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Article

Caputo Fractional Derivative Inequalities for Refined Modified (h, m)-Convex Functions

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Abstract: This paper introduces a novel type of convex function known as the refined modified $(h, m)-$convex function, which is a generalization of the traditional $(h, m)-$convex function. We establish Hadamard-type inequalities for this new definition by utilizing the Caputo $k-$fractional derivative. Specifically, we derive two integral identities that involve the nth order derivatives of given functions and use them to prove the estimation of Hadamard-type inequalities for the Caputo $k-$fractional derivatives of refined modified $(h, m)-$convex functions. The results obtained in this research demonstrate the versatility of the refined modified $(h, m)-$convex function and the usefulness of Caputo $k-$fractional derivatives in establishing important inequalities. Our work contributes to the existing body of knowledge on convex functions and offers insights into the applications of fractional calculus in mathematical analysis. The research findings have the potential to pave the way for future studies in the area of convex functions and fractional calculus, as well as in other areas of mathematical research.

Keywords: Caputo $k-$fractional derivatives; Refined modified $(h, m)-$convex function; Hadamard inequality
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1. Introduction

Fractional calculus is a branch of mathematics that deals with the concepts of fractional differentiation and integration. The history of fractional calculus dates back to the seventeenth century, when G. W. Leibniz and Marquis de l'Hospital initiated a discussion on semi-derivatives [1]. Since then, fractional calculus has been an active area of research that has found applications in many fields of science and engineering, including control theory, signal processing, and fluid mechanics.

The development of fractional calculus owes much to the contributions of Riemann and Liouville. In 1854, Riemann introduced the first integral operator, known as the Riemann-Liouville fractional integral operator [2]. This operator provides a natural extension of the classical concept of integration to non-integer orders. The Riemann-Liouville fractional derivative, which is obtained by applying the fractional integral operator to a function, was introduced by Liouville in 1832 [1]. The Caputo fractional derivative formula, which is a refinement of the Riemann-Liouville fractional derivative formula, was later introduced by Caputo [3]. The Caputo fractional derivative is now widely used in various fields of science and engineering, including fractional control theory and fractional differential equations.

In addition to the classical definitions of fractional derivatives and fractional integrals, numerous other definitions have been proposed by various authors. For example, the Grünwald-Letnikov fractional derivative, the Riesz fractional derivative, and the Marchaud fractional derivative are among the most widely used definitions. These different definitions have their own advantages and disadvantages and find different applications in different fields. The study of fractional derivatives and fractional integrals is an active area of research, and many open problems remain to be solved in this field [2,4].

The Caputo fractional derivative has a rich history dating back to the seventeenth century when G. W. Leibniz and Marquis de l'Hospital discussed semi-derivatives [3]. It is an essential tool in the field of fractional calculus, which is based on fractional differentiation and integration. The Riemann-Liouville fractional integral operator is the first integral operator in the field of fractional calculus, and the Riemann-Liouville fractional derivative was obtained using this operator. Caputo later improved the Riemann-Liouville fractional derivative formula, resulting in the well-known formula for Caputo fractional derivatives [3].

Definition 1. [3] *Let $f \in AC^n[a,b]$. The Caputo fractional derivatives of order $\alpha \in \mathbb{C}, Re(\alpha) > 0$ of f are defined as follows:*

$$ {}^C D^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \quad x > a, \tag{1}$$

$$ {}^C D^{\alpha}_{b^-} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, \quad x < b, \tag{2}$$

where $n = [\alpha] + 1$. If $\alpha = n \in \{1, 2, 3, ...\}$ and usual derivatives of order n exists, then Caputo fractional derivatives $({}^C D^{\alpha}_{a^+} f)(x)$ coincides with $f^{(n)}(x)$, whereas $({}^C D^{\alpha}_{b^-} f)(x)$ coincides with $f^{(n)}(x)$, with exactness to a constant multiplier $(-1)^n$.

In particular we have

$$ ({}^C D^0_{a^+} f)(x) = ({}^C D^0_{b^-} f)(x) = f(x), \tag{3}$$

where $n = 1$ and $\alpha = 0$.

The Caputo k-fractional derivative extends the classical Caputo fractional derivative to the k-analogues of the real and complex numbers [5]. In particular, it is defined for functions $f \in AC^n[a,b]$, where $n = [\alpha] + 1$ and $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$. The definition is given by Equations (1) and (2) for $x > a$ and $x < b$, respectively, where Γ is the Gamma function. When $\alpha = n$ and the usual derivatives of order n exist, the Caputo k-fractional derivatives coincide with the classical derivatives of order n, with an additional constant multiplier of $(-1)^n$ for $({}^C D^{\alpha}_{b^-} f)(x)$ [3].

Definition 2. [5] *Let $f \in AC^n[a,b]$. The Caputo $k-$fractional derivatives of order $\alpha \in \mathbb{C}, Re(\alpha) > 0$ of f are defined as follows:*

$$ {}^C D^{\alpha,k}_{a^+} f(x) = \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}} dt, \quad x > a, \tag{4}$$

$$ {}^C D^{\alpha,k}_{b^-} f(x) = \frac{(-1)^n}{k\Gamma_k(n-\frac{\alpha}{k})} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}} dt, \quad x < b, \tag{5}$$

where $k \geq 1$, $n = [\alpha] + 1$ and $\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{\frac{-t^k}{k}} dt$, $\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha)$.

Definition 3. [6] *The beta function of two variable u and v is defined as follows:*

$$ \beta(u,v) = \int_0^1 \phi^{u-1}(1-\phi)^{v-1} d\phi, $$

for $Re(u) > 0$, $Re(v) > 0$.

Convex functions have been a fundamental concept in mathematics for over a century, and have found numerous applications in fields ranging from optimization and statistics to economics and physics. A convex function is defined as a function whose graph lies above its secant lines, and has several important properties such as being differentiable almost everywhere, and having a unique minimum.

Classical convexity is the most commonly studied version of convexity, and requires a function to have a non-negative second derivative. A function that is twice differentiable and has a positive second derivative at every point is called strictly convex. Convex functions have a wide range of applications in optimization, as they guarantee the existence of a unique global minimum. Examples of classical convex functions include the exponential function, the logarithm function, and the quadratic function [7,8].

Another important version of convexity is quasi-convexity, which requires that a function's sub-level sets are convex. A function is quasi-convex if the set of points below its level set is convex. Quasi-convexity is a weaker form of convexity that is still useful in optimization and other areas of mathematics. Examples of quasi-convex functions include the absolute value function and the maximum function [7,9].

Another version of convexity is known as strongly convexity, which requires that a function's second derivative is bounded below by a positive constant. Strong convexity is a more stringent condition than classical convexity and has applications in optimization algorithms, such as gradient descent. Examples of strongly convex functions include the quadratic function with a positive-definite Hessian matrix [7,10].

Other versions of convexity include p-convexity, which requires a function to be a convex combination of its pth powers, and generalized convexity, which is defined by a broad class of convex functions including quasi-convex, pseudo-convex, and logarithmically convex functions. P-convexity has applications in mathematical economics, as it provides a way to model preferences of consumers with respect to income distributions [11–13]). Generalized convexity has applications in optimization, game theory, and statistics [14].

In conclusion, the study of convex functions and their various versions has been an active area of research for many years, and continues to be an important topic in mathematics and other fields. The different types of convexity provide a rich framework for understanding the behavior of functions in different settings, and their study has led to important insights and applications in many areas of science and engineering.

Inequality theory is a fundamental branch of mathematics that deals with the study of inequalities and their applications. Inequalities play a crucial role in many areas of mathematics, such as analysis, geometry, number theory, and optimization, as well as in other fields such as physics and economics. One of the most important inequalities in mathematics is the Hadamard inequality, which provides a bound on the determinant of a positive definite matrix in terms of its diagonal entries. The inequality is named after Jacques Hadamard, who first proved it in 1893. The Hadamard inequality has numerous applications in matrix theory, optimization, and statistics, and has been extended to many other settings, including complex matrices and operator theory [15–17].

The Hadamard inequality can also be generalized to functions, where it provides a bound on the product of the values of a convex function over an interval in terms of its integral over that interval. In this context, the inequality is known as the Hadamard-Fischer inequality, and has important applications in probability theory, functional analysis, and optimization [18,19].

In recent years, there has been growing interest in the study of fractional calculus and its applications in inequality theory. Fractional calculus provides a way to extend the classical calculus to non-integer orders, and has been used to develop new inequalities and bounds for various mathematical objects, such as functions, integrals, and differential equations. The study of fractional calculus and its applications in inequality theory has led to new insights and results in many areas of mathematics and physics [19, 20]. In conclusion, the study of inequality theory is an important and active area of research in mathematics and other fields. The Hadamard inequality, in particular, has been a fundamental result in matrix theory and has numerous applications in optimization and statistics. The generalization of the Hadamard inequality to functions and its extensions to fractional calculus have further enriched the field, and continue to be an active area of research.

The Hadamard inequality for convex function is stated in undermentioned theorem:

Theorem 1. [21] *If $f : J \to \mathbb{R}$ is a convex function defined on the interval J, then for $a, b \in J$, $a < b$ we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

The Hadamard inequality is a fundamental inequality that is a straightforward consequence of convexity for traditional convex functions, and has been extensively studied in the literature. However, the extension of the Hadamard inequality to the Caputo $k-$fractional derivatives of generalized convex functions has not been fully explored.

This paper presents a study of Caputo fractional derivatives for a generalized convex function, with the goal of exploring new notions and concepts motivated by analytic representation of convex functions. The study will focus on the Hadamard inequality for Caputo $k-$fractional derivatives of the refined modified $(h, m)-$convex function, a type of generalized convex function. The refined modified $(h, m)-$convex function is a type of generalized convex function that has been shown to have many useful properties in optimization and analysis. By studying the Hadamard inequality for Caputo $k-$fractional derivatives of this function, we aim to provide new insights into the behavior of generalized convex functions and their fractional derivatives.

This paper is structured as follows; In Section 2, we provide the necessary definitions and background information that will be used throughout the paper. In Section 3, we introduce two versions of the Hadamard inequality for refined modified $(h, m)-$convex functions using the Caputo $k-$fractional derivatives. We also present several special cases of these inequalities in the form of corollaries. Section 4 is devoted to establishing error estimates of Hadamard-type inequalities by utilizing two integral identities, including arbitrary order derivatives of function f. The derivations and analyses of these inequalities are presented in detail. In Section 5, we provide concluding remarks and summarize the main contributions of this paper. Furthermore, we identify some open problems for future research in this area. Overall, this paper contributes to the study of the Hadamard inequality for refined modified $(h, m)-$convex functions using fractional calculus techniques. The results presented in this paper have potential applications in various fields, including optimization, numerical analysis, and mathematical physics.

2. Basic Definitions

This section provides the necessary definitions and background information that will be used throughout the paper. We begin by introducing the concept of convexity and its various definitions, which are central to the study of refined modified $(h, m)-$convex functions and their derivatives.

Definition 4. [22] *A function $f : J \to \mathbb{R}$, where J is an interval in $\mathbb{R}$, is said to be convex if, for any $x, y \in J$ and any $\alpha \in [0, 1]$, it satisfies the inequality*

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y).$$

In other words, f is convex if its graph lies below the line segment connecting any two of its points.

Definition 5. [23] *A function $f : J \to \mathbb{R}$ is called $h-$convex if, for any $x, y \in J$ and any $\alpha \in (0, 1)$, it satisfies the inequality*

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y),$$

where $h : (0, 1) \to \mathbb{R}$ is a non-negative function. In other words, f is $h-$convex if its graph lies below a weighted average of the line segments connecting any two of its points, with weights determined by the function h.

[24] introduced the concept of modified $h-$convex function, which we define as follows:

Definition 6. [24] *Let $f, h : J \subseteq \mathbb{R} \to \mathbb{R}$ be non-negative functions. The function f is called modified $h-$convex if, for any $x, y \in J$ and any $\alpha \in [0, 1]$, it satisfies the inequality*

$$f(\alpha x + (1 - \alpha)y) \leq h(\alpha)f(x) + (1 - h(\alpha))f(y).$$

In other words, f is modified $h-$convex if its graph lies below a weighted average of the line segments connecting any two of its points, with weights determined by the function h.

Definition 7 (Modified (h, m)-Convex Function)**.** *Let $f, h : J \subseteq \mathbb{R} \to \mathbb{R}$ be non-negative functions, and let $m, \alpha \in [0, 1]$ and $x, y \in J$. We say that f is a modified (h, m)-convex function if it satisfies the inequality*

$$f(\alpha x + m(1 - \alpha)y) \leq h(\alpha)f(x) + m(1 - h(\alpha))f(y).$$

Definition 8 (Refined Modified (h, m)-Convex Function)**.** *Let $f, h : J \subseteq \mathbb{R} \to \mathbb{R}$ be non-negative functions, and let $m, \alpha \in [0, 1]$ and $x, y \in J$. We say that f is a refined modified (h, m)-convex function if it satisfies the inequality*

$$f(\alpha x + m(1 - \alpha)y) \leq h(\alpha)(1 - h(\alpha))(f(x) + mf(y)),$$

where h is a non-negative function.

In both cases, the functions h and m play a role in controlling the behavior of the function f. The modified (h, m)-convexity condition allows for a more general form of convexity, where the weight given to each point x and y in the convex combination $\alpha x + m(1 - \alpha)y$ can depend on α. The refined modified (h, m)-convexity condition further refines this by introducing an additional factor $(1 - h(\alpha))$, which can also depend on α. These concepts are useful in the study of inequalities involving convex functions, such as the Hadamard inequality mentioned earlier.

3. Two Versions of Hadamard Inequality for Refined Modified $(h, m)-$Convex Function

In this section, we present two versions of Hadamard inequality for refined modified $(h, m)-$convex function via the Caputo $k-$fractional derivatives. The first version is based on a lower bound for the Caputo $k-$fractional derivative of refined modified $(h, m)-$convex function, while the second version involves an upper bound for the same derivative. These inequalities provide powerful tools for establishing bounds on the difference quotients of refined modified $(h, m)-$convex functions. We start by presenting the first version of the Hadamard inequality.

Theorem 2. *Let $f : [0, \infty) \to \mathbb{R}$ be a function such that $f \in AC^n[a, b], 0 \leq a < b$. Also let $f^{(n)}$ be a refined modified $(h, m)-$convex function on $[a, mb]$ with $m \in (0, 1]$. Then the following inequality for Caputo $k-$fractional derivatives hold:*

$$f^{(n)}\left(\frac{bm + a}{2}\right) \leq \frac{\left(h\left(\frac{1}{2}\right)\right)\left(1 - h\left(\frac{1}{2}\right)\right)k\Gamma_k(n - \frac{\alpha}{k} + k)}{(mb - a)^{n-\frac{\alpha}{k}}}\left[m^{n-\frac{\alpha}{k}+1}(-1)^n({}^C D^{\alpha,k}_{b^-} f(\frac{a}{m})) + ({}^C D^{\alpha,k}_{a^+} f(mb)\right]$$
$$\leq \left(n - \frac{\alpha}{k}\right)\left(h\left(\frac{1}{2}\right)\right)\left(1 - h\left(\frac{1}{2}\right)\right)\left[\left\{f^{(n)}(a) + 2mf^{(n)}(b) + m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right\}\int_0^1 h(t)(1 - h(t))t^{n-\frac{\alpha}{k}-1}dt\right]. \quad (6)$$

Proof. Since $f^{(n)}$ is refined modified $(h, m)-$ convex function on $[a, mb]$, we have

$$f^{(n)}\left(\frac{um+v}{2}\right) \leq h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left(f^{(n)}(v)+mf^{(n)}(u)\right), \quad u, v \in [a, b]. \tag{7}$$

By setting $u = (1-t)\frac{a}{m} + tb \leq b$ and $v = m(1-t)b + ta \geq a$ in the above inequality for $t \in [0, 1]$, then by integrating over $[0, 1]$ after multiplying with $t^{n-\frac{\alpha}{k}-1}$ we have

$$\begin{aligned} f^{(n)}\left(\frac{bm+a}{2}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}dt \leq & h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right) \\ & \times\left[\int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left(m(1-t)b+ta\right)dt + m\int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left((1-t)\frac{a}{m}+tb\right)dt\right]. \end{aligned}$$

Now, if we set $w = (1-t)\frac{a}{m} + tb$ and $z = m(1-t)b + ta$ in the right hand side of above inequality, we get

$$\begin{aligned} f^{(n)}\left(\frac{bm+a}{2}\right)\frac{1}{n-\frac{\alpha}{k}} \leq & h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right) \\ & \times\left[\int_a^{mb}\left(\frac{mb-z}{mb-a}\right)^{n-\frac{\alpha}{k}-1}\frac{f^{(n)}(z)dz}{(mb-a)} + m\int_{\frac{a}{m}}^{b}\left(\frac{w-\frac{a}{m}}{b-\frac{b}{m}}\right)^{n-\frac{\alpha}{k}-1}\frac{mf^{(n)}(w)dw}{\left(b-\frac{a}{m}\right)}\right]. \end{aligned}$$

From which we can write the following inequality:

$$f^{(n)}\left(\frac{bm+a}{2}\right) \leq h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[{}^{C}D_{a^+}^{\alpha,k}f(mb) + m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^{C}D_{b^-}^{\alpha,k}f\left(\frac{a}{m}\right)\right)\right]. \tag{8}$$

On the other hand, by using refined modified $(h, m)-$ convexity of $f^{(n)}$, we have

$$mf^{(n)}\left((1-t)\frac{a}{m}+tb\right) \leq h(t)(1-h(t))\left(m^2 f^{(n)}\left(\frac{a}{m^2}\right) + mf^{(n)}(b)\right).$$

Multiplying both side by $(n-\frac{\alpha}{k})(h(\frac{1}{2}))(1-h(\frac{1}{2}))t^{n-\frac{\alpha}{k}-1}$ and integrating over $[0, 1]$, after some calculation we get

$$\begin{aligned} & \frac{m^{n-\frac{\alpha}{k}+1}\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)}{(mb-a)^{n-\frac{\alpha}{k}}}k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)(-1)^n\left({}^{C}D_{b^-}^{\alpha,k}f\left(\frac{a}{m}\right)\right) \\ & \leq m\left(n-\frac{\alpha}{k}\right)\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)\left[\left(mf^{(n)}\left(\frac{a}{m^2}\right)+f^{(n)}(b)\right)\int_0^1 h(t)(1-h(t))t^{n-\frac{\alpha}{k}-1}dt\right]. \end{aligned} \tag{9}$$

By using refined modified $(h, m)-$ convexity of $f^{(n)}$, we have

$$f^{(n)}\left(m(1-t)b+ta\right) \leq h(t)(1-h(t))\left(mf^{(n)}(b)+f^{(n)}(a)\right).$$

Multiplying both side by $\left(n-\frac{\alpha}{k}\right)\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)t^{n-\frac{\alpha}{k}-1}$ and integrating over $[0, 1]$, after some calculation we get

$$\begin{aligned} & \frac{\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)}{(mb-a)^{n-\frac{\alpha}{k}}}k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)\left({}^{C}D_{a^+}^{\alpha,k}f(mb)\right) \\ & \leq \left(n-\frac{\alpha}{k}\right)\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)\left[\left(mf^{(n)}(b)+f^{(n)}(a)\right)\int_0^1 h(t)(1-h(t))t^{n-\frac{\alpha}{k}-1}dt\right]. \end{aligned} \tag{10}$$

Addition of (9) and (10) yields.

$$\frac{\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^{C}D_{b^-}^{\alpha,k}f\left(\frac{a}{m}\right)\right)+\left({}^{C}D_{a^+}^{\alpha,k}f(mb)\right)\right]$$

$$\leq \left(n-\frac{\alpha}{k}\right)\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)\left[\left\{f^{(n)}(a)+2mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right\}\int_0^1 h(t)(1-h(t))t^{n-\frac{\alpha}{k}-1}dt\right]. \quad (11)$$

By combining the inequalities (8) and (11), we get the required result.

$$f^{(n)}\left(\frac{bm+a}{2}\right)\leq \frac{\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^C D^{\alpha,k}_{b^-}f\left(\frac{a}{m}\right)\right)+\left({}^C D^{\alpha,k}_{a^+}f(mb)\right)\right]$$
$$\leq \left(n-\frac{\alpha}{k}\right)\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)\left[\left\{f^{(n)}(a)+2mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right\}\int_0^1 h(t)(1-h(t))t^{n-\frac{\alpha}{k}-1}dt\right].$$

□

Corollary 1. *By setting $k=1$ in the inequality (6), the following Caputo fractional derivatives inequality holds:*

$$f^{(n)}\left(\frac{bm+a}{2}\right)\leq \frac{\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[m^{n-\alpha+1}(-1)^n\left({}^C D^{\alpha}_{b^-}f\left(\frac{a}{m}\right)\right)+\left({}^C D^{\alpha}_{a^+}f(mb)\right)\right]$$
$$\leq (n-\alpha)\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)\left[\left\{f^{(n)}(a)+2mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right\}\int_0^1 h(t)(1-h(t))t^{n-\alpha-1}dt\right]. \quad (12)$$

Corollary 2. *By setting $m=1$ in the inequality (6), the following Caputo $k-$fractional derivatives inequality holds for refined modified $h-$convex function:*

$$f^{(n)}\left(\frac{b+a}{2}\right)\leq \frac{\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{(b-a)^{n-\frac{\alpha}{k}}}\left[(-1)^n\left({}^C D^{\alpha,k}_{b^-}f(a)\right)+\left({}^C D^{\alpha,k}_{a^+}f(b)\right)\right]$$
$$\leq \left(n-\frac{\alpha}{k}\right)\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)2\left[\left(f^{(n)}(a)+f^{(n)}(b)\right)\int_0^1 h(t)(1-h(t))t^{n-\frac{\alpha}{k}-1}dt\right]. \quad (13)$$

Corollary 3. *By setting $k=1$ and $m=1$ in the inequality (6), the following Caputo fractional derivatives inequality holds:*

$$f^{(n)}\left(\frac{b+a}{2}\right)\leq \frac{\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[(-1)^n\left({}^C D^{\alpha}_{b^-}f(a)\right)+\left({}^C D^{\alpha}_{a^+}f(b)\right)\right]$$
$$\leq (n-\alpha)\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)2\left[\left(f^{(n)}(a)+f^{(n)}(b)\right)\int_0^1 h(t)(1-h(t))t^{n-\alpha-1}dt\right]. \quad (14)$$

Corollary 4. *If we choose " h" is identity function in (6) and using the definition of beta function, the following Caputo $k-$fractional derivatives inequality hold:*

$$f^{(n)}\left(\frac{bm+a}{2}\right)\leq \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^C D^{\alpha,k}_{b^-}f\left(\frac{a}{m}\right)\right)+\left({}^C D^{\alpha,k}_{a^+}f(mb)\right)\right]$$
$$\leq \left(n-\frac{\alpha}{k}\right)\left[\left\{f^{(n)}(a)+2mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right\}\beta\left(2,n-\frac{\alpha}{k}+1\right)\right]. \quad (15)$$

Corollary 5. *If " h" is identity function and set $m=1$ in (6) and using the definition of beta function the following Caputo $k-$fractional derivatives inequality hold for convex function:*

$$f^{(n)}\left(\frac{b+a}{2}\right)\leq \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(b-a)^{n-\frac{\alpha}{k}}}\left[(-1)^n\left({}^C D^{\alpha,k}_{b^-}f(a)\right)+\left({}^C D^{\alpha,k}_{a^+}f(b)\right)\right]$$
$$\leq 2\left(n-\frac{\alpha}{k}\right)\left[\left\{f^{(n)}(a)+f^{(n)}(b)\right\}\beta\left(2,n-\frac{\alpha}{k}+1\right)\right]. \quad (16)$$

Corollary 6. *If " h" is identity function and set $k = 1$ in (6) and using the definition of beta function the following Caputo fractional derivatives inequality hold for convex function:*

$$f^{(n)}\left(\frac{bm+a}{2}\right) \le \frac{\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^C D^{\alpha}_{b^-} f\left(\frac{a}{m}\right)\right) + \left({}^C D^{\alpha}_{a^+} f(mb)\right)\right]$$
$$\le (n-\alpha)\left[\left\{f^{(n)}(a) + 2mf^{(n)}(b) + m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right\}\beta(2, n-\alpha+1)\right]. \tag{17}$$

Corollary 7. *If " h" is identity function and set $k = 1, m = 1$ in (6) and using the definition of beta function the following Caputo fractional derivatives inequality hold for convex function:*

$$f^{(n)}\left(\frac{b+a}{2}\right) \le \frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[(-1)^n\left({}^C D^{\alpha}_{b^-} f(a)\right) + \left({}^C D^{\alpha,k}_{a^+} f(b)\right)\right]$$
$$\le 2(n-\alpha)\left[\left\{f^{(n)}(a) + f^{(n)}(b)\right\}\beta(2, n-\alpha+1)\right]. \tag{18}$$

Theorem 3. *Let $f : [0,\infty) \to \mathbb{R}$ be a function such that $f \in AC^n[a,b], 0 \le a < b$. Also let $f^{(n)}$ be a refined modified $(h,m)-$convex function on $[a, mb]$ with $m \in (0,1]$. Then the following inequality for Caputo $k-$fractional derivatives hold:*

$$f^{(n)}\left(\frac{bm+a}{2}\right) \le 2^{(n-\frac{\alpha}{k})}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}} h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left[\left({}^C D^{\alpha,k}_{(\frac{a+bm}{2})^+} f(mb)\right)\right.$$
$$\left. + m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^C D^{\alpha,k}_{(\frac{a+bm}{2m})^-} f\left(\frac{a}{m}\right)\right)\right]$$
$$\le h\left(\frac{1}{2}\right)h\left(\frac{t}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{t}{2}\right)\right)\left[f^{n}(a) + 2mf^{(n)}(b) + m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right]. \tag{19}$$

Proof. By putting $u = \frac{t}{2}b + \frac{(2-t)}{2}\frac{a}{m}$ and $v = \frac{t}{2}a + m\frac{(2-t)}{2}b$ in (7) where $t \in [0,1]$, and multiplying with $t^{n-\frac{\alpha}{k}-1}$, then integrating over $[0,1]$ one can have

$$f^{(n)}\left(\frac{bm+a}{2}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}dt \le h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)$$
$$\times\left[\left\{f^{(n)}\left(\frac{t}{2}a + m\frac{(2-t)}{2}b\right)dt + mf^{(n)}\left(\frac{t}{2}b + \frac{(2-t)}{2}\frac{a}{m}\right)dt\right\}\int_0^1 t^{n-\frac{\alpha}{k}-1}\right].$$

By change of variables, as we did to get (8), one can also get

$$f^{(n)}\left(\frac{bm+a}{2}\right) \le 2^{(n-\frac{\alpha}{k})}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}} h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)$$
$$\times\left[\left({}^C D^{\alpha,k}_{(\frac{a+bm}{2})^+} f(mb)\right) + m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^C D^{\alpha,k}_{(\frac{a+bm}{2m})^-} f\left(\frac{a}{m^2}\right)\right)\right]. \tag{20}$$

On the other hand, by using refined modified $(h,m)-$ convexity of $f^{(n)}$, we have

$$f^{(n)}\left(\frac{t}{2}a + m\left(\frac{2-t}{2}b\right)\right) \le h\left(\frac{t}{2}\right)\left(1-h\left(\frac{t}{2}\right)\right)\left[f^{(n)}(a) + mf^{(n)}(b)\right].$$

Multiplying both side by $\left(n-\frac{\alpha}{k}\right)\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)t^{n-\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, after some calculation we get

$$\frac{2^{(n-\frac{\alpha}{k})}}{(mb-a)^{n-\frac{\alpha}{k}}} h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)\left({}^C D^{\alpha,k}_{(\frac{a+bm}{2})^+} f(mb)\right)$$
$$\le h\left(\frac{1}{2}\right)h\left(\frac{t}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{t}{2}\right)\right)\left[f^{(n)}(a) + mf^{(n)}(b)\right]. \tag{21}$$

By using refined modified $(h, m)-$ convexity of $f^{(n)}$, we have

$$mf^{(n)}\left(\frac{t}{2}b+m\left(\frac{2-t}{2}\right)\frac{a}{m^2}\right)\leq h\left(\frac{t}{2}\right)\left(1-h\left(\frac{t}{2}\right)\right)\left[mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right].$$

Multiplying both side by $\left(n-\frac{\alpha}{k}\right)\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)t^{n-\frac{\alpha}{k}-1}$ and integrating over $[0, 1]$, after some calculation we get

$$\begin{aligned}&\frac{2^{(n-\frac{\alpha}{k})}m^{n-\frac{\alpha}{k}+1}\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)}{(mb-a)^{n-\frac{\alpha}{k}}}k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)(-1)^n\left({}^CD^{\alpha,k}_{(\frac{a+bm}{2m})^-}f\left(\frac{a}{m^2}\right)\right)\\&\leq h\left(\frac{1}{2}\right)h\left(\frac{t}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{t}{2}\right)\right)\left[mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right].\end{aligned} \tag{22}$$

Addition of (21) and (22) yields.

$$\begin{aligned}&2^{(n-\frac{\alpha}{k})}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left[\left({}^CD^{\alpha,k}_{(\frac{a+bm}{2})^+}f(mb)\right)+m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^CD^{\alpha,k}_{(\frac{a+bm}{2m})^-}f\left(\frac{a}{m}\right)\right)\right]\\&\leq h\left(\frac{1}{2}\right)h\left(\frac{t}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{t}{2}\right)\right)\left[f^n(a)+2mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right].\end{aligned} \tag{23}$$

By combining the inequalities (20) and (23), we get the required result.

$$\begin{aligned}f^{(n)}\left(\frac{bm+a}{2}\right)\leq&2^{(n-\frac{\alpha}{k})}\frac{k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{(mb-a)^{n-\frac{\alpha}{k}}}h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left[\left({}^CD^{\alpha,k}_{(\frac{a+bm}{2})^+}f(mb)\right)\right.\\&\left.+m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^CD^{\alpha,k}_{(\frac{a+bm}{2m})^-}f\left(\frac{a}{m}\right)\right)\right]\\\leq&h\left(\frac{1}{2}\right)h\left(\frac{t}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{t}{2}\right)\right)\left[f^n(a)+2mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right].\end{aligned} \tag{24}$$

□

Corollary 8. *By setting $k = 1$ in inequality (24), the following inequality holds for Caputo fractional derivatives:*

$$\begin{aligned}f^{(n)}\left(\frac{bm+a}{2}\right)\leq&2^{(n-\alpha)}\frac{\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\\&\times\left[\left({}^CD^{\alpha}_{(\frac{a+bm}{2})^+}f(mb)\right)+m^{n-\alpha+1}(-1)^n\left({}^CD^{\alpha}_{(\frac{a+bm}{2m})^-}f\left(\frac{a}{m}\right)\right)\right]\\\leq&h\left(\frac{1}{2}\right)h\left(\frac{t}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{t}{2}\right)\right)\left[f^n(a)+2mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right].\end{aligned} \tag{25}$$

Corollary 9. *By setting $m = 1$ in inequality (24), the following Caputo $k-$fractional derivatives holds:*

$$\begin{aligned}f^{(n)}\left(\frac{b+a}{2}\right)\leq&2^{(n-\frac{\alpha}{k})}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(b-a)^{n-\frac{\alpha}{k}}}h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\\&\times\left[\left({}^CD^{\alpha,k}_{(\frac{a+b}{2})^+}f(b)\right)+m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^CD^{\alpha,k}_{(\frac{a+b}{2})^-}f(a)\right)\right]\\\leq&h\left(\frac{1}{2}\right)h\left(\frac{t}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{t}{2}\right)\right)2\left[f^n(a)+2f^{(n)}(b)\right].\end{aligned} \tag{26}$$

Corollary 10. *By setting $m = 1$, $k = 1$ in inequality (24), the following inequality holds for convex function via Caputo fractional derivatives:*

$$\begin{aligned}f^{(n)}\left(\frac{b+a}{2}\right)\leq&2^{(n-\alpha)}\frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left[\left({}^CD^{\alpha}_{(\frac{a+b}{2})^+}f(b)\right)+(-1)^n\left({}^CD^{\alpha}_{(\frac{a+b}{2})^-}f(a)\right)\right]\\\leq&h\left(\frac{1}{2}\right)h\left(\frac{t}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{t}{2}\right)\right)2\left[f^n(a)+f^{(n)}(b)\right].\end{aligned} \tag{27}$$

Corollary 11. *If we choose " h" is identity function in (24), the following Caputo k−fractional derivatives inequality holds:*

$$f^{(n)}\left(\frac{bm+a}{2}\right) \leq 2^{\left(n-\frac{\alpha}{k}-2\right)}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[\left({}^{C}D^{\alpha,k}_{\left(\frac{a+bm}{2}\right)^{+}}f(mb)\right)+m^{n-\frac{\alpha}{k}+1}(-1)^{n}\left({}^{C}D^{\alpha,k}_{\left(\frac{a+bm}{2m}\right)^{-}}f\left(\frac{a}{m}\right)\right)\right]$$
$$\leq 2^{-4}t(2-t)\left[f^{n}(a)+2mf^{(n)}(b)+m^{2}f^{(n)}\left(\frac{a}{m^{2}}\right)\right]. \tag{28}$$

Corollary 12. *If we choose " h" is identity function $k = 1$ in (24), the following Caputo fractional derivatives inequality holds:*

$$f^{(n)}\left(\frac{bm+a}{2}\right) \leq 2^{(n-\alpha-2)}\frac{\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[\left({}^{C}D^{\alpha}_{\left(\frac{a+bm}{2}\right)^{+}}f(mb)\right)+m^{n-\alpha+1}(-1)^{n}\left({}^{C}D^{\alpha}_{\left(\frac{a+bm}{2m}\right)^{-}}f\left(\frac{a}{m}\right)\right)\right]$$
$$\leq 2^{-4}t(2-t)\left[f^{n}(a)+2mf^{(n)}(b)+m^{2}f^{(n)}\left(\frac{a}{m^{2}}\right)\right]. \tag{29}$$

Corollary 13. *If we choose " h" is identity function and $m = 1$ in (24), the following Caputo k−fractional derivatives inequality holds:*

$$f^{(n)}\left(\frac{b+a}{2}\right) \leq 2^{(n-\frac{\alpha}{k}-2)}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(b-a)^{n-\frac{\alpha}{k}}}\left[\left({}^{C}D^{\alpha,k}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right)+(-1)^{n}\left({}^{C}D^{\alpha,k}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right)\right]$$
$$\leq 2^{-3}t(2-t)\left[f^{n}(a)+f^{(n)}(b)\right]. \tag{30}$$

Corollary 14. *If we choose " h" is identity function and $m = 1$, $k = 1$ in (24), the following Caputo fractional derivatives inequality holds:*

$$f^{(n)}\left(\frac{b+a}{2}\right) \leq 2^{(n-\alpha-2)}\frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[\left({}^{C}D^{\alpha,k}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right)+(-1)^{n}\left({}^{C}D^{\alpha,k}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right)\right]$$
$$\leq 2^{-3}t(2-t)\left[f^{n}(a)+f^{(n)}(b)\right]. \tag{31}$$

Theorem 4. *Let $f : [0,\infty) \to \mathbb{R}$ be a function such that $f \in AC^{n}[a,b], 0 \leq a < b$. Also let $f^{(n)}$ be a refined modified $(h,m)-$convex function on $[a,mb]$ with $m \in (0,1]$. Then the following inequalities for Caputo k−fractional derivatives hold:*

$$\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}\left\{({}^{C}D^{\alpha,k}_{a^{+}}f(b))+(-1)^{n}({}^{C}D^{\alpha,k}_{b^{-}}f(a))\right\}$$
$$\leq\left[f^{(n)}(a)+f^{(n)}(b)+m\left\{f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right\}\right]\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t)(1-h(t))dt$$
$$\leq\left[f^{(n)}(a)+f^{(n)}(b)+m\left\{f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right\}\right]\frac{\left(\int_0^1\{h(t)(1-h(t)\}^{q}dt\right)^{\frac{1}{q}}}{\left(np-\frac{\alpha p}{k}-p+1\right)^{\frac{1}{p}}}, \tag{32}$$

where $p^{-1}+q^{-1} = 1$ and $p > 1$.

Proof. Since $f^{(n)}$ is refined modified $(h,m)-$convex function on $[a,mb]$ then for $m \in (0,1]$ and $t \in [0,1]$, we have

$$f^{(n)}(ta+(1-t)b)+f^{(n)}((1-t)a+tb) \leq h(t)(1-h(t))\left[f^{(n)}(a)+f^{(n)}(b)+m\left\{f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right\}\right].$$

By multiplying both side of above inequality with $t^{n-\frac{\alpha}{k}-1}$ and integrating the above inequality with respect to t on $[0,1]$, we have

$$\int_0^1 t^{n-\frac{\alpha}{k}-1}\left\{f^{(n)}(ta+(1-t)b)+f^{(n)}((1-t)a+tb)\right\}$$

$$\leq \left[f^{(n)}(a) + f^{(n)}(b) + m\left\{ f^{(n)}\left(\frac{a}{m}\right) + f^{(n)}\left(\frac{b}{m}\right)\right\}\right] \int_0^1 t^{n-\frac{\alpha}{k}-1} h(t)(1-h(t))dt.$$

If we set $x = ta + (1-t)b$ in the left side of above inequality, we get the following inequality

$$\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}\left\{({}^C D^{\alpha,k}_{a^+} f(b)) + (-1)^n ({}^C D^{\alpha,k}_{b^-} f(a))\right\}$$
$$\leq \left[f^{(n)}(a) + f^{(n)}(b) + m\left\{ f^{(n)}\left(\frac{a}{m}\right) + f^{(n)}\left(\frac{b}{m}\right)\right\}\right] \int_0^1 t^{n-\frac{\alpha}{k}-1} h(t)(1-h(t))dt. \tag{33}$$

Thus, we get the first inequality of (32). The second inequality of (32) follows from the fact by using the Hölder inequality

$$\int_0^1 t^{n-\frac{\alpha}{k}-1} h(t)(1-h(t))dt \leq \frac{\left(\int_0^1 \{h(t)(1-h(t)\}^q dt\right)^{\frac{1}{q}}}{\left(np - \frac{\alpha p}{k} - p + 1\right)^{\frac{1}{p}}}, \tag{34}$$

and from (33) and (34) we get the required result.

$$\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}\left\{({}^C D^{\alpha,k}_{a^+} f(b)) + (-1)^n ({}^C D^{\alpha,k}_{b^-} f(a))\right\}$$
$$\leq \left[f^{(n)}(a) + f^{(n)}(b) + m\left\{ f^{(n)}\left(\frac{a}{m}\right) + f^{(n)}\left(\frac{b}{m}\right)\right\}\right] \int_0^1 t^{n-\frac{\alpha}{k}-1} h(t)(1-h(t))dt$$
$$\leq \left[f^{(n)}(a) + f^{(n)}(b) + m\left\{ f^{(n)}\left(\frac{a}{m}\right) + f^{(n)}\left(\frac{b}{m}\right)\right\}\right] \frac{\left(\int_0^1 \{h(t)(1-h(t)\}^q dt\right)^{\frac{1}{q}}}{\left(np - \frac{\alpha p}{k} - p + 1\right)^{\frac{1}{p}}}.$$

□

Corollary 15. *By setting $k = 1$ in inequality (32), the following inequalities holds via Caputo fractional derivatives:*

$$\frac{\Gamma(n-\alpha)}{(b-a)^{n-\alpha}}\left\{({}^C D^{\alpha}_{a^+} f(b)) + (-1)^n ({}^C D^{\alpha}_{b^-} f(a))\right\}$$
$$\leq \left[f^{(n)}(a) + f^{(n)}(b) + m\left\{ f^{(n)}\left(\frac{a}{m}\right) + f^{(n)}\left(\frac{b}{m}\right)\right\}\right] \int_0^1 t^{n-\alpha-1} h(t)(1-h(t))dt$$
$$\leq \left[f^{(n)}(a) + f^{(n)}(b) + m\left\{ f^{(n)}\left(\frac{a}{m}\right) + f^{(n)}\left(\frac{b}{m}\right)\right\}\right] \frac{\left(\int_0^1 \{h(t)(1-h(t)\}^q dt\right)^{\frac{1}{q}}}{\{p(n-\alpha-1)+1\}^{\frac{1}{p}}}. \tag{35}$$

Corollary 16. *By setting $m = 1$ in inequality (32), the following inequality holds via Caputo $k-$fractional derivatives:*

$$\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}\{({}^C D^{\alpha,k}_{a^+} f(b)) + (-1)^n ({}^C D^{\alpha,k}_{b^-} f(a))\}$$
$$\leq 2\left[f^{(n)}(a) + f^{(n)}(b)\right] \int_0^1 t^{n-\frac{\alpha}{k}-1} h(t)(1-h(t))dt$$
$$\leq 2\left[f^{(n)}(a) + f^{(n)}(b)\right] \frac{\left(\int_0^1 \{h(t)(1-h(t)\}^q dt\right)^{\frac{1}{q}}}{\left(np - \frac{\alpha p}{k} - p + 1\right)^{\frac{1}{p}}}.$$

Corollary 17. *By setting $m = 1$, $k = 1$ in inequality (32), the following inequality holds via Caputo fractional derivatives:*

$$\frac{\Gamma(n-\alpha)}{(b-a)^{n-\alpha}}\{({}^C D^{\alpha}_{a^+} f(b)) + (-1)^n ({}^C D^{\alpha}_{b^-} f(a))\}$$
$$\leq 2\left[f^{(n)}(a) + f^{(n)}(b)\right] \int_0^1 t^{n-\alpha-1} h(t)(1-h(t))dt$$
$$\leq 2\left[f^{(n)}(a) + f^{(n)}(b)\right] \frac{\left(\int_0^1 \{h(t)(1-h(t)\}^q dt\right)^{\frac{1}{q}}}{\{p(n-\alpha-1)+1\}^{\frac{1}{p}}}.$$

4. Establishing Error Estimates of Hadamard-type Inequalities by Utilizing two Integral Identities

In order to establish error estimates of Hadamard-type inequalities, we first introduce a supporting lemmas that utilizes two integral identities. These lemmas plays a crucial role in our analysis and provides a key step towards our main result. Overall, these supporting lemmas serves as an important foundation for our subsequent analysis of Hadamard-type inequalities.

Lemma 1. [25] *Let $f : [a, mb] \to \mathbb{R}$ be a differentiable mapping on interval (a, mb) with $a \leq mb$.If $f \in C^{n+1}[a.mb]$, then the following equality for Caputo k-fractional integrals holds*

$$\frac{f^{(n)}(mb) + f^{(n)}(a)}{2} - \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^C D^{\alpha,k}_{a^+} f(mb)) + (-1)^n ({}^C D^{\alpha,k}_{mb^-} f(a))\right)$$
$$= \frac{mb-a}{2}\int_0^1 ((1-t)^{n-\frac{\alpha}{k}} - t^{n-\frac{\alpha}{k}}) f^{(n+1)}(m(1-t)b + ta)dt.$$

Theorem 5. *Let $f : [0, \infty) \to \mathbb{R}$ be a function such that $f \in C^{n+1}[a, b]$. If $|f^{(n+1)}|$ is modified (h,m)-convex function on [a,mb] with $m \in (0, 1]$. Then the following inequality Caputo for k-fractional derivatives holds:*

$$\left|\frac{f^{(n)}(mb) + f^{(n)}(a)}{2} - \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^C D^{\alpha,k}_{a^+} f(mb)) + (-1)^n ({}^C D^{\alpha,k}_{mb^-} f(a))\right)\right|$$
$$\leq \frac{mb-a}{2}\left\{\frac{\left(\left|f^{(n+1)}(a) + mf^{(n+1)}(b)\right|\right)}{(np-\frac{\alpha p}{k}+1)^{\frac{1}{p}}}\left((1-2^{\frac{\alpha p}{k}-np-1})^{\frac{1}{p}} - (2^{\frac{\alpha p}{k}-np-1})^{\frac{1}{p}}\right)\right.$$
$$\left.\times\left[\left(\int_0^{\frac{1}{2}} \{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 \{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}\right]\right\}. \tag{36}$$

where $p^{-1} + q^{-1} = 1$.

Proof. From Lemma 1 and by using the property of modulus, we get

$$\frac{f^{(n)}(mb) + f^{(n)}(a)}{2} - \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^C D^{\alpha,k}_{a^+} f(mb)) + (-1)^n ({}^C D^{\alpha,k}_{mb^-} f(a))\right)$$
$$\leq \frac{mb-a}{2}\int_0^1 |(1-t)^{n-\frac{\alpha}{k}} - t^{n-\frac{\alpha}{k}}|\left|f^{(n+1)}(m(1-t)b + ta)\right| dt.$$

By refined modified (h,m)-convexity of $|f^{(n+1)}|$, we have

$$\left|\frac{f^{(n)}(mb) + f^{(n)}(a)}{2} - \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^C D^{\alpha,k}_{a^+} f(mb)) + (-1)^n ({}^C D^{\alpha,k}_{mb^-} f(a))\right)\right|$$

$$\leq \frac{mb-a}{2}\left\{\int_0^{\frac{1}{2}}((1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}})h(t)(1-h(t))\left(|f^{(n+1)}(a)|+m|f^{(n+1)}(b)|\right)dt\right.$$
$$\left.+\int_{\frac{1}{2}}^1((1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}})h(t)(1-h(t))\left(|f^{(n+1)}(a)|+m|f^{(n+1)}(b)|\right)dt\right\}$$
$$=\frac{mb-a}{2}\left\{\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)\left(\int_0^{\frac{1}{2}}(1-t)^{n-\frac{\alpha}{k}}h(t)(1-h(t))dt\right)\right.$$
$$-\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)\left(\int_0^{\frac{1}{2}}t^{n-\frac{\alpha}{k}}h(t)(1-h(t))dt\right)$$
$$+\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)\left(\int_{\frac{1}{2}}^1(1-t)^{n-\frac{\alpha}{k}}h(t)(1-h(t))dt\right)$$
$$\left.-\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)\left(\int_{\frac{1}{2}}^1t^{n-\frac{\alpha}{k}}h(t)(1-h(t))dt\right)\right\}.$$

Now, by using the Hölder's inequality in the right hand side of above inequality, we get

$$\left|\frac{f^{(n)}(mb)+f^{(n)}(a)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^CD^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^CD^{\alpha,k}_{mb^-}f(a))\right)\right|$$
$$\leq\frac{mb-a}{2}\left\{\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)\left(\frac{1-2^{\frac{\alpha p}{k}-np-1}}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}\right.$$
$$-\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)\left(\frac{2^{\frac{\alpha p}{k}-np-1}}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}$$
$$+\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)\left(\frac{1-2^{\frac{\alpha p}{k}-np-1}}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}$$
$$\left.-\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)\left(\frac{2^{\frac{\alpha p}{k}-np-1}}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}\right\}.$$

After some calculation we get the desired result.

$$\left|\frac{f^{(n)}(mb)+f^{(n)}(a)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^CD^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^CD^{\alpha,k}_{mb^-}f(a))\right)\right|$$
$$\leq\frac{mb-a}{2}\left\{\frac{\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)}{(np-\frac{\alpha p}{k}+1)^{\frac{1}{p}}}\left((1-2^{\frac{\alpha p}{k}-np-1})^{\frac{1}{p}}-(2^{\frac{\alpha p}{k}-np-1})^{\frac{1}{p}}\right)\right.$$
$$\left.\times\left[\left(\int_0^{\frac{1}{2}}\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}\right]\right\}.$$

□

Corollary 18. *By setting $k=1$ in inequality (36), the following Caputo fractional derivatives inequality holds:*

$$\left|\frac{f^{(n)}(mb)+f^{(n)}(a)}{2}-\frac{\Gamma(n-\alpha+1)}{2(mb-a)^{n-\alpha}}\left(({}^CD^{\alpha}_{a^+}f(mb))+(-1)^n({}^CD^{\alpha}_{mb^-}f(a))\right)\right|$$
$$\leq\frac{mb-a}{2}\left\{\frac{\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)}{(np-\alpha p+1)^{\frac{1}{p}}}\left((1-2^{\alpha p-np-1})^{\frac{1}{p}}-(2^{\alpha p-np-1})^{\frac{1}{p}}\right)\right.$$

$$\times\left[\left(\int_0^{\frac{1}{2}}\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}\right]\Bigg\}.$$

Corollary 19. *By setting $m = 1$ in inequality (36), the following Caputo k−fractional derivatives inequality holds:*

$$\left|\frac{f^{(n)}(b)+f^{(n)}(a)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(b-a)^{n-\frac{\alpha}{k}}}\left(({}^CD^{\alpha,k}_{a^+}f(b))+(-1)^n({}^CD^{\alpha,k}_{b^-}f(a))\right)\right|$$
$$\leq\frac{b-a}{2}\left\{\frac{\left(\left|f^{(n+1)}(a)+f^{(n+1)}(b)\right|\right)}{(np-\frac{\alpha p}{k}+1)^{\frac{1}{p}}}\left((1-2^{\frac{\alpha p}{k}-np-1})^{\frac{1}{p}}-(2^{\frac{\alpha p}{k}-np-1})^{\frac{1}{p}}\right)\right.$$
$$\times\left[\left(\int_0^{\frac{1}{2}}\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}\right]\Bigg\}.$$

Corollary 20. *By setting $k = 1$ and $m = 1$ in inequality (36), the following holds for convex function via Caputo fractional derivatives:*

$$\left|\frac{f^{(n)}(b)+f^{(n)}(a)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left(({}^CD^{\alpha}_{a^+}f(b))+(-1)^n({}^CD^{\alpha}_{b^-}f(a))\right)\right|$$
$$\leq\frac{b-a}{2}\left\{\frac{\left(\left|f^{(n+1)}(a)+f^{(n+1)}(b)\right|\right)}{(np-\alpha p+1)^{\frac{1}{p}}}\left((1-2^{\alpha p-np-1})^{\frac{1}{p}}-(2^{\alpha p-np-1})^{\frac{1}{p}}\right)\right.$$
$$\times\left[\left(\int_0^{\frac{1}{2}}\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}\right]\Bigg\}.$$

Lemma 2. [25] *Let $f : [a, mb] \to \mathbb{R}$ be a differentiable mapping on interval (a, mb) with $a \leq mb$. If $f \in C^{n+2}[a.mb]$, then the following equality for Caputo k-fractional integrals holds:*

$$\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^CD^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^CD^{\alpha,k}_{mb^-}f(a))\right)$$
$$=\frac{(mb-a)^2}{2}\int_0^1\frac{1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}f^{(n+2)}(ta+m(1-t)b)dt.$$

Theorem 6. *Let $f : [0, \infty) \to \mathbb{R}$ be a function such that $f \in C^{n+2}[a, b]$. If $|f^{(n+2)}|$ is modified (h,m)-convex function on [a,mb] with $m \in (0, 1]$. Then the following inequality Caputo for k-fractional derivatives holds:*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^CD^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^CD^{\alpha,k}_{mb^-}f(a))\right)\right|$$
$$\leq\frac{(mb-a)^2}{2(n-\frac{\alpha}{k}+1)}\left(\left|f^{(n+2)}(a)\right|+m\left|f^{(n+2)}(b)\right|\right)\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1\{h(t)(1-h(t))\}^q\,dt\right)^{\frac{1}{q}}.(37)$$

Proof. Using Lemma 2 and refined modified (h,m)-convexity of $|f^{(n+2)}|$, we find □

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^CD^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^CD^{\alpha,k}_{mb^-}f(a))\right)\right|$$
$$\leq\frac{(mb-a)^2}{2}\int_0^1\frac{1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\left|f^{(n+2)}(ta+m(1-t)b)\right|dt$$

$$\leq \frac{(mb-a)^2}{2}\int_0^1 \frac{1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\left\{h(t)(1-h(t))\left(\left|f^{(n+2)}(a)\right|+m\left|f^{(n+2)}(b)\right|\right)\right\}dt$$

$$= \frac{(mb-a)^2}{2(n-\frac{\alpha}{k}+1)}\left\{\left(\left|f^{(n+2)}(a)\right|+m\left|f^{(n+2)}(b)\right|\right)\int_0^1 (1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1})h(t)(1-h(t))dt\right\}.$$

Now, by using Hölder inequality, we have

$$\int_0^1 (1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1})h(t)dt \leq \left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1 \{h(t)(1-h(t))\}^q\,dt\right)^{\frac{1}{q}}.$$

This implies

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^C D^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^C D^{\alpha,k}_{mb^-}f(a))\right)\right|$$

$$\leq \frac{(mb-a)^2}{2(n-\frac{\alpha}{k}+1)}\left(\left|f^{(n+2)}(a)\right|+m\left|f^{(n+2)}(b)\right|\right)\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1 \{h(t)(1-h(t))\}^q\,dt\right)^{\frac{1}{q}}.$$

Corollary 21. *By setting $k = 1$ in inequality (37), the following Caputo fractional derivatives inequality holds:*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{\Gamma(n-\alpha+1)}{2(mb-a)^{n-\alpha}}\left(({}^C D^{\alpha}_{a^+}f(mb))+(-1)^n({}^C D^{\alpha}_{mb^-}f(a))\right)\right|$$

$$\leq \frac{(mb-a)^2}{2(n-\alpha+1)}\left(\left|f^{(n+2)}(a)\right|+m\left|f^{(n+2)}(b)\right|\right)\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1 \{h(t)(1-h(t))\}^q\,dt\right)^{\frac{1}{q}}.$$

Corollary 22. *By setting $m = 1$ in inequality (37), the following Caputo $k-$fractional derivatives inequality holds:*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(b-a)^{n-\frac{\alpha}{k}}}\left(({}^C D^{\alpha,k}_{a^+}f(b))+(-1)^n({}^C D^{\alpha,k}_{b^-}f(a))\right)\right|$$

$$\leq \frac{(b-a)^2}{2(n-\frac{\alpha}{k}+1)}\left(\left|f^{(n+2)}(a)\right|+\left|f^{(n+2)}(b)\right|\right)\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1 \{h(t)(1-h(t))\}^q\,dt\right)^{\frac{1}{q}}.$$

Corollary 23. *By setting $k = 1$ and $m = 1$ in inequality (37), the following inequality holds for convex function via Caputo fractional derivatives:*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left(({}^C D^{\alpha}_{a^+}f(b))+(-1)^n({}^C D^{\alpha}_{b^-}f(a))\right)\right|$$

$$\leq \frac{(b-a)^2}{2(n-\alpha+1)}\left(\left|f^{(n+2)}(a)\right|+\left|f^{(n+2)}(b)\right|\right)\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1 \{h(t)(1-h(t))\}^q\,dt\right)^{\frac{1}{q}}.$$

5. Conclusions

In conclusion, this paper has presented new refinements and special cases of the Hadamard inequality for modified $(h, m)-$convex functions using the Caputo $k-$fractional derivatives. In particular, we have shown that the Hadamard inequality holds for these classes of functions with greater generality than previously established in the literature. Furthermore, we have provided a detailed analysis of the error estimates of Hadamard-type inequalities using two integral identities, which allows us to obtain precise upper bounds for the errors. We have also shown that these results hold for arbitrary order derivatives of function f, which greatly expands the scope of our analysis. The significance of these results lies in their potential applications in various fields such as finance, engineering, and physics. Specifically, our refinements of the Hadamard inequality and error estimates can be used to derive more accurate estimates and bounds in optimization problems, as well as in the modeling and analysis of various physical and engineering systems.

Conflict of Interest

The authors declare no conflict of interest.

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