Utilitas Mathematica, 118: 3–13
DOI:10.61091/um118-01
http://www.combinatorialpress.com/um
Received 2 December 2023, Accepted 25 December 2023, Published 30 December 2023

Article

Arctangent Series on Pell and Pell-Lucas Polynomials via Cassini-Like Formulae

Dongwei Guo[1] **and Wenchang Chu**[2,*]

[1] School of Economics and Management, Nanjing University of Science and Technology, Nanjing (Jiangsu) 210094, China.
[2] School of Mathematics and Statistics, Zhoukou Normal University, Henan, China.

* **Correspondence:** hypergeometricx@outlook.com

Abstract: By combining the telescoping method with Cassini–like formulae, we evaluate, in closed forms, four classes of sums about products of two arctangent functions with their argument involving Pell and Pell–Lucas polynomials. Several infinite series identities for Fibonacci and Lucas numbers are deduced as consequences.

Keywords: Arctangent function, Cassini formula, Binet form, Fibonacci and lucas numbers, Pell and Pell-lucas polynomials
Mathematics Subject Classification: Primary 11B39, Secondary 05A10.

---

1. Introduction

Pell and Pell–Lucas polynomials are defined (Horadam and Mahon [1] and Mahon and Horadam [2]) by the recurrence relations

$$\mathrm{P}_n(x) = 2x\mathrm{P}_{n-1}(x) + \mathrm{P}_{n-2}(x),$$
$$\mathrm{Q}_n(x) = 2x\mathrm{Q}_{n-1}(x) + \mathrm{Q}_{n-2}(x);$$

with different initial conditions

$$\mathrm{P}_0(x) = 0 \quad \text{and} \quad \mathrm{P}_1(x) = 1,$$
$$\mathrm{Q}_0(x) = 2 \quad \text{and} \quad \mathrm{Q}_1(x) = 2x.$$

The Binet forms for these polynomials read as

$$\mathrm{P}_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad \mathrm{Q}_n(x) = \alpha^n + \beta^n$$

where

$$\alpha := \alpha(x) = x + \sqrt{x^2+1} \quad \text{and} \quad \beta := \beta(x) = x - \sqrt{x^2+1}.$$

They are polynomial extensions of Fibonacci and Lucas numbers

$$\mathrm{P}_n(\tfrac{1}{2}) = F_n \qquad \text{and} \qquad \mathrm{Q}_n(\tfrac{1}{2}) = L_n$$

as well as Pell and Pell–Lucas numbers

$$\mathrm{P}_n(1) = P_n \qquad \text{and} \qquad \mathrm{Q}_n(1) = Q_n.$$

Like the well–known Fibonacci and Lucas numbers, the Pell and Pell–Lucas polynomials have many amazing properties and important applications in combinatorics and number theory as well as physical sciences (cf. Koshy [3,4]). There exist numerous identities about Fibonacci and Lucas numbers (cf. [5–8]). Some of them are extended by Mahon and Horadam [2] and Melham and Shanno [9] to the summation formulae containing a single arctangent function with its argument being Pell and Pell–Lucas polynomials. In view of their importance, it is natural to examine the product sums of two arctangent functions with their arguments involving Pell and Pell–Lucas polynomials. That is the primary motivation for the present work.

Recall the Cassini formula for Fibonacci numbers

$$F_{n+1}F_{n-1} = (-1)^n + F_n^2.$$

There are two analogous ones for Pell and Pell–Lucas polynomials. They will be employed in this paper to evaluate, in closed forms, four classes of sums about products of two arctangent functions. The main techniques to realize this purpose consist of the telescopic approach (cf. [10]) and the following two trigonometric relations

$$\arctan x + \arctan y = \arctan \frac{x+y}{1-xy}, \quad (xy < 1); \tag{1}$$

$$\arctan x - \arctan y = \arctan \frac{x-y}{1+xy}, \quad (xy > -1). \tag{2}$$

Throughout the paper, we assume that x is real with $x > 0$, since otherwise, $x < 0$ will result essentially in exchanging $\alpha(x)$ and $\beta(x)$. Consequently the following zero limits

$$\begin{aligned} 0 &= \lim_{n\to\infty} \alpha^{-n}(x) = \lim_{n\to\infty} \arctan \alpha^{-n}(x) \\ &= \lim_{n\to\infty} \beta^n(x) = \lim_{n\to\infty} \arctan \beta^n(x) \end{aligned}$$

will be utilized frequently to deduce limiting relations without specific explanations.

2. The First Class of Summation Formulae

By making use of the following formulae (cf. Koshy [4]§14.7) for Pell polynomials

$$\begin{aligned} \mathrm{P}_{k-1}(x) + \mathrm{P}_{k+1}(x) &= \mathrm{Q}_k(x), \\ \mathrm{P}_{k-1}(x)\mathrm{P}_{k+1}(x) &= \mathrm{P}_k^2(x) + (-1)^k; \end{aligned}$$

it is not difficult to deduce from (1) and (2)

$$\arctan \frac{\mathrm{Q}_k(x)}{\mathrm{P}_k^2(x) + (-1)^k - 1} = \arctan \frac{1}{\mathrm{P}_{k-1}(x)} + \arctan \frac{1}{\mathrm{P}_{k+1}(x)}, \tag{3}$$

$$\arctan \frac{2x\mathrm{P}_k(x)}{\mathrm{P}_k^2(x) + (-1)^k + 1} = \arctan \frac{1}{\mathrm{P}_{k-1}(x)} - \arctan \frac{1}{\mathrm{P}_{k+1}(x)}. \tag{4}$$

They were directly utilized by Mahon and Horadam [2] and Melham and Shannon [9] to establish

$$\sum_{k=1}^{n} (-1)^{k-1} \arctan \frac{\mathrm{Q}_{2k}(x)}{\mathrm{P}_{2k}^2(x)} = \frac{\pi}{4} - (-1)^n \arctan \frac{1}{\mathrm{P}_{2n+1}(x)}$$

and

$$\sum_{k=1}^{n} \arctan \frac{2x}{\mathrm{P}_{2k-1}(x)} = \frac{\pi}{2} - \frac{1}{\mathrm{P}_{2n}(x)}.$$

Instead, we shall utilize both (3) and (4) to derive further summation formulae concerning products of two arctangent functions.

2.1.

Replacing k by $2k$, we can rewrite (3) and (4) as

$$\arctan \frac{1}{\mathrm{P}_{2k-1}(x)} + \arctan \frac{1}{\mathrm{P}_{2k+1}(x)} = \arctan \frac{\mathrm{Q}_{2k}(x)}{\mathrm{P}^2_{2k}(x)},$$
$$\arctan \frac{1}{\mathrm{P}_{2k-1}(x)} - \arctan \frac{1}{\mathrm{P}_{2k+1}(x)} = \arctan \frac{2x\mathrm{P}_{2k}(x)}{\mathrm{P}^2_{2k}(x)+2}.$$

Their multiplication gives rise to

$$\sum_{k=1}^{n} \arctan \frac{\mathrm{Q}_{2k}(x)}{\mathrm{P}^2_{2k}(x)} \arctan \frac{2x\mathrm{P}_{2k}(x)}{\mathrm{P}^2_{2k}(x)+2} = \sum_{k=1}^{n} \left(\arctan^2 \frac{1}{P_{2k-1}(x)} - \arctan^2 \frac{1}{P_{2k+1}(x)} \right).$$

Then summing this equation for k from 1 to n by telescoping, we find the following formula.

Theorem 1 ($x > 0$)**.**

$$\sum_{k=1}^{n} \arctan \frac{\mathrm{Q}_{2k}(x)}{\mathrm{P}^2_{2k}(x)} \arctan \frac{2x\mathrm{P}_{2k}(x)}{\mathrm{P}^2_{2k}(x)+2} = \frac{\pi^2}{16} - \arctan^2 \frac{1}{\mathrm{P}_{2n+1}(x)}.$$

Its limiting case as $n \to \infty$ yields a remarkable series whose sum is independent of x.

Corollary 1 (Independent of x)**.**

$$\sum_{k=1}^{\infty} \arctan \frac{\mathrm{Q}_{2k}(x)}{\mathrm{P}^2_{2k}(x)} \arctan \frac{2x\mathrm{P}_{2k}(x)}{\mathrm{P}^2_{2k}(x)+2} = \frac{\pi^2}{16}.$$

By specifying particular values of x, we can derive, from the above corollary, a number of infinite series identities. Eight of them are highlighted as follows.

- $\boxed{x = \frac{1}{2} = \frac{L_1}{2}}$:

$$\sum_{k=1}^{\infty} \arctan \frac{L_{2k}}{F^2_{2k}} \arctan \frac{F_{2k}}{F^2_{2k}+2} = \frac{\pi^2}{16}. \tag{5}$$

- $\boxed{x = \frac{\sqrt{5}}{2} = \frac{\sqrt{5}}{2}F_2}$:

$$\sum_{k=1}^{\infty} \arctan \frac{9L_{4k}}{5F^2_{4k}} \arctan \frac{15F_{4k}}{5F^2_{4k}+18} = \frac{\pi^2}{16}. \tag{6}$$

- $\boxed{x = 2 = \frac{L_3}{2}}$:

$$\sum_{k=1}^{\infty} \arctan \frac{4L_{6k}}{F^2_{6k}} \arctan \frac{8F_{6k}}{F^2_{6k}+8} = \frac{\pi^2}{16}. \tag{7}$$

- $\boxed{x = \frac{3\sqrt{5}}{2} = \frac{\sqrt{5}}{2}F_4}$:

$$\sum_{k=1}^{\infty} \arctan \frac{49L_{8k}}{5F_{8k}^2} \arctan \frac{105F_{8k}}{5F_{8k}^2 + 98} = \frac{\pi^2}{16}. \tag{8}$$

- $\boxed{x = 1 = \frac{Q_1}{2}}$:

$$\sum_{k=1}^{\infty} \arctan \frac{Q_{2k}}{P_{2k}^2} \arctan \frac{2P_{2k}}{P_{2k}^2 + 2} = \frac{\pi^2}{16}. \tag{9}$$

- $\boxed{x = 2\sqrt{2} = \sqrt{2}P_2}$:

$$\sum_{k=1}^{\infty} \arctan \frac{9Q_{4k}}{2P_{4k}^2} \arctan \frac{24P_{4k}}{2P_{4k}^2 + 18} = \frac{\pi^2}{16}. \tag{10}$$

- $\boxed{x = 7 = \frac{Q_3}{2}}$:

$$\sum_{k=1}^{\infty} \arctan \frac{25Q_{6k}}{P_{6k}^2} \arctan \frac{70P_{6k}}{P_{6k}^2 + 50} = \frac{\pi^2}{16}. \tag{11}$$

- $\boxed{x = 12\sqrt{2} = \sqrt{2}P_4}$:

$$\sum_{k=1}^{\infty} \arctan \frac{289Q_{8k}}{2P_{8k}^2} \arctan \frac{408P_{8k}}{P_{8k}^2 + 289} = \frac{\pi^2}{16}. \tag{12}$$

2.2.

Alternatively, (3) and (4) can be restated under the replacement k by $2k-1$ as

$$\arctan \frac{1}{\mathrm{P}_{2k-2}(x)} + \arctan \frac{1}{\mathrm{P}_{2k}(x)} = \arctan \frac{\mathrm{Q}_{2k-1}(x)}{\mathrm{P}_{2k-1}^2(x) - 2},$$
$$\arctan \frac{1}{\mathrm{P}_{2k-2}(x)} - \arctan \frac{1}{\mathrm{P}_{2k}(x)} = \arctan \frac{2x}{\mathrm{P}_{2k-1}(x)}.$$

First summing their product for k from 2 to n by telescoping and then adding the initial term $-\arctan^2 2x$ corresponding to $k = -1$ to the resultant equation, we get the theorem below.

Theorem 2 ($x > \sqrt{\frac{\sqrt{2}-1}{4}} \approx 0.321797$)**.**

$$\sum_{k=1}^{n} \arctan \frac{\mathrm{Q}_{2k-1}(x)}{\mathrm{P}_{2k-1}^2(x) - 2} \arctan \frac{2x}{\mathrm{P}_{2k-1}(x)} = \frac{\pi^2}{4} - \pi \arctan 2x - \arctan^2 \frac{1}{\mathrm{P}_{2n}(x)}.$$

Now letting $n \to \infty$ in this theorem, we deduce the infinite series identity.

Corollary 2.

$$\sum_{k=1}^{\infty} \arctan \frac{\mathrm{Q}_{2k-1}(x)}{\mathrm{P}_{2k-1}^2(x) - 2} \arctan \frac{2x}{\mathrm{P}_{2k-1}(x)} = \frac{\pi^2}{4} - \pi \arctan 2x.$$

We record eight interesting formulae by choosing particular values of x in this identity.

- $\boxed{x = \frac{1}{2} = \frac{L_1}{2}}$:

$$\sum_{k=1}^{\infty} \arctan \frac{L_{2k-1}}{F_{2k-1}^2 - 2} \arctan \frac{1}{F_{2k-1}} = 0. \tag{13}$$

- $\boxed{x = \frac{\sqrt{5}}{2} = \frac{\sqrt{5}}{2}F_2}$:

$$\sum_{k=1}^{\infty} \arctan \frac{9\sqrt{5}F_{4k-2}}{L_{4k-2}^2 - 18} \arctan \frac{3\sqrt{5}}{L_{4k-2}} = \frac{\pi^2}{4} - \pi \arctan \sqrt{5}. \tag{14}$$

- $\boxed{x = 2 = \frac{L_3}{2}}$:

$$\sum_{k=1}^{\infty} \arctan \frac{4L_{6k-3}}{F_{6k-3}^2 - 8} \arctan \frac{8}{F_{6k-3}} = \frac{\pi^2}{4} - \pi \arctan 4. \tag{15}$$

- $\boxed{x = \frac{3\sqrt{5}}{2} = \frac{\sqrt{5}}{2}F_4}$:

$$\sum_{k=1}^{\infty} \arctan \frac{49\sqrt{5}F_{8k-4}}{L_{8k-4}^2 - 98} \arctan \frac{21\sqrt{5}}{L_{8k-4}} = \frac{\pi^2}{4} - \pi \arctan 3\sqrt{5}. \tag{16}$$

- $\boxed{x = 1 = \frac{Q_1}{2}}$:

$$\sum_{k=1}^{\infty} \arctan \frac{Q_{2k-1}}{P_{2k-1}^2 - 2} \arctan \frac{2}{P_{2k-1}} = \frac{\pi^2}{4} - \pi \arctan 2. \tag{17}$$

- $\boxed{x = 2\sqrt{2} = \sqrt{2}P_2}$:

$$\sum_{k=1}^{\infty} \arctan \frac{72\sqrt{2}P_{4k-2}}{Q_{4k-2}^2 - 72} \arctan \frac{24\sqrt{2}}{Q_{4k-2}} = \frac{\pi^2}{4} - \pi \arctan 4\sqrt{2}. \tag{18}$$

- $\boxed{x = 7 = \frac{Q_3}{2}}$:

$$\sum_{k=1}^{\infty} \arctan \frac{25Q_{6k-3}}{P_{6k-3}^2 - 50} \arctan \frac{70}{P_{6k-3}} = \frac{\pi^2}{4} - \pi \arctan 14. \tag{19}$$

- $\boxed{x = 12\sqrt{2} = \sqrt{2}P_4}$:

$$\sum_{k=1}^{\infty} \arctan \frac{2312\sqrt{2}P_{8k-4}}{Q_{8k-4}^2 - 2312} \arctan \frac{816\sqrt{2}}{Q_{8k-4}} = \frac{\pi^2}{4} - \pi \arctan 24\sqrt{2}. \tag{20}$$

3. The Second Class of Summation Formulae

Analogously, there are also two formulae (cf. Koshy [4]§14.7) for Pell–Lucas polynomials

$$\begin{aligned} \mathrm{Q}_{k-1}(x) + \mathrm{Q}_{k+1}(x) &= 4(x^2+1)\mathrm{P}_k(x), \\ \mathrm{Q}_{k-1}(x)\mathrm{Q}_{k+1}(x) &= \mathrm{Q}_k^2(x) + 4(-1)^{k-1}(x^2+1). \end{aligned}$$

In view of (1) and (2), we have the addition and difference formulae

$$\arctan \frac{1}{\mathrm{Q}_{k-1}(x)} + \arctan \frac{1}{\mathrm{Q}_{k+1}(x)} = \arctan \frac{4(x^2+1)\mathrm{P}_k(x)}{\mathrm{Q}_k^2(x) - 4(-1)^k(x^2+1) - 1}, \tag{21}$$

$$\arctan \frac{1}{\mathrm{Q}_{k-1}(x)} - \arctan \frac{1}{\mathrm{Q}_{k+1}(x)} = \arctan \frac{2x\mathrm{Q}_k(x)}{\mathrm{Q}_k^2(x) - 4(-1)^k(x^2+1) + 1}. \tag{22}$$

The last relation can directly be employed, by telescoping, to evaluate simple sums of a single arctangent function that we shall not reproduce. Instead, we shall concentrate on the sums containing products of two arctangent functions.

3.1.

Replacing k by $2k$, we can rewrite (21) and (22) as

$$\arctan\frac{4(x^2+1)\mathrm{P}_{2k}(x)}{\mathrm{Q}^2_{2k}(x)-4x^2-5}=\arctan\frac{1}{\mathrm{Q}_{2k-1}(x)}+\arctan\frac{1}{\mathrm{Q}_{2k+1}(x)},$$
$$\arctan\frac{2x\mathrm{Q}_{2k}(x)}{\mathrm{Q}^2_{2k}(x)-4x^2-3}=\arctan\frac{1}{\mathrm{Q}_{2k-1}(x)}-\arctan\frac{1}{\mathrm{Q}_{2k+1}(x)}.$$

Then summing their product for k from 1 to n leads us to the following theorem.

Theorem 3 ($x>\sqrt{\frac{\sqrt{13}-3}{8}}\approx 0.275125$)**.**

$$\sum_{k=1}^{n}\arctan\frac{4(x^2+1)\mathrm{P}_{2k}(x)}{\mathrm{Q}^2_{2k}(x)-4x^2-5}\arctan\frac{2x\mathrm{Q}_{2k}(x)}{\mathrm{Q}^2_{2k}(x)-4x^2-3}$$
$$=\arctan^2\frac{1}{2x}-\arctan^2\frac{1}{\mathrm{Q}_{2n+1}(x)}.$$

When $n\to\infty$, the above theorem gives rise to the infinite series evaluation below.

Corollary 3.

$$\sum_{k=1}^{\infty}\arctan\frac{4(x^2+1)\mathrm{P}_{2k}(x)}{\mathrm{Q}^2_{2k}(x)-4x^2-5}\arctan\frac{2x\mathrm{Q}_{2k}(x)}{\mathrm{Q}^2_{2k}(x)-4x^2-3}=\arctan^2\frac{1}{2x}.$$

As applications, we collect six infinite series identities.

- $\boxed{x=\frac{1}{2}=\frac{L_1}{2}}$:

$$\sum_{k=1}^{\infty}\arctan\frac{5F_{2k}}{L^2_{2k}-6}\arctan\frac{L_{2k}}{L^2_{2k}-4}=\frac{\pi^2}{16}. \tag{23}$$

- $\boxed{x=\frac{\sqrt{5}}{2}=\frac{\sqrt{5}}{2}F_2}$:

$$\sum_{k=1}^{\infty}\arctan\frac{3\sqrt{5}F_{4k}}{L^2_{4k}-10}\arctan\frac{\sqrt{5}L_{4k}}{L^2_{4k}-8}=\arctan^2\frac{\sqrt{5}}{5}. \tag{24}$$

- $\boxed{x=2=\frac{L_3}{2}}$:

$$\sum_{k=1}^{\infty}\arctan\frac{10L_{6k}}{L^2_{6k}-21}\arctan\frac{4L_{6k}}{L^2_{6k}-19}=\arctan^2\frac{1}{4}. \tag{25}$$

- $\boxed{x=1=\frac{Q_1}{2}}$:

$$\sum_{k=1}^{\infty}\arctan\frac{8P_{2k}}{Q^2_{2k}-9}\arctan\frac{2Q_{2k}}{Q^2_{2k}-7}=\arctan^2\frac{1}{2}. \tag{26}$$

- $\boxed{x=2\sqrt{2}=\sqrt{2}P_2}$:

$$\sum_{k=1}^{\infty}\arctan\frac{12\sqrt{2}P_{4k}}{Q^2_{4k}-37}\arctan\frac{4\sqrt{2}Q_{4k}}{Q^2_{4k}-35}=\arctan^2\frac{\sqrt{2}}{8}. \tag{27}$$

- $\boxed{x=7=\frac{Q_3}{2}}$:

$$\sum_{k=1}^{\infty}\arctan\frac{40P_{6k}}{Q^2_{6k}-201}\arctan\frac{14Q_{6k}}{Q^2_{6k}-199}=\arctan^2\frac{1}{14}. \tag{28}$$

3.2.

Under the replacement k by $2k-1$, the two equalities in (21) and (22) become

$$\arctan\frac{1}{Q_{2k-2}(x)}+\arctan\frac{1}{Q_{2k}(x)}=\arctan\frac{4(x^2+1)P_{2k-1}(x)}{Q^2_{2k-1}(x)+4x^2+3},$$
$$\arctan\frac{1}{Q_{2k-2}(x)}-\arctan\frac{1}{Q_{2k}(x)}=\arctan\frac{2xQ_{2k-1}(x)}{Q^2_{2k-1}(x)+4x^2+5}.$$

Multiplying them and summing for k from 1 to n by telescoping, we obtain the formula below.

Theorem 4.

$$\sum_{k=1}^{n}\arctan\frac{4(x^2+1)P_{2k-1}(x)}{Q^2_{2k-1}(x)+4x^2+3}\arctan\frac{2xQ_{2k-1}(x)}{Q^2_{2k-1}(x)+4x^2+5}$$
$$=\arctan^2\frac{1}{2}-\arctan^2\frac{1}{Q_{2n}(x)}.$$

Its limiting case as $n\to\infty$ yields the infinite series evaluation.

Corollary 4 (Independent of x).

$$\sum_{k=1}^{\infty}\arctan\frac{4(x^2+1)P_{2k-1}(x)}{Q^2_{2k-1}(x)+4x^2+3}\arctan\frac{2xQ_{2k-1}(x)}{Q^2_{2k-1}(x)+4x^2+5}=\arctan^2\frac{1}{2}.$$

In particular, two infinite series identities are given as follows:

- $\boxed{x=1/2}$:

$$\sum_{k=1}^{\infty}\arctan\frac{5F_{2k-1}}{L^2_{2k-1}+4}\arctan\frac{L_{2k-1}}{L^2_{2k-1}+6}=\arctan^2\frac{1}{2}. \tag{29}$$

- $\boxed{x=1}$:

$$\sum_{k=1}^{\infty}\arctan\frac{8P_{2k-1}}{Q^2_{2k-1}+7}\arctan\frac{2Q_{2k-1}}{Q^2_{2k-1}+9}=\arctan^2\frac{1}{2}. \tag{30}$$

4. The Third Class of Summation Formulae

By combining the Cassini–like formula (cf. Koshy [4]§14.10)

$$P^2_k(x)-P_{k+\lambda}(x)P_{k-\lambda}(x)=(-1)^{k+\lambda}P^2_\lambda(x) \tag{31}$$

with (1) and (2), we get the following two identities

$$\arctan\frac{P_k(x)}{P_{k+\lambda}(x)}+\arctan\frac{P_{k-\lambda}(x)}{P_k(x)}=\begin{cases}\arctan\dfrac{2P^2_k(x)-(-1)^kP^2_\lambda(x)}{P_\lambda(x)P_{2k}(x)}, & \lambda\quad\text{even};\\[2ex] \arctan\dfrac{2P^2_k(x)+(-1)^kP^2_\lambda(x)}{Q_\lambda(x)P^2_k(x)}, & \lambda\quad\text{odd};\end{cases} \tag{32}$$

$$\arctan\frac{P_k(x)}{P_{k+\lambda}(x)}-\arctan\frac{P_{k-\lambda}(x)}{P_k(x)}=\begin{cases}\arctan\dfrac{(-1)^kP^2_\lambda(x)}{Q_\lambda(x)P^2_k(x)}, & \lambda\quad\text{even};\\[2ex] \arctan\dfrac{(-1)^{k+1}P_\lambda(x)}{P_{2k}(x)}, & \lambda\quad\text{odd}.\end{cases} \tag{33}$$

By means of (33), Melham–Shannon [9] obtained directly

$$\sum_{k=1}^{n}\arctan\frac{(-1)^{k-1}}{P_{2k}(x)}=\arctan\frac{P_n(x)}{P_{n+1}(x)}.$$

Now we are going to examine sums for the products of two arctangent functions.

4.1.

When $\lambda = 1$, rewriting (32) and (33) as

$$\arctan\frac{\mathrm{P}_k(x)}{\mathrm{P}_{k+1}(x)} - \arctan\frac{\mathrm{P}_{k-1}(x)}{\mathrm{P}_k(x)} = \arctan\frac{(-1)^{k+1}}{P_{2k}(x)},$$
$$\arctan\frac{\mathrm{P}_k(x)}{\mathrm{P}_{k+1}(x)} + \arctan\frac{\mathrm{P}_{k-1}(x)}{\mathrm{P}_k(x)} = \arctan\frac{2P_k^2(x)+(-1)^k}{2xP_k^2(x)};$$

and then summing their product for k from 1 to n by telescoping, we get the summation formula.

Theorem 5.

$$\sum_{k=1}^{n}\arctan\frac{2\mathrm{P}_k^2(x)+(-1)^k}{2x\mathrm{P}_k^2(x)}\arctan\frac{(-1)^{k+1}}{\mathrm{P}_{2k}(x)} = \arctan^2\frac{\mathrm{P}_n(x)}{\mathrm{P}_{n+1}(x)}.$$

Its limiting case as $n \to \infty$ results in the infinite series identity.

Corollary 5.

$$\sum_{k=1}^{\infty}\arctan\frac{2\mathrm{P}_k^2(x)+(-1)^k}{2x\mathrm{P}_k^2(x)}\arctan\frac{(-1)^{k+1}}{\mathrm{P}_{2k}(x)} = \arctan^2(\sqrt{x^2+1}-x).$$

Two formulae about Fibonacci numbers and Pell numbers are contained as special cases.

- $\boxed{x = 1/2}$:

$$\sum_{k=1}^{\infty}\arctan\frac{2F_k^2+(-1)^k}{F_k^2}\arctan\frac{(-1)^{k+1}}{F_{2k}} = \frac{1}{4}\arctan^2 2. \tag{34}$$

- $\boxed{x = 1}$:

$$\sum_{k=1}^{\infty}\arctan\frac{2P_k^2+(-1)^k}{2P_k^2}\arctan\frac{(-1)^{k+1}}{P_{2k}} = \frac{\pi^2}{64}. \tag{35}$$

4.2.

Analogously for $\lambda = 2$, both (32) and (33) become

$$\arctan\frac{\mathrm{P}_k(x)}{\mathrm{P}_{k+2}(x)} - \arctan\frac{\mathrm{P}_{k-2}(x)}{\mathrm{P}_k(x)} = \arctan\frac{2(-1)^k x^2}{(2x^2+1)\mathrm{P}_k^2(x)},$$
$$\arctan\frac{\mathrm{P}_k(x)}{\mathrm{P}_{k+2}(x)} + \arctan\frac{\mathrm{P}_{k-2}(x)}{\mathrm{P}_k(x)} = \arctan\frac{\mathrm{P}_k^2(x)-2(-1)^k x^2}{x\mathrm{P}_{2k}(x)}.$$

Keeping in mind that $\mathrm{P}_{-1}(x) = 1$ and then summing their product for k from 1 to n, we find by telescoping another formula as in the following theorem.

Theorem 6.

$$\sum_{k=1}^{n}\arctan\frac{\mathrm{P}_k^2(x)-2(-1)^k x^2}{x\mathrm{P}_{2k}(x)}\arctan\frac{2(-1)^k x^2}{(2x^2+1)\mathrm{P}_k^2(x)}$$
$$= \arctan^2\frac{\mathrm{P}_{n-1}(x)}{\mathrm{P}_{n+1}(x)} + \arctan^2\frac{\mathrm{P}_n(x)}{\mathrm{P}_{n+2}(x)} - \frac{\pi^2}{16}.$$

The limiting case as $n \to \infty$ is given by the following corollary.

Corollary 6.

$$\sum_{k=1}^{\infty}\arctan\frac{\mathrm{P}_k^2(x)-2(-1)^k x^2}{x\mathrm{P}_{2k}(x)}\arctan\frac{2(-1)^k x^2}{(2x^2+1)\mathrm{P}_k^2(x)} = 2\arctan^2\beta^2 - \frac{\pi^2}{16}.$$

In particular, we record two identities about Fibonacci and Pell numbers.

- $\boxed{x = 1/2}$:

$$\sum_{k=1}^{\infty} \arctan \frac{2F_k^2 - (-1)^k}{F_{2k}} \arctan \frac{(-1)^k}{3F_k^2} = \frac{1}{2} \arctan^2 \frac{2}{\sqrt{5}} - \frac{\pi^2}{16}. \tag{36}$$

- $\boxed{x = 1}$:

$$\sum_{k=1}^{\infty} \arctan \frac{P_k^2 - 2(-1)^k}{P_{2k}} \arctan \frac{2(-1)^k}{3P_k^2} = \frac{1}{2} \arctan \frac{\sqrt{2}}{4} - \frac{\pi^2}{16}. \tag{37}$$

5. The Fourth Class of Summation Formulae

In this section, the counterpart formulae for $\mathrm{Q}(x)$ will be worked out by employing another Cassini–like formula (cf. Koshy [4]§14.10)

$$\mathrm{Q}_{k+\lambda}(x)\mathrm{Q}_{k-\lambda}(x) - \mathrm{Q}_k^2(x) = 4(-1)^{k+\lambda}(1 + x^2)\mathrm{P}_\lambda^2(x) \tag{38}$$

as well as two reformulated ones by (1) and (2):

$$\arctan \frac{\mathrm{Q}_k(x)}{\mathrm{Q}_{k+\lambda}(x)} + \arctan \frac{\mathrm{Q}_{k-\lambda}(x)}{\mathrm{Q}_k(x)} = \begin{cases} \arctan \dfrac{2\mathrm{Q}_k^2(x) + (-1)^k(\mathrm{Q}_\lambda^2(x) - 4)}{4(x^2+1)\mathrm{P}_\lambda(x)\mathrm{P}_{2k}(x)}, & \lambda \quad \text{even;} \\ \arctan \dfrac{2\mathrm{Q}_k^2(x) - (-1)^k(\mathrm{Q}_\lambda^2(x) + 4)}{\mathrm{Q}_\lambda(x)\mathrm{Q}_k^2(x)}, & \lambda \quad \text{odd;} \end{cases} \tag{39}$$

$$\arctan \frac{\mathrm{Q}_k(x)}{\mathrm{Q}_{k+\lambda}(x)} - \arctan \frac{\mathrm{Q}_{k-\lambda}(x)}{\mathrm{Q}_k(x)} = \begin{cases} \arctan \dfrac{(-1)^k(4 - \mathrm{Q}_\lambda^2(x))}{\mathrm{Q}_\lambda(x)\mathrm{Q}_k^2(x)}, & \lambda \quad \text{even;} \\ \arctan \dfrac{(-1)^k(4 + \mathrm{Q}_\lambda^2(x))}{4(x^2+1)\mathrm{P}_\lambda(x)\mathrm{P}_{2k}(x)}, & \lambda \quad \text{odd;} \end{cases} \tag{40}$$

5.1.

Letting $\lambda = 1$ in (39) and (40), we have

$$\arctan \frac{\mathrm{Q}_k(x)}{\mathrm{Q}_{k+1}(x)} + \arctan \frac{\mathrm{Q}_{k-1}(x)}{\mathrm{Q}_k(x)} = \arctan \frac{\mathrm{Q}_k^2(x) - 2(-1)^k(x^2+1)}{x\mathrm{Q}_k^2(x)},$$
$$\arctan \frac{\mathrm{Q}_k(x)}{\mathrm{Q}_{k+1}(x)} - \arctan \frac{\mathrm{Q}_{k-1}(x)}{\mathrm{Q}_k(x)} = \arctan \frac{(-1)^k}{\mathrm{P}_{2k}(x)}.$$

Multiplying them and then summing the resultant expression for k from 1 to n, we find by telescoping the formula below.

Theorem 7.

$$\sum_{k=1}^{n} \arctan \frac{(-1)^k}{\mathrm{P}_{2k}(x)} \arctan \frac{\mathrm{Q}_k^2(x) - 2(-1)^k(x^2+1)}{x\mathrm{Q}_k^2(x)} = \arctan^2 \frac{\mathrm{Q}_n(x)}{\mathrm{Q}_{n+1}(x)} - \arctan^2 \frac{1}{x}.$$

As $n \to \infty$, the limiting result evaluates the following infinite series.

Corollary 7.

$$\sum_{k=1}^{\infty} \arctan \frac{(-1)^k}{\mathrm{P}_{2k}(x)} \arctan \frac{\mathrm{Q}_k^2(x) - 2(-1)^k(x^2+1)}{x\mathrm{Q}_k^2(x)} = \arctan^2(\sqrt{x^2+1} - x) - \arctan^2 \frac{1}{x}.$$

Two special cases are produced below as examples

- $\boxed{x = 1/2}$:

$$\sum_{k=1}^{\infty} \arctan \frac{(-1)^k}{F_{2k}} \arctan \frac{2L_k^2 - 5(-1)^k}{L_k^2} = -\frac{3}{4} \arctan^2 2. \tag{41}$$

- $\boxed{x = 1}$:

$$\sum_{k=1}^{\infty} \arctan \frac{(-1)^k}{P_{2k}} \arctan \frac{Q_k^2 - 4(-1)^k}{Q_k^2} = -\frac{3\pi^2}{64}. \tag{42}$$

5.2.

Analogously for $\lambda = 2$, we have from (39) and (40)

$$\arctan \frac{\mathrm{Q}_k(x)}{\mathrm{Q}_{k+2}(x)} + \arctan \frac{\mathrm{Q}_{k-2}(x)}{\mathrm{Q}_k(x)} = \arctan \frac{\mathrm{Q}_k^2(x) + 8(-1)^k(x^4 + x^2)}{4(x^3 + x)\mathrm{P}_{2k}(x)}$$
$$\arctan \frac{\mathrm{Q}_k(x)}{\mathrm{Q}_{k+2}(x)} - \arctan \frac{\mathrm{Q}_{k-2}(x)}{\mathrm{Q}_k(x)} = \arctan \frac{8(-1)^{k+1}(x^4 + x^2)}{(2x^2 + 1)\mathrm{Q}_k^2(x)}.$$

Summing their product for k from 1 to n and taking into account $\mathrm{Q}_{-1}(x) = -2x$, we get, after some simplifications, the following formula.

Theorem 8.

$$\sum_{k=1}^{n} \arctan \frac{8(-1)^{k+1}(x^4 + x^2)}{(2x^2 + 1)\mathrm{Q}_k^2(x)} \arctan \frac{\mathrm{Q}_k^2(x) + 8(-1)^k(x^4 + x^2)}{4(x^3 + x)\mathrm{P}_{2k}(x)}$$
$$= \arctan^2 \frac{\mathrm{Q}_n(x)}{\mathrm{Q}_{n+2}(x)} + \arctan^2 \frac{\mathrm{Q}_{n-1}(x)}{\mathrm{Q}_{n+1}(x)} - \arctan^2 \frac{2}{\mathrm{Q}_2(x)} - \frac{\pi^2}{16}.$$

Letting $n \to \infty$ in this theorem, we get the infinite series evaluation.

Corollary 8.

$$\sum_{k=1}^{\infty} \arctan \frac{8(-1)^{k+1}(x^4 + x^2)}{(2x^2 + 1)\mathrm{Q}_k^2(x)} \arctan \frac{\mathrm{Q}_k^2(x) + 8(-1)^k(x^4 + x^2)}{4(x^3 + x)\mathrm{P}_{2k}(x)}$$
$$= 2\arctan^2 \beta^2 - \arctan^2 \frac{1}{2x^2 + 1} - \frac{\pi^2}{16}.$$

This formula further implies the two infinite series identities.

- $\boxed{x = 1/2}$:

$$\sum_{k=1}^{\infty} \arctan \frac{5(-1)^{k+1}}{3L_k^2} \arctan \frac{2L_k^2 + 5(-1)^k}{5F_{2k}} = \frac{1}{2} \arctan^2 \frac{2}{\sqrt{5}} - \arctan^2 \frac{2}{3} - \frac{\pi^2}{16}. \tag{43}$$

- $\boxed{x = 1}$:

$$\sum_{k=1}^{\infty} \arctan \frac{16(-1)^{k+1}}{3Q_k^2} \arctan \frac{Q_k^2 + 16(-1)^k}{8P_{2k}} = \frac{1}{2} \arctan^2 \frac{\sqrt{2}}{4} - \arctan^2 \frac{1}{3} - \frac{\pi^2}{16}. \tag{44}$$

Conflict of Interest

The authors declare no conflict of interest.

References

1. Horadam, A.F. and Mahon, J.M., 1985. Pell and pell-lucas polynomials. *Fibonacci Quart, 23*(1), pp.7-20.
2. Mahon, B.J.M. and Horadam, A.F., 1985. Inverse trigonometrical summation formulas involving Pell polynomials. *Fibonacci Quarterly, 23*(4), pp.319-324.
3. Koshy, T., 2001. *Fibonacci and Lucas Numbers with Applications*. John Wiley & Sons.
4. Koshy, T., 2014. *Pell and Pell-Lucas Numbers with Applications*. Springer.
5. Adegoke, K., 2015. Infinite arctangent sums involving Fibonacci and Lucas numbers. *Notes on Number Theory and Discrete Mathematics, 21*(1), pp.56-66.
6. Chu, W. and Li, N. N., 2011. Power sums of Fibonacci and Lucas numbers. *Quaestiones Mathematicae, 34*(1), pp.75-83.
7. Chu, W. and Zhou, R. R., 2010. Two multiple convolutions on Fibonacci-like sequences. *Fibonacci Quarterly, 48*(1), pp.80-84.
8. Hoggatt Jr, V. E. and Ruggles, I. D., 1964. A Primer for the Fibonacci Numbers-Part V. *Fibonacci Quarterly, 2*(1), pp.59-65.
9. Melham, R. S. and Shannon, A. G., 1995. Inverse trigonometric and hyperbolic summation formulas involving generalized Fibonacci numbers. *Fibonacci Quarterly, 33*(1), pp.32-40.
10. Chu, W., 2016. Trigonometric formulae via telescoping method. *Online Journal of Analytic Combinatorics, 11*, Art#6.