THE EXPECTED SHAPE OF RANDOM DOUBLY ALTERNATING BAXTER PERMUTATIONS

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ABSTRACT. Guibert and Linusson introduced the family of *doubly alternating Baxter permutations*, i.e. Baxter permutations $\sigma \in S_n$, such that σ and σ^{-1} are alternating. They proved that the number of such permutations in S_{2n} and S_{2n+1} is the Catalan number C_n. In this paper we compute the expected limit shape of such permutations, following the approach by Miner and Pak.

1. INTRODUCTION

A *Catalan structure* is a family of combinatorial objects whose number is the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

There is a staggering amount of literature on various Catalan structures, the list of which is ever growing (see [Gou, Pak, Slo, S2, S3]). This multitude and diversity of Catalan structures is in part a consequence of their different nature, and in part a misperception, as many such structures are essentially equivalent, via a "nice bijection". In the latter case, this can happen despite the apparently different geometric representations of the Catalan objects (think polygon triangulations vs. binary trees). Discerning different combinatorial structures can be difficult and even harder to formalize (as in, how do you prove non-existence of a "nice bijection"?).

In this paper we study the asymptotic behavior of doubly alternating Baxter permutations $\sigma \in S_n$. Following [MP], we compute the *expected limit shape* of $\mathscr{B}_n$ by viewing permutations as 0-1 matrices, and averaging them. When scaled appropriately the resulting distribution on $[0,1]^2$ converges to a *limit surface* $\Phi(x, y)$ with an explicit formula that seems new in the literature, thus differentiating this Catalan structure from others (see Theorem 1.1 below).

Baxter permutations are defined to be $\sigma \in S_n$, such that there are no indices $1 \le i < j < k \le n$, which satisfy $\sigma(j+1) < \sigma(i) < \sigma(k) < \sigma(j)$ or $\sigma(j) < \sigma(k) < \sigma(i) < \sigma(j+1)$. They were introduced and enumerated by Baxter in 1964 [Bax] (see also [CGHK, Vie]), and recently became a popular subject (see §6.2).

The *alternating permutations* are defined to be $\sigma \in S_n$, such that $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \ldots$. They were defined by André in 1879, and extensively studied over the years (see [S1, S2]). Their number is known as the *Euler–Bernoulli number* with the exponential generating function $\tan(x) + \sec(x)$, and they have numerous connections to other fields.

A Baxter permutation $\sigma \in S_n$ is called *doubly alternating* if both σ and σ^{-1} are alter-nating. Guibert and Linusson showed in [GL] that the set $\mathscr{B}_n$ of such permutations is a Catalan structure:

$$\left|\mathscr{B}_{2n}\right| = \left|\mathscr{B}_{2n+1}\right| = C_n.$$

Let $P(m, i, j)$ denote the probability that a random $\sigma \in \mathscr{B}_{2m}$ has $\sigma(i) = j$. Here is our main result.

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Theorem 1.1. *Let* $0 < \alpha < \beta < 1-\alpha$. *We have:*

$$P(m, \lfloor 2\alpha m\rfloor, \lfloor 2\beta m\rfloor) \sim \frac{\varphi(\alpha,\beta)}{m} \quad \text{as} \quad m \to \infty,$$

where

$$\varphi(\alpha,\beta) = \frac{1}{8\pi}\int_0^{\alpha}\int_0^{\alpha-y} \frac{dx\,dy}{[(x+y)(\beta-x)(1-\beta-y)]^{3/2}}.$$

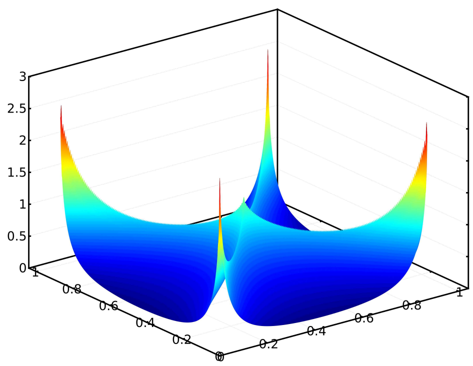


FIGURE 1. The graph of the limit surface $\Phi(x, y)$.

In other words, when the distribution $P(m,i,j)$ is scaled appropriately, it converges to the limit surface $\Phi(x, y)$ as shown in Figure 1. Here the surface $\Phi : (0,1)^2 \to \mathbb{R}$ is obtained from $\varphi : \{0 < x < y < 1,\ x + y < 1\} \to \mathbb{R}$ by reflection across both diagonals.

There are a few things to note. First, the surface $\Phi(x, y)$ has the symmetry group of a square. Curiously, most of symmetries do not appear in the actual numbers $B(m,i,j)$. We discuss this in further detail in Section 6. Second, most of the surface is "under water", i.e. we have $\Phi(x, y) < 1$ everywhere except near the corners and at the center spike $\Phi(\frac{1}{2},\frac{1}{2}) = \frac{3}{2}$. The reason is simple: the peaks in the corners actually diverge (see Theorem 5.1).[1] Finally, the limit shape for $\mathscr{B}_n$ is fundamentally different from those in [MP], where the limit surfaces are degenerate. We give more details on these in Section 6.

The rest of the paper is structured as follows. In the next section we introduce the basic notations. In Section 3, we present the main lemma (Lemma 3.1), giving an explicit triple summation formula for $P(m,i,j)$. Section 4 contains the proof of the main lemma. We then use the main lemma to prove Theorem 1.1 in Section 5. We conclude with final remarks and open problems in Section 6.

2. BASIC DEFINITIONS AND NOTATION

For a permutation $\sigma \in S_n$, its *complement* σ^c is defined pointwise by $\sigma^c(i) = n+1-\sigma(i)$. Given $\sigma = a_1 \cdots a_n \in S_n$ and permutations $\tau_1,\ldots,\tau_n$, we define the *inflation* $\sigma[\tau_1,\ldots,\tau_n]$ to be the permutation obtained by replacing a_i with a copy of τ_i, shifted to be in the same relative position as a_i.

We say that $a_n \sim b_n$, or a_n is asymptotically equivalent to b_n if $a_n/b_n \to 1$ as $n \to \infty$. Recall Stirling's formula,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

[1] Our graph in Figure 1 has been truncated.

It gives the following asymptotic formula for the Catalan numbers:

$$C_n \sim \frac{1}{\sqrt{\pi}\, n^{3/2}}\, 4^n\,. \tag{1}$$

In Section 5, we will make heavy use of this approximation.

3. Main Lemma

As in the introduction, let $\mathscr{B}_n$ denote the set of doubly alternating Baxter permutations of length n. Recall that $|\mathscr{B}_{2m}|$ is equal to the m-th Catalan number C_m. It was proved in [GL] that for $\sigma \in \mathscr{B}_n$ with n odd, we have $\sigma = 12[1, \tau^c]$, for some $\tau \in \mathscr{B}_{n-1}$. Therefore, the limit shape of $\mathscr{B}_{2m+1}$ is that of $\mathscr{B}_{2m}$. From this point on, we restrict our attention to sets $\mathscr{B}_{2m}$.

It was further proved [GL], that elements of $\mathscr{B}_{2m}$ can be described by a standard recursive Catalan structure. Specifically, if $\sigma(1) = 2k+1$ (and $\sigma(1)$ must always be odd), then we have $\sigma = 2341[1, \tau^c, 1, \omega]$ with $\omega \in \mathscr{B}_{2k}$ and $\tau \in \mathscr{B}_{2(m-k-1)}$. Figure 2 illustrates this decomposition.

$\sigma =$

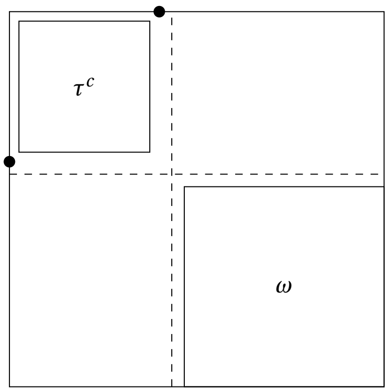


Figure 2. A recursive description for $\sigma \in \mathscr{B}_{2m}$.

For $r, s \geq 1$, let $p_{r,s}$ be the number of Dyck paths of length $2(r+s-1)$, which first revisit the horizontal axis by step $2s$ at the latest. Then $p_{r,s}$ is given by a partial Catalan convolution:

$$p_{r,s} = \sum_{k=1}^{s} C_{r+s-(k+1)} \cdot C_{k-1} = C_{r+s-2}C_0 + \ldots + C_{r-1}C_{s-1}\,.$$

Let $B(m,i,j)$ be the number of permutations $\sigma \in \mathscr{B}_{2m}$ such that $\sigma(i) = j$. We use the recursive structure above to obtain the following summation formula for $B(m,i,j)$, in terms of the Catalan numbers.

Main Lemma 3.1. *Let $B(m,i,j)$ and $p_{r,s}$ be defined as above, and let $a = \lceil i/2 \rceil$, $b = \lfloor j/2 \rfloor$. Suppose further that $i \leq j \leq 2m-i+1$. Then we have:*

$$B(m,i,j) = \begin{cases} C_b C_{m-b-1} \ + \displaystyle\sum_{r=1}^{a-1}\sum_{s=1}^{a-r} p_{r,s} \cdot C_{b-r+1} \cdot C_{m-b-s-1} & \textit{for } j \textit{ odd}, \\ \displaystyle\sum_{r=1}^{a-1}\sum_{s=1}^{a-r} p_{r,s} \cdot C_{b-r} \cdot C_{m-b-s} & \textit{for } j \textit{ even}. \end{cases} \tag{$*$}$$

We prove the main lemma by induction in the next section, and use it in Section 5 to prove Theorem 1.1.

4. Proof of Main Lemma

4.1. **The symmetries.** We observe the following easy properties of $B(m,i,j)$.

Proposition 4.1. *We have*

(*i*) $B(m,i,j) = B(m,j,i) = B(m,2m+1-j,2m+1-i)$.

(*ii*) *For* $i=1$, *we have*

$$B(m,1,j) = \begin{cases} 0 & \text{if } j \text{ is even}, \\ C_b \cdot C_{m-b-1} & \text{if } j = 2b+1 \text{ is odd}. \end{cases}$$

(*iii*) *Let* $i>1$ *and* $j<2m$. *Then*

$$B(m,i,j) = \sum_{k=1}^{\lceil i/2\rceil -1} C_{k-1}\cdot B(m-k,i-2k,j) + \sum_{k=m-\lfloor j/2\rfloor+1}^{m} C_{m-k}\cdot B(k-1,i-1,2m-j).$$

Proof. The equalities in (i) correspond to reflection over the main diagonal and main antidiagonal, respectively. These reflections preserve both the Baxter and doubly alternating properties.

The equality in (ii) follows immediately from the recursive structure of $\mathscr{B}_{2m}$ given in Figure 2. As a special case, note that $B(m,1,1) = C_{m-1}$.

The recurrence relation in (iii) follows from conditioning on $\sigma(2k) = 2m$, and checking the resulting restrictions on $\sigma(i) = j$. Note that for the second sub-sum, we require $2(k-1) \ge 2m-j$ to satisfy the required constraints. □

The first part of this proposition shows that our Main Lemma is sufficient to calculate $B(m,i,j)$ for *all* i,j, by using reflections.

4.2. **Preliminary calculations.** Throughout the rest of this section, we assume that $i \le j \le 2m-i$. Let us further refine the recurrence in equation in (iii). We need to ensure that all terms $B(n,p,q)$ in the above satisfy the inequalities $p \le q \le 2n-p$. The first sum requires no modification. The second sum must be split however, and a symmetry applied to one group of terms. Since we need $i-1 \le 2m-j \le 2k-i-1$, it follows that the terms with $k \ge m+\frac{1}{2}(i+1-j)$ are already of the desired form. For summands preceding this cut-off point, we will apply an antidiagonal reflection to the terms $B(*,*,*)$. For notational convenience, let $h = \left\lfloor \frac{1}{2}(j-i-1)\right\rfloor$ and let a, b be as in the Main Lemma. Then our triple sum becomes

$$\begin{aligned} B(m,i,j) = &\sum_{k=1}^{a-1} C_{k-1}\cdot B(m-k,i-2k,j) \\ &+ \sum_{k=m-b+1}^{m-h-1} C_{m-k}\cdot B(k-1,j-1-2(m-k),2k-i) \\ &+ \sum_{k=m-h}^{m} C_{m-k}\cdot B(k-1,i-1,2m-j). \end{aligned}$$

We reindex the second two sums of the above, by letting $k \leftarrow m-b+\ell$:

$$\begin{aligned} B(m,i,j) = &\sum_{k=1}^{a-1} C_{k-1}\cdot B(m-k,i-2k,j) \\ &+ \sum_{\ell=1}^{b-h-1} C_{b-\ell}\cdot B(m-b+\ell-1,j-1-2(b-\ell),2(m-b+\ell)-i) \\ &+ \sum_{\ell=b-h}^{b} C_{b-\ell}\cdot B(m-b+\ell-1,i-1,2m-j). \end{aligned} \tag{2}$$

We also state a simple lemma:

Lemma 4.2. *Let $r,s \geq 1$. Then*

$$p_{r,s} = C_{r+s-2} + \sum_{k=1}^{s-1} C_{k-1} \cdot p_{r,s-k}\,.$$

Proof. This follows from expanding each $p_{r,s-k}$ term and changing the order of summation. Specifically, we use the substitution $m \leftarrow j+k+2$. □

4.3. **Proof of Main Lemma.** Suppose our formula $(*)$ holds for all $B(n,p,q)$ satisfying $n < m$, and $p,q \in \{1,\ldots,2n\}$ with $p \leq q \leq 2n-p$. To show that the result holds for $B(m,i,j)$, we prove it separately for each of the four choices on the parity of i and j. Let $i = 2a$ and $j = 2b$. In this case, we have

$$a = \left\lceil \frac{i}{2} \right\rceil, \qquad b = \left\lfloor \frac{j}{2} \right\rfloor, \quad \text{and} \quad h = \left\lfloor \frac{(j-i-1)}{2} \right\rfloor = b-a-1\,.$$

Thus, the triple sum in equation (2) becomes

$$\begin{aligned} B(m,2a,2b) = &\sum_{k=1}^{a-1} C_{k-1} \cdot B\big(m-k, 2(a-k), 2b\big) \\ &+ \sum_{\ell=1}^{a} C_{b-\ell} \cdot B\big(m-b+\ell-1, 2(\ell-1)+1, 2(m-b-a+\ell)\big) \\ &+ \sum_{\ell=a+1}^{b} C_{b-\ell} \cdot B\big(m-b+\ell-1, 2a-1, 2(m-b)\big)\,. \end{aligned}$$

By induction, this gives

$$\begin{aligned} B(m,2a,2b) = &\sum_{k=1}^{a-1} C_{k-1} \cdot \sum_{c=1}^{a-k-1} \sum_{d=1}^{a-k-c} p_{c,d} \cdot C_{b-c} \cdot C_{m-k-b-d} \\ &+ \sum_{\ell=1}^{a} C_{b-\ell} \cdot \sum_{c=1}^{\ell-1} \sum_{d=1}^{\ell-c} p_{c,d} \cdot C_{m-b-a+\ell-c} \cdot C_{a-1-d} \\ &+ \sum_{\ell=a+1}^{b} C_{b-\ell} \cdot \sum_{c=1}^{a-1} \sum_{d=1}^{a-c} p_{c,d} \cdot C_{m-b-c} \cdot C_{\ell-1-d}\,. \end{aligned} \tag{3}$$

For a fixed r and s, let c_1, c_2, c_3 be the coefficients on $C_{b-r} \cdot C_{m-b-s}$ in the first, second and third sums in the above equation. Looking at the first sum we see that c_1 is only nonzero when $r \leq a-s$. In such a case, by considering $c = r$ and $k+d = s$ we see that the coefficient is given by

$$c_1 = \sum_{k=1}^{s-1} p_{r,s-k} \cdot C_{k-1} = p_{r,s} - C_{r+s-2}\,,$$

with equality following from Lemma 4.2.

For the second sum, we consider $\ell = r$ and $a+c-r = s$. Since $c \geq 1$, the second sum contributes nothing for the cases when $r+s \leq a$. Thus for $r+s \leq a$ we have $c_2 = 0$, and for $r+s > a$ we instead get

$$c_2 = \sum_{j=1}^{a-s} p_{r+s-a,j} \cdot C_{a-j-1}\,.$$

Our third sum requires some manipulation. We rewrite the third sum in (3) as

$$\sum_{c=1}^{a-1} \sum_{d=1}^{a-c} p_{c,d} \cdot C_{m-b-c} \cdot \left(\sum_{\ell=a+1}^{b} C_{b-\ell} \cdot C_{\ell-1-d} \right),$$

and here the inner sum can be re-expressed as a subtraction, through the standard Catalan convolution:

$$\sum_{c=1}^{a-1} \sum_{d=1}^{a-c} p_{c,d} \cdot C_{m-b-c} \cdot (C_{b-d} - C_{b-d-1}C_0 - \ldots - C_{b-a}C_{a-d-1})\,.$$

If we consider those summands where $c = s$, then for $r+s \le a$ we have $c_3 = C_{r+s-2}$ (by Lemma 4.2), and otherwise

$$c_3 = -\left(\sum_{j=1}^{a-s} p_{s,j} \cdot C_{r-j-1}\right).$$

Combining these terms, we see that

$$c_1 + c_2 + c_3 = \begin{cases} p_{r,s} & \text{for } r+s \le a, \\ \sum_{j=1}^{a-s} \left(p_{r+s-a,j} \cdot C_{a-j-1} - p_{s,j} \cdot C_{r-j-1}\right) & \text{for } r+s > a. \end{cases}$$

Expand and rearrange each $p_{*,*}$ term in this latter sum, and observe that it is equal to 0. This completes the proof for the case when i and j are both even. The other three cases are similar and their proof is omitted.[2] □

5. Asymptotic estimates

5.1. **Preliminaries.** Let $P(m,i,j)$ be the probability that $\sigma \in \mathscr{B}_{2m}$ satisfies $\sigma(i) = j$. Then $P(m,i,j) = B(m,i,j)/C_m$.

Throughout the section, it will be convenient to pretend that $2\alpha m, 2\beta m$ are integers. Further, we assume that $2\alpha m, 2\beta m$ are even, in order to restrict to one version of the formulas in the equation $(*)$. Therefore, we write:

$$(**) \qquad P(m, 2\alpha m, 2\beta m) = \sum_{r=1}^{\alpha m-1} \sum_{s=1}^{\alpha m-r} \frac{p_{r,s} \cdot C_{\beta m-r} \cdot C_{m-\beta m-s}}{C_m}.$$

Theorem 5.1. *We have the following corner probabilities:*

$$P(m,1,1),\, P(m,1,2m-1),\, P(m,2,2m-1) \to \frac{1}{4},$$
$$P(m,1,2) = P(m,1,2m) = P(m,2,2) = 0,$$

and the remaining corner cases are given by symmetries.

Proof. It follows at once from the formulas in the Main Lemma that the probabilities in these cases are either 0, or $C_{m-1}/C_m \to 1/4$. □

Before proving Theorem 1.1, we will need to take a closer look at the partial Catalan convolutions $p_{r,s}$. Note that $C_{r+s-2} \le p_{r,s} \le C_{r+s-1}$. In fact, the following stronger result holds.

Lemma 5.2. *Fix* $\rho \in (2/3, 1)$. *Suppose that* k *satisfies*

$$n^\rho < k < n - n^\rho.$$

Then, for every $\varepsilon > 0$,

$$(1-\varepsilon)C_{n-1} \le 2 \cdot p_{n-k,k} \le (1+\varepsilon)C_{n-1},$$

for all n *large enough.*

Letting $k = s, r = n-k-1$ gives this above inequality. The proof converts the summation $p_{n-k,k}$ into an integral, using the asymptotic (1). We omit the details.

[2]Full proofs of these cases will appear in the first author's Ph.D. thesis [Dok].

5.2. **Partitioning the summation.** Observe that the double sum in equation $(**)$ is over a triangular set of points T, with $(r,s) \in T$ satisfying

$$1 \le r \le \alpha m - 1 \qquad \text{and} \quad 1 \le s \le \alpha m - r\,.$$

We partition T into the four sets A, B, C and D as follows (see Figure 3).

$$\begin{aligned}
\mathrm{A} &= \{(r,s) : (r+s) \le \alpha m^{\delta}\}\\
\mathrm{B} &= T \setminus (A \cup C \cup D)\\
\mathrm{C} &= \{(r,s) : s \le (r+s)^{\rho}\}\\
\mathrm{D} &= \{(r,s) : s \ge r+s-(r+s)^{\rho}\}
\end{aligned}$$

We show that the sum of terms in $(**)$ corresponding to each of the sets A, C and D are dominated by the those from set B. This will allow us to apply Lemma 5.2 and approximation in (1) to terms whose indices lie in set B.

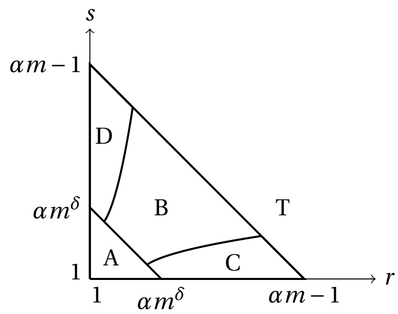


FIGURE 3. The summation points, partitioned into four sets.

We reduce the problem of summation over T to an integral over the right triangle Δ, given by $x,y \ge 0$ and $x+y<\alpha$. For that we similarly define the regions **A**, **B**, **C** and **D** to be all (x,y) in Δ so that $(\lceil mx \rceil, \lceil my \rceil)$ is a point in A, B, C or D, respectively.

5.3. **Estimates for set** A**.** Each of the three sets A, C and D contains a relatively small number of terms. We will show that the sum of terms in $(**)$ corresponding to these sets grows slowly, so that their contribution is dwarfed by that of set B.

Lemma 5.3. *Let $\delta < 1/4$. Then the sum of terms in $(**)$ corresponding to* A *is $o\left(\frac{1}{m}\right)$.*

Proof. The largest individual term in $(**)$ is obtained when $r = s = 1$. We get:

$$\frac{2C_{\beta m-1}C_{m-\beta m-1}}{C_m} \sim \frac{1}{\beta(1-\beta)8\sqrt{\pi}} \cdot \frac{1}{m^{3/2}}\,.$$

We bound each of the summands from A by this, so that their total is at most

$$\begin{aligned}
|\mathrm{A}| \cdot \frac{2C_{\beta m-1}C_{m-\beta m-1}}{C_m} &= (\alpha m^{\delta})^2 \cdot \frac{C_{\beta m-1}C_{m-\beta m-1}}{C_m}\\
\sim \frac{1}{8\sqrt{\pi}\beta(1-\beta)} \cdot \frac{(\alpha m^{\delta})^2}{m^{3/2}} &= \frac{\alpha^2}{8\sqrt{\pi}\beta(1-\beta)} \cdot \frac{1}{m^{3/2-2\delta}} = o\left(\frac{1}{m}\right),
\end{aligned}$$

where the last equality follows from $\delta < 1/4$. $\square$

5.4. **Estimates for sets** C *and* D**.**

Lemma 5.4. *Let* $\rho \in (2/3, 1)$*. Then the sum of terms in* $(**)$ *corresponding to set* C *is* $o\big(\frac{1}{m}\big)$*. The same also holds for the set* D*.*

Proof. Choose a constant M large enough so that

$$C_n \le M \cdot \frac{4^n}{(n+1)^{3/2}}, \qquad \text{for all } n \ge 0.$$

Denote by P_{C} the sum of terms in $(**)$ corresponding to C, i.e.

$$P_{\mathrm{C}} = \sum_{(r,s)\in \mathrm{C}} \frac{p_{r,s} \cdot C_{\beta m - r} \cdot C_{m-\beta m - s}}{C_m} \le \sum_{(r,s)\in \mathrm{C}} \frac{C_{r+s-1} \cdot C_{\beta m - r} \cdot C_{m-\beta m - s}}{C_m},$$

where the last inequality follows from $p_{r,s} \le C_{r+s-1}$. We use formula (1) as well as our choice of M to obtain

$$\frac{4P_{\mathrm{C}}}{\sqrt{\pi} M^3} \le \sum_{(r,s)\in \mathrm{C}} \frac{m^{3/2}}{[(r+s)(\beta m - r + 1)((1-\beta)m - s + 1)]^{3/2}}.$$

And as $m \to \infty$ this last sum is asymptotically equivalent to

$$\frac{1}{m} \iint_{\mathbf{C}} \frac{dx\, dy}{[(x+y)(\beta - x)(1-\beta-y)]^{3/2}}.$$

Now observe that the above integral converges, and that $Area(\mathbf{C}) \to 0$ as $m \to \infty$. Thus, $P_{\mathrm{C}} = o\big(\frac{1}{m}\big)$, as desired. The proof for D follows verbatim from the above argument and the bound $p_{r,s} \le C_{r+s-1}$. □

Remark 5.5. *At the beginning of this section, we assumed that* $2\alpha m, 2\beta m$ *are even integers. For the other choices of parity, the sum* $(**)$ *differs in the range of indices* (r,s)*, and, depending on the chosen parity of* $2\alpha m$*, the addition of a* $O(m^{-3/2})$ *term, coming from the first term in the summation* $(*)$ *from the main lemma. The above arguments and the following proof of Theorem 1.1 can be adapted with little difficulty to handle these differences.*

5.5. **Proof of Theorem 1.1.** Denote by P_{B} the sum of terms from $(**)$ corresponding to set B, i.e.

$$P_{\mathrm{B}} = \sum_{(r,s)\in \mathrm{B}} \frac{p_{r,s} \cdot C_{\beta m - r} \cdot C_{m-\beta m - s}}{C_m}.$$

We need to prove that $P_{\mathrm{B}} \sim \frac{1}{m}\varphi(\alpha,\beta)$. This follows from the construction of B. First, note that for $(r,s) \in \mathrm{B}$, the indices $r+s$, $\beta m - r$ and $(1-\beta)m - s \to \infty$ as $m \to \infty$. Thus, by Lemma 5.2 we have

$$P_{\mathrm{B}} \sim \frac{1}{2} \sum_{(r,s)\in \mathrm{B}} \frac{C_{r+s-1} \cdot C_{\beta m - r} \cdot C_{m-\beta m - s}}{C_m}.$$

From formula (1) we obtain

$$P_{\mathrm{B}} \sim \frac{1}{8\pi} \sum_{(r,s)\in \mathrm{B}} \frac{m^{3/2}}{[(r+s)(\beta m - r)((1-\beta)m - s)]^{3/2}}$$
$$\sim \frac{1}{8\pi m} \iint_{\mathbf{B}} \frac{dx\, dy}{[(x+y)(\beta - x)(1-\beta-y)]^{3/2}}.$$

As $m \to \infty$, the region $\mathbf{B}$ expands to fill the entire triangle Δ, so we obtain

$$P_{\mathrm{B}} \sim \frac{1}{8\pi m} \int_0^{\alpha} \int_0^{\alpha - y} \frac{dx\, dy}{[(x+y)(\beta - x)(1-\beta-y)]^{3/2}} = \frac{\varphi(\alpha,\beta)}{m}.$$

Finally, by Lemmas 5.3 and 5.4, the probability $P(m, 2\alpha m, 2\beta m) \sim P_{\mathrm{B}}$. This completes the proof. □

6. Final remarks and open problems

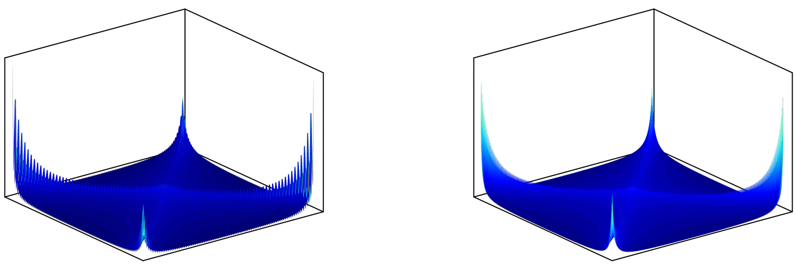
6.1. There is a large literature on the asymptotic behavior of Catalan structures, too large for us to summarize. We refer to [FS, Odl, PW] for a comprehensive introduction to asymptotic methods, and to [Pit] for strongly related discrete probability results.

6.2. The choice of doubly alternating Baxter permutations is a part of a general program of study of pattern-avoiding permutations. Baxter permutations are defined to avoid two (generalized) patterns, and widely studied in the literature (see e.g. [AGP, BBF, DG1, FFNO], in part due to their connection to plane bipolar orientations, tilings [Korn], and other combinatorial objects (see [Kit, §2.2] for the introduction and references). As mentioned in the introduction, the alternating permutations are classical in the own right; we refer to [S1] for a comprehensive survey on their role in Enumerative Combinatorics, to [DW] for a recent detailed probabilistic study, and to [Kit, §6.1] for connections to other pattern-avoiding permutations.

6.3. The study of random pattern avoiding permutations by means of the limit shape was recently introduced by Miner and the second author [MP]. The authors obtained highly detailed information for **123** and **213** patterns, each a classical Catalan structure. In both cases the limit surface is degenerate as most permutation matrix entries concentrate around the anti-diagonal, and the interesting behavior occurs in the micro-scale near the anti-diagonal with various phase transitions. We refer to [AM, ML] for strongly related results, and some interesting extensions for larger patterns.

6.4. Following the ideas in [ML] and [MP], one can ask whether for a given (generalized) pattern there is always a limit surface, and whether it is degenerate. Since the set of bistochastic matrices is a compact (in fact, the *Birkhoff polytope*), the answer to the former is likely yes. But the latter question seems difficult and currently out of reach.

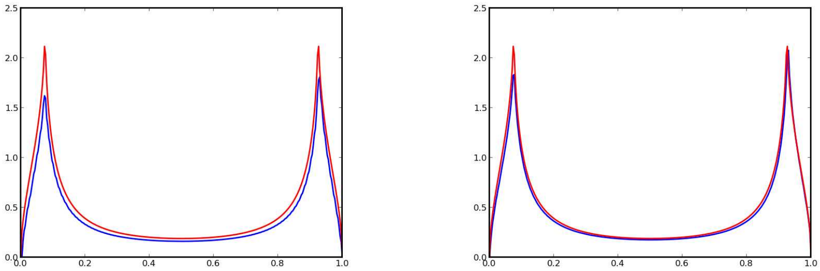
6.5. The fact that the limit surface $\Phi(x,y)$ is highly symmetric and piecewise smooth came as a surprise, as in the initial experiments for averages of all $\sigma \in \mathscr{B}_n$, the graphs exhibited fewer symmetries and numerous small spikes, see Figure 4. Of course, these spiked are due to the the parity differences in formula $(*)$ in the Main Lemma, and as n grows, they gradually disappear. Curiously, the asymmetry across the main diagonal persisted until very large n, suggesting a rather slow convergence in Theorem 1.1.



(A) (B)

Figure 4. Averages of all $\sigma \in \mathscr{B}_n$, where $n = 200$ and 800.

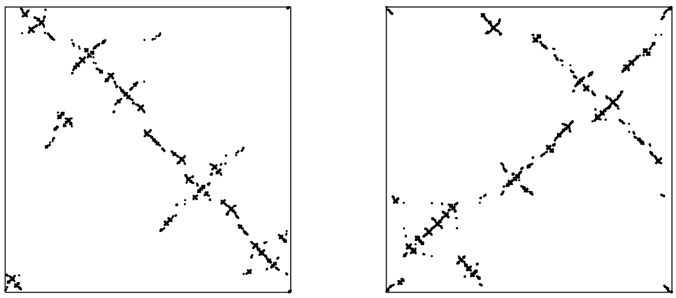
To illustrate this phenomenon, note that the limit shape has an extra degree of symmetry, which the actual numbers $B(m,i,j)$ do not have. The following two graphs are for $\alpha = 3/40$, and the horizontal axis is β. The red line is $\varphi(\alpha,\beta)$, the blue line is $m \cdot P(m, 2\alpha m, 2\beta m)$, for $m = 500$ on the left and $m = 2000$ on the right. For the emphasis, recall that in the notation above, we have $n = 2m$.



(A) (B)

FIGURE 5. Horizontal slices of the limit surface.

6.6. Let us note that for the case of size 3 patterns mentioned in §6.3, the random restricted permutations do in fact resemble their limit shapes, due to exponentially small decay of probabilities away from the anti-diagonal. However, in the the case of random $\sigma \in \mathscr{B}_n$, permutations tend to exhibit a high degree or structure, and do not resemble the limit surface $\Phi(x, y)$. This suggests that computing square-to-square correlations would an interesting problem. While we expect this to be doable, this goes outside the scope of our project.



(A) $\sigma \in \mathscr{B}_{2000}$ (B) $\sigma \in \mathscr{B}_{4000}$

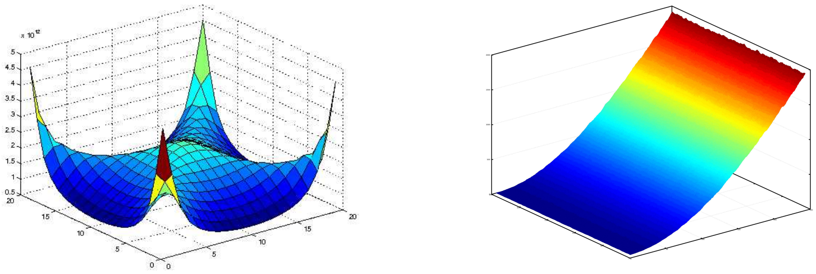
FIGURE 6. Randomly sampled permutations $\sigma \in \mathscr{B}_{2000}$ and $\mathscr{B}_{4000}$.

6.7. It would be interesting to compute the limit shape of random Baxter permutations. While they can be sampled exactly uniformly (see e.g. [FFNO, Vie]), the underlying bijection does not seem to give any useful formulas. In fact, a special effort is required to sample beyond $n = 20$. Still, the apparent connection to $\Phi(x, y)$ is undeniable (see Figure 7).

A possible explanation lies in the "flat structure" of alternating permutations. As evident from Figure 7 for $n = 500$, there seem to be a limit surface of alternating permutations, with no spikes except at the boundary. In fact, the boundary is given by the asymptotics of Entringer numbers given in [DW].[3] This suggests that all spikes in $\Phi(\cdot, \cdot)$ come from the "Baxter condition".

6.8. It is known that the number of (singly) alternating Baxter permutations in S_{2n} and S_{2n+1}, is C_n^2 and $C_n C_{n+1}$, respectively [DG1]. It would be interesting to compute the expected limit shape of such permutations. Note the similarity of the shape in Figure 8 with the shape of the other Baxter families.

[3] We plan to return to this problem in [Dok].



(A) (B)

FIGURE 7. Averages of Baxter permutations ($n = 20$) and alternating permutations ($n = 500$).

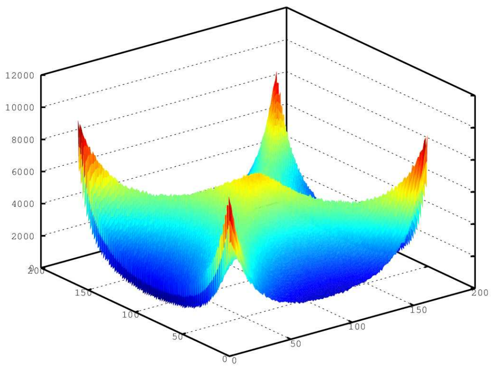


FIGURE 8. Averages of alternating Baxter permutations ($n = 200$).

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