Utilitas Mathematica, 118: 27–32
DOI:10.61091/um118-03
http://www.combinatorialpress.com/um
Received 22 October 2023, Accepted 25 December 2023, Published 8 January 2024

Article

EKFN-Modules

Mankagna Albert DIOMPY[1,*], **Ousseynou BOUSSO**[1], **Remy Diaga DIOUF**[1] **and Oumar DIANKHA**[1]

[1] Département de Mathématiques et Informatique, Faculté des Sciences et Techniques, Université Cheikh Anta Diop, 5005 Dakar (Senegal).

* **Correspondence:** albertdiompy@yahoo.fr

Abstract: In this paper, we utilize the σ category to introduce $EKFN$-modules, which extend the concept of the $EKFN$-ring. After presenting some properties, we demonstrate, under certain hypotheses, that if M is an $EKFN$-module, then the following equivalences hold: the class of uniserial modules coincides with the class of cu-uniserial modules; $EKFN$-modules correspond to the class of locally noetherian modules; and the class of CD-modules is a subset of the $EKFN$-modules.

Keywords: Noetherian, endo-noetherian, $EKFN$-module, Uniserial, cu-uniserial, Virtually uniserial, Locally noetherian and CD-module
Mathematics Subject Classification: 13C60, 16P20, 16P40.

1. Introduction

Throughout this paper, R denotes a commutative, associative ring with an identity $1 \neq 0$, satisfying the ascending chain condition (ACC) on annihilators. Furthermore, all modules considered are unitary left R-modules. The category of all left R-modules is denoted by R-MOD, with σ representing the full subcategory of R-MOD, whose objects are isomorphic to a submodule of an M-generated module.

A module M is termed noetherian (or artinian) if every ascending (or descending) chain of its submodules becomes stationary. A module M is classified as endo-noetherian (or endo-artinian) if, for any family $(f_i)_{i\geq 1}$ of endomorphisms of M, the sequence $\{Ker(f_1) \subseteq Ker(f_2) \subseteq \cdots\}$ (or $\{\Im(f_1) \supseteq \Im(f_2) \supseteq \cdots\}$, respectively) stabilizes. It is immediately evident that every noetherian ring is endo-noetherian, although the converse is not universally true. For instance, $\mathbb{Q}_{\mathbb{Z}}$ is an endo-noetherian module but not noetherian, as evidenced by the increasing sequence:

$$\mathbb{Z} \subset \frac{1}{2}\mathbb{Z} \subset \frac{1}{4}\mathbb{Z} \subset \cdots$$

In [1], C. T. Gueye et al. introduced the class of $EKFN$-rings. These rings satisfy the following condition: every endo-noetherian module is noetherian. The aim of this paper is to extend the notion of $EKFN$-rings to the σ category. This new class of modules within this category is termed $EKFN$-modules, signifying that every endo-noetherian object in σ is noetherian.

Recall that an R-module M is defined as uniserial if its submodules are linearly ordered by inclusion. Moreover, M is described as serial if it is a direct sum of uniserial modules. The R-module M is classified as cyclic-uniform uniserial (cu-uniserial) if, for every non-zero finitely generated submodule $K \subseteq M$, the quotient $K/\mathrm{Rad}(K)$ is both cyclic and uniform; here, $\mathrm{Rad}(K)$ denotes the intersection of all maximal submodules of M. A module M is termed cyclic-uniform serial (cu-serial) if it is a direct sum of cyclic-uniform uniserial modules. An R-module M is virtually uniserial if, for every non-zero finitely generated submodule $K \subseteq M$, the quotient $K/\text{Rad}(K)$ is virtually simple. Similarly, an R-module M is called virtually serial if it is a direct sum of virtually uniserial modules. M is considered locally noetherian if every finitely generated submodule of M is noetherian. A submodule N of M is defined as small in M if, for every proper submodule L of M, the sum $N + L \neq M$. A module M is described as hollow if every proper submodule of M is small in M.

A module P is projective if and only if for every surjective module homomorphism $f : N \to M \to 0$ and every homomorphism $g : P \to M$, there exists a module homomorphism $h : P \to N$ such that $f \circ h = g$. A submodule N of M is termed superfluous if the only submodule T of M satisfying $N + T = M$ is M itself. A morphism $f : A \to B$ of modules is considered minimal provided that $\ker(f)$ is a superfluous submodule of A. A module N is designated a small cover of a module M if there exists a minimal epimorphism $f : N \to M$. N is called a flat cover of M if N is a small cover of M and N is flat. A module A is a projective cover of B provided that A is projective and there exists a minimal epimorphism $A \to B$. It is noted that the projective cover of a module does not always exist but is unique when it does. A ring is termed perfect if every module over this ring possesses a projective cover. The singular submodule $Z(M)$ of a module M is the set of elements $m \in M$ such that $mI = 0$ for some essential right ideal I of R.

We define $\overline{Z}_M(N)$, as a dual of the singular submodule, by $\overline{Z}_M(N) = \bigcap\{Ker(f); f : N \to U, U \in \Gamma\}$, where Γ denotes the class of all M-small modules. A module M is termed *discrete* if it satisfies conditions (D_1) and (D_2):

(D_1): For every submodule N of M, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $M_2 \cap N$ is small in M_2.

(D_2): For any summand K of M, every exact sequence $M \to K \to 0$ splits.

The structure of our paper is outlined as follows: Initially, we present preliminary results on $EKFN$-rings and some fundamental properties of the σ category, particularly when M is finitely generated. Subsequently, we characterize the class of $EKFN$-modules in rings that satisfy the ACC on annihilators.

2. Preliminary results

Lemma 1. *(Theorem 3.10 of [1]) Let R be a commutative ring. These conditions are equivalent:*

1. *R is an EKFN-ring.*
2. *R is a artinian principal ideal ring.*

Lemma 2. *(From 15.4 of [2]) Let R be a ring and M a R-module. These conditions are verified:*

1. *If M is finitely generated as a module over $S = End(M)$, then $\sigma = R/Ann(M)$-Mod.*
2. *If R is commutative, then for every finitely generated R-module M, we have $\sigma = R/Ann(M)$-Mod*

Proof. 1. For a generated set $m_1, m_2, \cdots m_k$ of M_S, let's consider the map

$$\varphi : R \longrightarrow R(m_1, m_2, \cdots m_k) \subset M^k r \longmapsto r(m_1, m_2, \cdots m_k)$$

We have $ker(\varphi) = \bigcap_{i \leq k} Ann(m_i) = Ann(M)$, then $R/Ann(M) \simeq Im(\varphi) \subset M^k$.

2. The second point is a consequence of (1) since we have $R/Ann(M) \subset S = End(M)$ canonically.

□

Lemma 3. *(From 15.2 of [2]) For two R-modules M, N the following are equivalent:*

1. *N is a subgenerator in σ;*

2. *$\sigma = \sigma[N]$;*
3. *$N \in \sigma$ and $M \in \sigma[N]$.*

Recall a module M is a S-module if every hopfian object of σ is noetherian.

Proposition 1. *Let M be a R-module. If M is a S-module then M is an $EKFN$-module.*

Proof. Let N be an endo-noetherian module in σ. N is also hopfian because every endo-noetherian module is strongly hopfian and every strongly hopfian module is hopfian. As M is a S-module then N is noetherian. □

3. Aims results

Proposition 2. *Let M be an $EKFN$-module. Then the homomorphic image of every endo-noetherian module of σ is endo-noetherian.*

Proof. Let N be an endo-noetherian module in σ; as M is an $EKFN$-module, then N is noetherian. Assume that $f : N \longrightarrow f(N) = K$ is an homomorphism image of N. It's well-known that homomorphism image of noetherian module is noetherian. Thus K is noetherian and therefore K is endo-notherian. □

Remark 1. *In general, neither a submodule nor a quotient of endo-noetherian module is endo-noetherian. For example:*

- *Let R be the free ring over $\mathbf{Z}$ generated by $\{x_n, n \in N\}$. Then, R is left endo-noetherian but the left ideal I generated by $\{x_n, n \in N\}$ infinite direct sum of left ideals I_n generate by $\{x_n\}$, therefore the R-module I is not endo-noetherian.*
- *$\mathbb{Q}$ is an endo-noetherian $\mathbb{Z}$-module but $\mathbb{Q}/\mathbb{Z} = \bigoplus_{p\in\mathbb{P}} \mathbf{Z}(p^\infty)$ is not endo-noetherian.*

Proposition 3. *Let M be an $EKFN$-module. Then these conditions are verified.*

1. *Every submodule of an endo-noetherian module of σ is endo-noetherian.*
2. *Every quotient of an endo-noetherian module of σ is endo-noetherian.*

Proof. Let M be an $EKFN$-module.

1. Let N be an endo-noetherian module in σ and P a submodule of N. As M is an $EKFN$-module, then N is noetherian and so P because every submodule of a noetherian module is noetherian. Therefore P is endo-noetherian.
2. If N is an endo-noetherian module of σ then N is noetherian and it is known that every quotient of a noetherian module is neotherian. Therefore every quotient of N is endo-noetherian.

□

Proposition 4. *Let R be a commutative ring. We suppose M is finitely generated over $S = End(M)$. If M is an $EKFN$-module then these conditions are verified:*

1. *$S = End(M)$ has stable range 1 and $codim(End(M)) \leq dim(M) + codim(M)$.*
2. *There are, up to isomorphim, only many finitely indecomposable projective $End(M)$-modules.*

Proof.

1. M finitely generated $EKFN$-module over $S = End(M)$ implies M is artinian, hence M has a finite Goldie dimension. In addition M is cohopfian means every injective endomorphism of M is bijective. Referring to Theorem 4.3 of [3], then the endomorphism ring $S = End(M)$ is semilocal. Therefore $S = End(M)$ has stable range 1.
2. As $End(M)$ is semilocal then by theorem 4.10 of [3] we have the result.

□

Corollary 1. *Let M be a $R-EKFN$-module. If B and C two arbitrary R-modules and $A \oplus B \simeq A \oplus C$, then $B \simeq C$.*

Proof. We prove this corollary by referring to Proposition 4 and Theorem 4.5 (Evans) of [3]. □

Proposition 5. *Let R be a commutative ring and M a finitely generated R-module. If M is a $EKFN$-module, then every object of σ has a projective cover.*

Proof. M finitely generated over commutative ring, by referring to Lemma 2,
$\sigma = R/Ann(M)$-MOD meaning that every object of σ is a $R/Ann(M)$-module. In addition, by Lemma 1, $R/Ann(M)$ is artinian principal ideal ring. So M is artinian. It is well known that every artinian module is perfect. By definition of perfect ring, every object of σ have projective covers. □

Proposition 6. *Let R be a ring and M a finitely generated module over $S = End(M)$. Assume every simple S-module has a flat cover. If M is an $EKFN$-module then M is a finite direct sum of S-modules with local endomorphism ring.*

Proof. M finitely generated over $S = End(M)$ implies that $M \simeq R/Ann(M)$ and M $EKFN$-module implies that M is artinian. Hence $S = End(M)$ is semilocal. In addition, since every simple S-module has a flat cover, by referring on Theorem 3.8 of [4], $S = End(M)$ is semiperfect. By Proposition 3.14 of [3] M is a direct sum of R-modules with local endomorphism ring. □

Theorem 1. *Let R be a commuataive ring. If M is a finitely generated $EKFN$-module then the following are equivalent:*

1. *Every object of σ is cu-uniserial.*
2. *Every object of σ is uiserial.*

Proof. (2) $\Rightarrow$ (1) obvious
(1) $\Rightarrow$ (2) M a finitely generated $EKFN$-module implies $M \simeq R/Ann(M)$ is a artinian principal ideal ring. Hence $R/Ann(M)$ is a finite product of commutative artinian local rings. Suppose $R/Ann(M) = R_1/Ann(M) \times R_2/Ann(M)$ with R_1 and R_2 are local rings.
Let K be a *cu*-uniserial object of σ and L a non-zero finitely generated submodule of K . Then K is uniform and Bezout by Theorem 2.3 of [5]. This means that L is cyclic and hence there exist left ideal $I_1 = K_1/Ann(M)$ with $Ann(M) \subset K_1$ and $I_2 = K_2/Ann(M)$ with $Ann(M) \subset K_2$ such that $L \simeq (R_1 \times R_2)/I_1 \times I_2 \simeq (R_1/I_1) \times (R_2/I_2)$. Since L is uniform, either $L \simeq (R_1/I_1)$ or $L \simeq (R_2/I_2)$. Therefore L is a local submodule. Hence $L/Rad(L)$ is simple submodule of M thus K is uniserial. □

Theorem 2. *Let R be a ring and M a finitely generated R-module. We suppose $M/Rad(M)$ is a subgenerator in σ. Then the following statements are equivalent.*

1. *M is an $EKFN$-module.*
2. *M is a local cu-uniserial module.*
3. *M is a semilocal cu-uniserial module.*
4. *M is a virtually uniserial module.*

Proof. (1) $\Rightarrow$ (2) M finitely generated and $EKFN$-module implies that $M \simeq R/Ann(M)$ is an artinian principal ideal ring. By [6], modules over commutative artinian principal ideal ring and uniseriel modules coincide. And it is easy to see that uniserial modules are *cu*-uniserial. In addition, M artinian implies M is a finite product of artinian local submodules . Hence M is local.
(2) $\Rightarrow$ (3) Obvious
(1) $\Leftrightarrow$ (4) It follows from theorem 2.10 of [7]
(3) $\Rightarrow$ (1) Let N be an endo-noetherian object of σ. M semilocal, hence $M/Rad(M)$ is a semisimple artinian module. In addition $M/Rad(m)$ subgenerator in σ implies $\sigma = \sigma[M/Ann(M)]$ meaning that every object of $\sigma[M]$ is an object of $\sigma[M/Ann(M)]$. Hence $N \in \sigma[M/Ann(M)]$. As $M/Ann(M)$ is

semisimple then every object of $\sigma[M/Ann(M)]$ is semisimple therefore N is semisimple. It is well known that for a semisimple module, endo-noetherian and noetherian coincide. In conclusion M is an $EKFN$-module. □

Recall M is locally noetherian if every finitely generated submodule of M is noetherian.

Lemma 4. *(From 27.3 of [2])*
For an R-module M the following assertions are equivalent:

1. *M is locally noetherian;*
2. *Every finitely generated module in σ is noetherian;*

Theorem 3. *Let M be a finitely generated module. The following conditions are equivalent:*

1. *M is EKFN-module;*
2. *M is locally noetherian;*

Proof. 1) $\implies$ 2) Let M an $EKFN$-module and N a finitely generated module $\in \sigma$. M finitely generated implies $\sigma = R/Ann(M)$-MOD i.e. every object of σ is a $R/Ann(M)$ -module. In addition M $EKFN$-module implies by referring to Lemma 2 $M \simeq R/Ann(M)$ is an artinian and so noetherian. We know that any finitely generated module over noetherian ring is noetherian; therefore N is noetherian. By Lemma 4 we can deduce that M is locally noetherian.
2) $\implies$ 1) Let $N \in \sigma$ an endo-noetherian module. Since M is locally noetherian, by Corollary 2.3 of [8]; $R/Ann(M)$ is an noetherian ring and $\sigma = R/Ann(M)$-MOD. Hence N is a $R/Ann(M)$-module. $M \simeq R/Ann(M)$ is finitely generated and noetherian. Morever $N \in \sigma$ implies N is an ideal of $R/Ann(M)$; hence a submodule of M. It's well know over noetherian ring, every submodule of finitely generated module is finitely generated. So N is noetherian because over noetherian ring, finitely generated and noetherian coincide. Therefore M is an $EKFN$-module. □

Corollary 2. *For an R-module M the following assertions are equivalent:*

1. *M is EKFN-module;*
2. *every injective module in σ is a direct sum of indecomposable. modules;*

Proof. It results from Theorem 3 and 27.5 of [2]. □

Recall a R is a CD-ring if every consigular R-module is discrete, and M is a CD-module if every M-cosingular module in σ is discrete.

Lemma 5. *(Theorem 2.23 of [9]) The following are equivalent for a CD-module M.*

1. *M has finite hollow dimension;*
2. *M is semilocal and finitely generated*

Lemma 6. *(Proposition 2.26 of [9]) Let R be a commutative domain. Then the following are equivalent.*

1. *R is a CD-ring;*
2. *Every consigular R-module is projective;*
3. *R is a field.*

Theorem 4. *Let M be a R-module with finite hollow dimension then the class of EKFN-modules contains the class of CD-modules.*

Proof. Suppose that M is a CD-module. Let $N \in \sigma$ an endo-noetherian module. Since M is CD-module with finite hollow dimension referring to Lemma 5 M is finitely gererated and so $M \simeq R/Ann(M)$. As M is CD-module, $R/Ann(M)$ is a CD-ring and by Lemma 6 $R/Ann(M)$ is a field.

Hence $R/Ann(M)$ is noetherian; $M \simeq R/Ann(M)$ is finitely generated and noetherian. Morever $N \in \sigma$ implies N is an ideal of $R/Ann(M)$; hence a submodule of M. It's well know over noetherian ring, every submodule of finitely generated module is finitely generated. So N is noetherian because over noetherian ring, finitely generated and noetherian coincide. Therefore M is an $EKFN$-module. □

Conflict of Interest

The authors declare no conflict of interest.

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