $y \in V_x \subseteq V_x \cup V_z$.
If $y < x$. Then $y \in V_x \subseteq V_x \cup V_z$.

Similarly, for the relation between z and y, one has $y \in V_x \cup V_z$.

Assume that $y||(x \vee z)$.
This means $y \nleq x$ and $y \nleq z$. Hence $y \in V_x$ and $y \in V_z$, and so $y \in V_x \cup V_z$.

Summing up the above, $V_{x\vee z} = V_x \cup V_z$.

By the mathematical induction, one has $V_{\bigvee_{i=1}^{n} x_i} = \bigcup\limits_{i=1}^{n} V_{x_i}$.

(V) The result in (IV) implies that $V_{x\vee z} = V_x \vee V_z$ is effective.

(VI) The following will prove $V_{\bigvee_{\beta\in\mathscr{B}} x_\beta} = \bigcup_{\beta\in\mathscr{B}} V_{x_\beta}$ for $x_\beta \in L$ $(\forall \beta \in \mathscr{B})$.

It is easy to see that $\bigcup_{\beta\in\mathscr{B}} V_{x_\beta} \subseteq V_{\bigvee_{\beta\in\mathscr{B}} x_\beta}$. The following will prove the converse. Let $p = \bigvee_{\beta\in\mathscr{B}} x_\beta$ and $Q = \{q : p \text{ covers } q \text{ in } L\}$.

Assume that $|Q| = 1$. $p = \bigvee_{\beta\in\mathscr{B}} x_\beta = (\bigvee_{\beta\in\mathscr{B}-Q} x_\beta) \vee q$ causes that there is $\beta_0 \in \mathscr{B}$ satisfying $V_{\beta_0} = V_p$, and so, $V_p \subseteq V_{\beta_0} \cup (\bigcup_{\beta\in\mathscr{B}, x_\beta\neq p} V_{x_\beta}) \subseteq \bigcup_{\beta\in\mathscr{B}} V_{x_\beta}$ because $V_{x_\beta} \subseteq V_p$ for any $\beta \in \mathscr{B}$.

Assume that $2 \le |Q|$. If there is $\beta_0 \in \mathscr{B}$ satisfying $\beta_0 = p$, then similarly to the discussion for $|Q| = 1$, the need is obtained. The other cases are proved as follows.

Since $V_p = \{s : p \nleq s\} = \{s : s < p\} \cup \{s : s || p\}$.

If $s||p$. Then, because of $q_1 \vee q_2 = p$ for any $q_1, q_2 \in Q$, it follows that there is $q_p \in Q$ satisfying $q_p||s$, and so, $s \in V_{q_p} \subseteq \bigcup_{q\in Q} V_q \subseteq \bigcup_{\beta\in\mathscr{B}} V_{x_\beta}$.

If $s < p$ and $s \neq x_\beta$ for any $\beta \in \mathscr{B}$.

Suppose $s < q_s$ for some $q_s \in Q$. This brings about $s \in V_{q_s} \subseteq \bigcup_{q\in Q} V_q \subseteq \bigcup_{\beta\in\mathscr{B}} V_{x_\beta}$.

Suppose $s \not< q$ for any $q \in Q$. This means $s||q$ for any $q \in Q$, and further, $s \in V_q \subseteq \bigcup_{q\in Q} V_q \subseteq \bigcup_{\beta\in\mathscr{B}} V_{x_\beta}$.

If $s < p$ and $s = x_\beta$ for some $\beta \in \mathscr{B}$, then $s \in \bigcup_{\beta\in\mathscr{B}} V_{x_\beta}$ is evident.

Step 2. We write $\mathbb{B}$ in detailed as $\mathbb{B} = \{b_j : j \in \mathscr{J}\}$. Let F be an infinite field with $0, 1$ as the zero and unit element in F respectively. The map $g : b_j \mapsto a_j = (0, \dots, 1, 0, \dots)$, where a_j is a vector whose entries are all zero except for a 1 in the jth position. Let $V_F = \{x = \sum_{\alpha\in\mathscr{A}_x} k_\alpha a_\alpha : k_\alpha \in F\}$. We prove easily that V_F satisfies V1-V8 in [6], p. 2, Definition 1.4, and hence, V_F is a vector space over F. g is evidently a bijection between $\mathbb{B}$ and $\{a_j : j \in \mathscr{J}\}$. We say that there is no difference between a_j and b_j whenever it is clear from the context.

For $\forall z \in \mathbb{B}$, one has $V_z = \{y \in L : z \nleq y\} = \{a_\alpha : z \nleq y, a_\alpha = g(y), \alpha \in \mathscr{Z}_z\}$. Simply say $V_z = \{a_\alpha : \alpha \in \mathscr{Z}_z\}$. Select one $S_z = \sum_{\alpha\in\mathscr{Z}_z} k_\alpha a_\alpha$ where $k_\alpha \neq 0$ for $\forall \alpha \in \mathscr{Z}_z$. Let $V = \mathbb{B} \cup \{S_z : z \in \mathbb{B}\}$, or say, $V = \{a_j : j \in \mathscr{J}\} \cup \{S_z : z \in \mathbb{B}\}$. Then $V \subseteq V_F$.

Step 3. (I) Let $\mathscr{I}_F = \{X = \{x_\alpha : \alpha \in \mathscr{A}_X\} \subseteq V_F : \sum_{\alpha\in\mathscr{A}_X} k_\alpha x_\alpha = 0 \Leftrightarrow k_\alpha = 0$ for all $\alpha \in \mathscr{A}_X$ where $k_\alpha \in F$ $(\alpha \in \mathscr{A}_X)\}$. One asserts that $(V_F, \mathscr{I}_F)$ is a pre-independence space.

Since $\mathscr{I}_F \ni \{a_j\} \subseteq V_F$ for $j \in \mathscr{J}$ is evident, one has $\mathscr{I}_F \neq \emptyset$, i.e. (i1) holds for $\mathscr{I}_F$.

Let $V_F \supseteq Y \subseteq X = \{x_\alpha : \alpha \in \mathscr{A}_X\} \in \mathscr{I}_F$. If $Y = \{y_\alpha : \alpha \in \mathscr{A}_Y\} \notin \mathscr{I}_F$. Then there exist nonzero scalars $p_\alpha \in F$, $(\alpha \in \mathscr{A}_Y)$ such that $\sum_{\alpha\in\mathscr{A}_Y} p_\alpha y_\alpha = 0$. Since $Y \subseteq X$ tells us $X = Y \cup \{x_\beta : \beta \in (\mathscr{A}_X - \mathscr{A}_Y)\}$. Put $q_\beta = 0$ for $\beta \in \mathscr{A}_X - \mathscr{A}_Y$ and $q_\alpha = p_\alpha$ for $\alpha \in \mathscr{A}_Y$. Then $\sum_{\alpha\in\mathscr{A}_Y} q_\alpha y_\alpha + \sum_{\beta\in\mathscr{A}_X-\mathscr{A}_Y} q_\beta x_\beta = 0$. Herein, $X \notin \mathscr{I}_F$, a contradiction. Thus (i2) holds.

Let $I_1, I_2 \in \mathscr{I}_F$ and $|I_1| < |I_2| < \infty$. Suppose $|I_1 \cap I_2| = k$. That is
$I_1 = \{x_1, \dots, x_k, x_{k+1}, \dots, x_n\}$, $I_2 = \{x_1, \dots, x_k, y_{k+1}, \dots, y_n, y_{n+1}, \dots, y_m\}$.

If $k = n-1, m = n+1$, then $I_1 = \{x_1, \dots, x_{n-1}, x_n\}$, $I_2 = \{x_1, \dots, x_{n-1}, y_n, y_{n+1}\}$. Suppose $I_1 \cup y_n, I_1 \cup y_{n+1} \notin \mathscr{I}_F$, i.e. $y_n = \sum_{i=1}^{n} f_i x_i = \sum_{i=1}^{n-1} f_i x_i + f_n x_n$, $y_{n+1} = \sum_{i=1}^{n} g_i x_i = \sum_{i=1}^{n-1} g_i x_i + g_n x_n$ $(f_i, g_i \in$

$F; i = 1, \ldots, n$). Since $I_2 \in \mathscr{I}_F$ and (i2), it follows $f_n, g_n \neq 0$, and so, $x_n = \frac{1}{f_n} y_n + \sum_{i=1}^{n-1} (-\frac{f_i}{f_n}) x_i$, further $y_{n+1} = \sum_{i=1}^{n-1} g_i x_i + \frac{g_n}{f_n} y_n + \sum_{i=1}^{n-1} (-\frac{f_i}{f_n} g_n) x_i = \sum_{i=1}^{n-1} (g_i - \frac{f_i}{f_n} g_n) x_i + \frac{g_n}{f_n} y_n$, a contradiction to $I_2 \in \mathscr{I}_F$.

By the mathematical induction, we have that (i3) holds.

Therefore, $(V_F, \mathscr{I}_F)$ is a pre-independence space, denoted it by M_p.

(II) Next to prove that M_p satisfies that $\mathscr{C}(M_p)$ exists and for any $Y \subseteq V_F$, if $Y \notin \mathscr{I}_F$, then there is $X \subseteq Y$ satisfying $X \in \mathscr{C}(M_p)$.

Let $a_{j_0} = (1, 0, \ldots)$ and $k \in F - 0$. Then it is easy to see that $\{k a_{j_0}, a_{j_0}\}$ is a circuit. Namely, $\mathscr{C}(M_p)$ exists.

$Y = \{y_\alpha : \alpha \in \mathscr{A}_Y\} \notin \mathscr{I}_F$ means that there are nonzero scalars $k_\alpha \in F$ $(\alpha \in \mathscr{A}_Y)$ satisfying $\sum_{\alpha \in \mathscr{A}_Y} k_\alpha y_\alpha = 0$. The set of subscripts relative with all nonzero elements in $\{k_\alpha : \alpha \in \mathscr{A}_Y\}$ is in notation D_Y. Then $\sum_{\alpha \in D_Y} k_\alpha y_\alpha = 0$. Besides, one should notice that for any $X_Y \subset D_Y$, $\sum_{\alpha \in X_Y} k_\alpha y_\alpha \neq 0$ according to the background of D_Y. Thus, we may say $\{y_\alpha : \alpha \in D_Y\} \in \mathscr{C}(M_p)$.

(III) The closure operator σ_p of M_p satisfying (I_B1) is shown in the following.

Let $T \in \mathscr{I}_F$ and $T \subseteq X \subseteq S$.

Assume that $|X| < \infty$. If $X - T = \{x\}$, then $T \cup x \in \mathscr{I}_F$ or not depends only on the property of x. Namely, if $T \cup x \in \mathscr{I}_F$, then $T \cup x$ is a maximal $\mathscr{I}_F$-subset of X, otherwise, T is the maximal one. All these results are found quite easily. By the mathematical induction, when $|X| < \infty$, we obtain the needed result.

Assume that $|X| \not< \infty$. According to the Well Order Principle in [8], p. 14, there is a well order $\le$ on $X - T$. Suppose $\{c : c < d\}$, $(d, c \in X - T)$, there is a maximal $\mathscr{I}_F$-subset I_T of $T \cup \{c : c < d\}$. It is easy to determine $I_T \cup d \in \mathscr{I}_F$ or $I_T \cup d \notin \mathscr{I}_F$ in view of the definition of $\mathscr{I}_F$. We may equivalently say, let $I_T = \{x_i : i \in \mathscr{T}_c\}$ be a maximal $\mathscr{I}_F$-subset of $T \cup \{c : c < d\}$, if $d = \sum_{i \in \mathscr{T}_c} k_i x_i$ $(k_i \in F, i \in \mathscr{T}_c)$, then $I_T \cup d \notin \mathscr{I}_F$; otherwise, $I_T \cup d \in \mathscr{I}_F$. Thus, if $I_T \cup d \in \mathscr{I}_F$, then $I_T \cup d$ is a maximal $\mathscr{I}_F$-subset of $T \cup \{c : c \le d\}$; if not, then I_T is a maximal one. Finally, by the Principle of Transfinite Induction (cf.[8], p. 14, Theorem 7.1), the needed result follows.

(IV) That σ_p satisfying (cl3) is proved as follows.

By (III), we have that there exists a basis $B = \{c_i : i \in \mathscr{B}_B\}$ of X for any $X = \{x_i : i \in \mathscr{S}_X\} \subseteq S$. We prove that $\sigma_p(X) = \sigma_p(B)$.

In virtue of (cl2), $\sigma_p(B) \subseteq \sigma_p(X)$. Because $B \in \mathscr{I}_F$, one has $\sigma_p(B) = B \cup \{q : B \cup q \notin \mathscr{I}_F\} = B \cup \{q \in V_F : q = \sum_{\delta \in \Delta_q} e_\delta c_\delta$ and $e_\delta \in F, \delta \in \Delta_q \subseteq \mathscr{B}_B\}$. Obviously, $q \neq 0$ since $c_\delta \neq 0, (\delta \in \Delta_q)$ and $B \in \mathscr{I}_F$. Besides $\sigma_p(X) = X \cup \{y : y = \sum_{i \in \mathscr{Y}_y} f_i x_i,\ (f_i \in F, i \in \mathscr{Y}_y)\}$. Since B is a basis of X, it follows that $x_i = \sum_{\gamma \in \mathscr{R}_{x_i}} (t_\gamma c_\gamma) \in \sigma_p(B), (t_\gamma \in F, \gamma \in \mathscr{R}_{x_i} \subseteq \mathscr{B}_B)$, and so, for $y \in \sigma_p(X) - X$, $y = \sum_{i \in \mathscr{Y}_y} f_i (\sum_{\gamma \in \mathscr{R}_{x_i}} (t_\gamma c_\gamma)) = \sum_{i \in \mathscr{Y}_y, \gamma \in \mathscr{R}_{x_i}} (f_i t_\gamma) c_\gamma$, and hence $y \in \sigma_p(B)$.

One gets $\sigma_p(X) = \sigma_p(B)$ and B is a basis of $\sigma_p(X)$. Furthermore, $\sigma_p(\sigma_p(X)) = \sigma_p(B)$, and so (cl3) holds for σ_p.

(V) By Theorem 2.3, $L(M_p)$ exists.

(VI) In view of Property 2.1, $M_p|V = (V, \mathscr{I} = \mathscr{I}_F|V)$ is a pre-independence space and $\mathscr{C}(M_p|V)$ exists. We may easily observe that $L(M_p|V)$ exists by Property 2.1 and Theorem

2.3.

Step 4. (I) It is obviously proven $V_z \cup S_z \in \mathscr{C}(M_p|V)$ from the definition of $\mathscr{I}$, where $\mathscr{C}(M_p|V)$ is the set of all the circuits of $M_p|V$. Furthermore, $V_z \cup \{S_x : x \leq z\}$ is the union of circuits because $V_x \subseteq V_z$ for $x \leq z$. Next, one proves $\mathscr{C}(M_p|V) = \{V_z \cup S_z : z \in \mathbb{B}\}$.

Suppose $C \in \mathscr{C}(M_p|V) - \{V_z \cup S_z : z \in \mathbb{B}\}$. By (i1), $C \neq \emptyset$. In virtue of (i2) and $C \notin \mathscr{I}$, one gets $C \cap \mathbb{B} \neq \emptyset$ and $C \cap \{S_z : z \in \mathbb{B}\} \neq \emptyset$. Let $S_z \in C$. By the definition of S_z and $C \notin \mathscr{I}$, one has $V_z \cup S_z \subseteq C$. By the minimality of C and $V_z \cup S_z \in \mathscr{C}(M_p|V)$, one gets $C = V_z \cup S_z$. This means that $V_z \cup S_z$ is the one and only single circuit containing S_z in $M_p|V$.

(II) We will prove that $V_z \cup \{S_x : x \leq z\}$ is a flat of $M_p|V$.

Let σ_{pV} be the closure operator of $M_p|V$. Because V_z is a maximal $\mathscr{I}$-subset in V_z, one has $\sigma_{pV}(V_z) = V_z \cup \{y : V_z \cup y \notin \mathscr{I}\}$. In addition, evidently, for any $I \subseteq V_z$ and $y \in \mathbb{B}$, it has $I \cup y \in \mathscr{I}$. Hence $y \in \sigma_{pV}(V_z) - V_z$ follows $y \in V - \mathbb{B} = \{S_z : z \in \mathbb{B}\}$. Moreover, $y = S_x$ for some $x \in \mathbb{B}$. If $x \leq z$, then $V_x \subseteq V_z$, and so, by the above step 4 (I), $V_x \cup S_x \in \mathscr{C}(M_p|V)$, and hence $V_z \cup S_x \notin \mathscr{I}$. If $x > z$, then $S_x = \sum_{\substack{a_i \in V_z \\ k_i \neq 0}} k_i a_i + \sum_{\substack{a_i \in V_x - V_z \\ t_i \neq 0}} t_i a_i$, and so $V_z \cup S_x \in \mathscr{I}$. If $x || z$, then $z \in V_x$. Certainly, $z \notin V_z$. Hence there is $a_z = g(z)$, where g is defined as in Step 2. Since $S_x = \sum_{\substack{a_i \in V_x = \{a_i : i \in \mathscr{H}_x\} \\ i \in \mathscr{H}_x, k_i \in F - 0}} k_i a_i = k_z a_z + \sum_{a_i \in V_x - z} k_i a_i$, one follows that S_x could not be represented by all the elements in V_z, and so $V_z \cup S_x \in \mathscr{I}$.

Consequently, $\sigma_{pV}(V_z) = V_z \cup \{S_x : x \leq z\}$.

(III) In view of $\mathbb{B} = \bigcup_{z \in \mathbb{B}} V_z$, one has $V = (\bigcup_{z \in \mathbb{B}} V_z) \cup \{S_z : z \in \mathbb{B}\} = \bigcup_{z \in \mathbb{B}} (V_z \cup S_z) = \bigcup_{z \in \mathbb{B}} (V_z \cup \{S_x : x \leq z\})$, and so, V is a cyclic flat of $M_p|V$. Therefore, the greatest and least elements in $L(M_p|V)$ are V and $\emptyset$ respectively.

(IV) One proves: for any cyclic flat $X \neq V$ of $M_p|V$, $X \in \{(V_z \cup \{S_x : x \leq z\}) : z \in \mathbb{B}\}$.

For $S_z \in X$, since $V_z \cup S_z \in \mathscr{C}(M_p|V)$ and the above closely (I), one has $V_z \cup S_z \subseteq X$.

For $v \in X \cap \mathbb{B}$, since $v \in X$ and X is a cyclic flat, one obtains that there is a circuit $C_v \subseteq X$ satisfying $v \in C_v$. By the proof in Step 4 (I), $C_v \cap \{S_z : z \in \mathbb{B}\} \neq \emptyset$, and so there is some $y \in \mathbb{B}$ satisfying $S_y \in C_v$. Furthermore, $C_v = V_y \cup S_y$.

Thus, X is the closure of $\cup\{V_z : S_z \in X - \mathbb{B}\}$.

By Step 1 (VI), one has $X = \sigma_{pV}(V_{z'})$ where $z' = \vee\{z : S_z \in X - \mathbb{B}\}$. Thus the cyclic flats of $M_p|V$ are the sets $V_z \cup \{S_x : x \leq z\}$.

Step 5. Let $h : 1_L \mapsto V, z \mapsto V_z \cup \{S_x : x \leq z\}$ for $z \in \mathbb{B}$. Then combining with the results from Step 1 to Step 4, one has that L is isomorphic to $L(M_p|V)$.

□

Combining Theorem 2.3 with Theorem 3.1, we get the following result.

Corollary 3.2. *There is a bijection between the set of a bounded lattice and the set of a pre-independence space $M_p = (S, \mathscr{I})$ in which $\mathscr{C}(M_p)$ exists and if $X \notin \mathscr{I}$, then it has $C \in \mathscr{C}(M_p)$ satisfying $C \subseteq X$, additionally, σ_p satisfies (cl3).*

According to [3], p. 388, for any independence space $\mathscr{M}_I(S)$ with σ as its closure operator, then σ satisfies (cl1)–(cl3), and besides, when considered with [2] and [3], all of (2I)–(2IV) are correct for $\mathscr{M}_I$. In addition, by [3], pp. 386–388, one gets that $\mathscr{M}_I(S))$ exists and for $X \notin \mathscr{I}$, it has a circuit $C \in \mathscr{C}(\mathscr{M}_I(S))$ satisfying $C \subseteq X$. Thus, all the results in Section 2 and Section 3 are true for independence spaces. That is, the consequences provided for are a generalization of that in [10].

At the same time, we note that [1], Theorem 3.2, gives a method to formulate the definition of a finite matroid using cyclic flats and its ranks. For infinite matroids, we foresee a similar proof.

4. Cyclic Width

We may ask: do the properties of $L(M_p)$ imply infinite matroid theoretic properties of a pre-independence space $M_p(S) = (S, \mathscr{I}_p)$? We know that the three of dual operation, restriction operation and contraction operation are basic operations for finite matroids. But by Remark 2.2, $(S, \mathscr{I}_p^*)$ and $(T, \mathscr{I}_p.T)$ for $T \subseteq S$ may fail to exist as a pre-independence space. Even though, [2] informs us that B-matroids is closed under the dual operation, or restriction operation or contraction operation respectively. Hence, this section aims to discuss some properties for a B-matroid $M_B(S) = (S, \mathscr{I})$ by the properties of $L(M_B)$.

A B-matroid N on T is called a *minor* of $M_B(S)$ if N is obtained by any finite combinations of restrictions and contractions of $M_B(S)$. Besides by [2], $(M_B|T)^* = M_B^*.T$ and $(M_B^*(S))^* = M_B(S)$. One notices that by [2] and Theorem 2.3, $L(M_B)$ indeed exists. Recall in [5], p. 98 & [4] that the *width* of a poset is the maximum number of elements in a set of incomparable elements. Hence, the *cyclic width* of $M_B(S)$ is defined as the width of $L(M_B)$.

In what follows, $CW(k)$ represents the family of all B-matroids whose cyclic width is at most k. One first shows that $CW(k)$ is closed under duals and minors.

Theorem 4.1. *The class $CW(k)$ is closed under duals and minors.*

Proof. By the dual operation of $M_B(S) \in CW(k)$, it follows the closure under duals.

In light of [2],§3.2, we only need to prove the closure under the restriction operation.

Let σ be the closure operator of $M_B(S)$ and $T \subseteq S$.

If $k \not< \infty$, then obviously, $M_B|T \in CW(k)$.

If $k < \infty$. By the discussion in [2], pp. 82–83, one has $\sigma_T(Y) = \sigma(Y) \cap T$ for all $Y \subseteq T$ where σ_T is the closure operator of $M_B|T$. Suppose $M_B|T \in CW(k_T)$. Before proceeding, we recall that the family of dependent sets of $M_B|T$ is $\{Y : Y \subseteq T, Y \notin \mathscr{I}\}$, and further, $\mathscr{C}(M_B|T) = \{Y : Y \subseteq T, Y \in \mathscr{C}(M_B(S))\}$.

Let $\{Y_\alpha : \alpha \in \mathscr{D}_T\}$ be a maximal antichain in $L(M_B|T)$ and the cardinality $|\mathscr{D}_T|$ of $\mathscr{D}_T$ is k_T. Let $B_\alpha \subseteq Y_\alpha$ be a basis of Y_α in $M_B|T$. Then B_α is also a basis of Y_α in M_B, and further, $\sigma_T(Y_\alpha) = \sigma_T(B_\alpha)$. Furthermore, $\sigma(B_\alpha) = B_\alpha \cup \{x : B_\alpha \cup x \notin \mathscr{I}\} = Y_\alpha \cup \{x \in S - Y_\alpha : B_\alpha \cup x \notin \mathscr{I}\}$. Since $B_\alpha \cup x \notin \mathscr{I}$ and $B_\alpha \in \mathscr{I}$ together implies that there exists a circuit C_x of M_B satisfying $x \in C_x \subseteq B_\alpha \cup x$, and hence $\sigma(B_\alpha) = Y_\alpha \cup \bigcup_{x \in C_x \subseteq B_\alpha \cup x \notin \mathscr{I}} C_x$. Let $X_\alpha = \sigma(B_\alpha)$. One has that X_α is a cyclic flat and $Y_\alpha = X_\alpha \cap T$.

One sees that $\{X_\alpha : \alpha \in \mathscr{D}_T\}$ is a set of incomparable elements in $L(M_B)$ because $\{Y_\alpha : \alpha \in \mathscr{D}_T\}$ is an antichain in $L(M_B|T)$. Herein, $k_T \le k$, and further, $M_B|T \in CW(k)$.

□

Example Let S be the edge set of the infinite graph G shown in Figure 1. $\mathscr{I}_G = \{X \subseteq S : X$ does not contain the edge set of any cycle in $G\}$. It is a straightforward to prove that $(S, \mathscr{I}_G)$ is a B-matroid, and obviously, $\mathscr{C}((S, \mathscr{I}_G)) = \{\{e_1, a\}\}$. Let σ be the closure operator of $(S, \mathscr{I}_G)$. Then $\sigma(e_1) = \sigma(a) = \{e_1, a\}$ as the one and only one cyclic flat in $(S, \mathscr{I}_G)$. Hence, $(S, \mathscr{I}_G) \in CW(1)$, and so, the width of $L((S, \mathscr{I}_G))$ is 1.

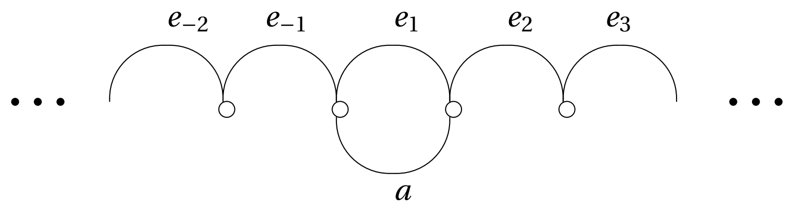

Figure 1

Let $H_i = S - e_i$ $(i = 3, 4, \ldots)$. Routine verification posits that $(H_i, \mathscr{I}_G|H_i)$ is a B-matroid and $(H_i, \mathscr{I}_G|H_i)$ is isomorphic to $(S, \mathscr{I}_G)$, and additionally, $(H_i, \mathscr{I}_G|H_i)$ is a minor of $(S, \mathscr{I}_G)$ and the width of $L((H_i, \mathscr{I}_G|H_i))$ is 1. $(i = 3, 4, \ldots)$.

Remark 4.2. As finite matroids (cf.[1]), a class of B-matroids is *well-quasi-ordered* if it contains no infinite antichain in the minor order. [1] tells that for finite matroids, $CW(1)$ is well-quasi-ordered. But the above example shows that $CW(1)$ is not a well-quasi-ordered class of B-matroids. This is also a difference between finite matroids and infinite matroids. More study is needed.

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