Article

EKFN-Modules

Mankagna Albert DIOMPI1,*, Ousseynou BOUSSO1, Remy Diaga DIOUF1 and Oumar DIANKHA1

1 Département de Mathématiques et Informatique, Faculté des Sciences et Techniques, Université Cheikh Anta Diop, 5005 Dakar (Senegal).

* Correspondence: albertdiompy@yahoo.fr

Abstract: In this paper, we utilize the $\sigma$ category to introduce EKFN-modules, which extend the concept of the EKFN-ring. After presenting some properties, we demonstrate, under certain hypotheses, that if $M$ is an EKFN-module, then the following equivalences hold: the class of uniserial modules coincides with the class of $cu$-uniserial modules; EKFN-modules correspond to the class of locally noetherian modules; and the class of $CD$-modules is a subset of the EKFN-modules.

Keywords: Noetherian, endo-noetherian, EKFN-module, Uniserial, $cu$-uniserial, Virtually uniserial, Locally noetherian and $CD$-module

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1. Introduction

Throughout this paper, $R$ denotes a commutative, associative ring with an identity $1 \neq 0$, satisfying the ascending chain condition (ACC) on annihilators. Furthermore, all modules considered are unitary left $R$-modules. The category of all left $R$-modules is denoted by $R$-MOD, with $\sigma$ representing the full subcategory of $R$-MOD, whose objects are isomorphic to a submodule of an $M$-generated module.

A module $M$ is termed noetherian (or artinian) if every ascending (or descending) chain of its submodules becomes stationary. A module $M$ is classified as endo-noetherian (or endo-artinian) if, for any family $(f_i)_{i \geq 1}$ of endomorphisms of $M$, the sequence $\{\text{Ker}(f_1) \subseteq \text{Ker}(f_2) \subseteq \cdots\}$ (or $\{\mathcal{I}(f_1) \supseteq \mathcal{I}(f_2) \supseteq \cdots\}$, respectively) stabilizes. It is immediately evident that every noetherian ring is endo-noetherian, although the converse is not universally true. For instance, $\mathbb{Q}_\mathbb{Z}$ is an endo-noetherian module but not noetherian, as evidenced by the increasing sequence:

$$\mathbb{Z} \subset \frac{1}{2}\mathbb{Z} \subset \frac{1}{4}\mathbb{Z} \subset \cdots$$

In [1], C. T. Gueye et al. introduced the class of EKFN-rings. These rings satisfy the following condition: every endo-noetherian module is noetherian. The aim of this paper is to extend the notion of EKFN-rings to the $\sigma$ category. This new class of modules within this category is termed EKFN-modules, signifying that every endo-noetherian object in $\sigma$ is noetherian.

Recall that an $R$-module $M$ is defined as uniserial if its submodules are linearly ordered by inclusion. Moreover, $M$ is described as serial if it is a direct sum of uniserial modules. The $R$-module $M$ is classified as cyclic-uniform uniserial (cu-uniserial) if, for every non-zero finitely generated submodule $K \subseteq M$, the quotient $K/\text{Rad}(K)$ is both cyclic and uniform; here, $\text{Rad}(K)$ denotes the intersection
of all maximal submodules of $M$. A module $M$ is termed cyclic-uniform serial (cu-serial) if it is a direct sum of cyclic-uniform uniserial modules. An $R$-module $M$ is virtually uniserial if, for every non-zero finitely generated submodule $K \subseteq M$, the quotient $K/\text{Rad}(K)$ is virtually simple. Similarly, an $R$-module $M$ is called virtually serial if it is a direct sum of virtually uniserial modules. $M$ is considered locally noetherian if every finitely generated submodule of $M$ is noetherian. A submodule $N$ of $M$ is defined as small in $M$ if, for every proper submodule $L$ of $M$, the sum $N + L \neq M$. A module $M$ is described as hollow if every proper submodule of $M$ is small in $M$.

A module $P$ is projective if and only if for every surjective module homomorphism $f : N \to M \to 0$ and every homomorphism $g : P \to M$, there exists a module homomorphism $h : P \to N$ such that $f \circ h = g$. A submodule $N$ of $M$ is termed superfluous if the only submodule $T$ of $M$ satisfying $N + T = M$ is $M$ itself. A morphism $f : A \to B$ of modules is considered minimal provided that $\ker(f)$ is a superfluous submodule of $A$. A module $N$ is designated a small cover of a module $M$ if there exists a minimal epimorphism $f : N \to M$. $N$ is called a flat cover of $M$ if $N$ is a small cover of $M$ and $N$ is flat. A module $A$ is a projective cover of $B$ provided that $A$ is projective and there exists a minimal epimorphism $A \to B$. It is noted that the projective cover of a module does not always exist but is unique when it does. A ring is termed perfect if every module over this ring possesses a projective cover. The singular submodule $Z(M)$ of a module $M$ is the set of elements $m \in M$ such that $mI = 0$ for some essential right ideal $I$ of $R$.

We define $\bar{Z}_M(N)$, as a dual of the singular submodule, by $\bar{Z}_M(N) = \cap \{\ker(f); f : N \to U, U \in \Gamma\}$, where $\Gamma$ denotes the class of all $M$-small modules. A module $M$ is termed discrete if it satisfies conditions $(D_1)$ and $(D_2)$:

$(D_1)$: For every submodule $N$ of $M$, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $M_2 \cap N$ is small in $M_2$.

$(D_2)$: For any summand $K$ of $M$, every exact sequence $M \to K \to 0$ splits.

The structure of our paper is outlined as follows: Initially, we present preliminary results on $EKFN$-rings and some fundamental properties of the $\sigma$ category, particularly when $M$ is finitely generated. Subsequently, we characterize the class of $EKFN$-modules in rings that satisfy the ACC on annihilators.

2. Preliminary results

**Lemma 1.** (Theorem 3.10 of [1]) Let $R$ be a commutative ring. These conditions are equivalent:

1. $R$ is an $EKFN$-ring.
2. $R$ is a artinian principal ideal ring.

**Lemma 2.** (From 15.4 of [2]) Let $R$ be a ring and $M$ a $R$-module. These conditions are verified:

1. If $M$ is finitely generated as a module over $S = \text{End}(M)$, then $\sigma = R/\text{Ann}(M)$-$\text{Mod}$.
2. If $R$ is commutative, then for every finitely generated $R$-module $M$, we have $\sigma = R/\text{Ann}(M)$-$\text{Mod}$

**Proof.** 1. For a generated set $m_1, m_2, \ldots m_k$ of $M$, let’s consider the map

$$\varphi : R \to R(m_1, m_2, \ldots m_k) \subset M^k \mapsto r(m_1, m_2, \ldots m_k)$$

We have $\ker(\varphi) = \cap_{i \in \mathbb{K}} \text{Ann}(m_i) = \text{Ann}(M)$, then $R/\text{Ann}(M) \simeq \text{Im}(\varphi) \subset M^k$.

2. The second point is a consequence of (1) since we have $R/\text{Ann}(M) \subset S = \text{End}(M)$ canonically. $\square$

**Lemma 3.** (From 15.2 of [2]) For two $R$-modules $M$, $N$ the following are equivalent:

1. $N$ is a subgenerator in $\sigma$;
2. $\sigma = \sigma[N]$;
3. $N \in \sigma$ and $M \in \sigma[N]$.

Recall a module $M$ is a $S$-module if every hopfian object of $\sigma$ is noetherian.

**Proposition 1.** Let $M$ be a $R$-module. If $M$ is a $S$-module then $M$ is an EKFN-module.

**Proof.** Let $N$ be an endo-noetherian module in $\sigma$. $N$ is also hopfian because every endo-noetherian module is strongly hopfian and every strongly hopfian module is hopfian. As $M$ is a $S$-module then $N$ is noetherian. \hfill \Box

### 3. Aims results

**Proposition 2.** Let $M$ be an EKFN-module. Then the homomorphic image of every endo-noetherian module of $\sigma$ is endo-noetherian.

**Proof.** Let $N$ be an endo-noetherian module in $\sigma$; as $M$ is an EKFN-module, then $N$ is noetherian. Assume that $f : N \rightarrow f(N) = K$ is an homomorphism image of $N$. It’s well-known that homomorphism image of noetherian module is noetherian. Thus $K$ is noetherian and therefore $K$ is endo-noetherian. \hfill \Box

**Remark 1.** In general, neither a submodule nor a quotient of endo-noetherian module is endo-noetherian. For example:

- Let $R$ be the free ring over $\mathbb{Z}$ generated by $\{x_n, n \in N\}$. Then, $R$ is left endo-noetherian but the left ideal $I$ generated by $\{x_n, n \in N\}$ infinite direct sum of left ideals $I_n$ generate by $\{x_n\}$, therefore the $R$-module $I$ is not endo-noetherian.
- $\mathbb{Q}$ is an endo-noetherian $\mathbb{Z}$-module but $\mathbb{Q}/\mathbb{Z} = \bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p^\infty)$ is not endo-noetherian.

**Proposition 3.** Let $M$ be an EKFN-module. Then these conditions are verified.

1. Every submodule of an endo-noetherian module of $\sigma$ is endo-noetherian.
2. Every quotient of an endo-noetherian module of $\sigma$ is endo-noetherian.

**Proof.** Let $M$ be an EKFN-module.

1. Let $N$ be an endo-noetherian module in $\sigma$ and $P$ a submodule of $N$. As $M$ is an EKFN-module, then $N$ is noetherian and so $P$ because every submodule of a noetherian module is noetherian. Therefore $P$ is endo-noetherian.
2. If $N$ is an endo-noetherian module of $\sigma$ then $N$ is noetherian and it is known that every quotient of a noetherian module is noetherian. Therefore every quotient of $N$ is endo-noetherian. \hfill \Box

**Proposition 4.** Let $R$ be a commutative ring. We suppose $M$ is finitely generated over $S = \text{End}(M)$. If $M$ is an EKFN-module then these conditions are verified:

1. $S = \text{End}(M)$ has stable range 1 and $\text{codim}(\text{End}(M)) \leq \dim(M) + \text{codim}(M)$.
2. There are, up to isomorphism, only many finitely indecomposable projective $\text{End}(M)$-modules.

**Proof.**

1. $M$ finitely generated EKFN-module over $S = \text{End}(M)$ implies $M$ is artinian, hence $M$ has a finite Goldie dimension. In addition $M$ is cohopfian means every injective endomorphism of $M$ is bijective. Referring to Theorem 4.3 of [3], then the endomorphism ring $S = \text{End}(M)$ is semilocal. Therefore $S = \text{End}(M)$ has stable range 1.
2. As $\text{End}(M)$ is semilocal then by theorem 4.10 of [3] we have the result. \hfill \Box
Corollary 1. Let \( M \) be a \( R - \text{EKFN} \)-module. If \( B \) and \( C \) two arbitrary \( R \)-modules and \( A \oplus B \cong A \oplus C \), then \( B \cong C \).

Proof. We prove this corollary by referring to Proposition 4 and Theorem 4.5 (Evans) of [3]. □

Proposition 5. Let \( R \) be a commutative ring and \( M \) a finitely generated \( R \)-module. If \( M \) is a EKFN-module, then every object of \( \sigma \) has a projective cover.

Proof. \( M \) finitely generated over commutative ring, by referring to Lemma 2, \( \sigma = R/\text{Ann}(M)\)-MOD meaning that every object of \( \sigma \) is a \( R/\text{Ann}(M) \)-module. In addition, by Lemma 1, \( R/\text{Ann}(M) \) is artinian principal ideal ring. So \( M \) is artinian. It is well known that every artinian module is perfect. By definition of perfect ring, every object of \( \sigma \) have projective covers. □

Proposition 6. Let \( R \) be a ring and \( M \) a finitely generated module over \( S = \text{End}(M) \). Assume every simple \( S \)-module has a flat cover. If \( M \) is an EKFN-module then \( M \) is a finite direct sum of \( S \)-modules with local endomorphism ring.

Proof. \( M \) finitely generated over \( S = \text{End}(M) \) implies that \( M \cong R/\text{Ann}(M) \) and \( M \) EKFN-module implies that \( M \) is artinian. Hence \( S = \text{End}(M) \) is semilocal. In addition, since every simple \( S \)-module has a flat cover, by referring on Theorem 3.8 of [4], \( S = \text{End}(M) \) is semiperfect. By Proposition 3.14 of [3] \( M \) is a direct sum of \( R \)-modules with local endomorphism ring. □

Theorem 1. Let \( R \) be a commutative ring. If \( M \) is a finitely generated EKFN-module then the following are equivalent:

1. Every object of \( \sigma \) is cu-uniserial.
2. Every object of \( \sigma \) is uliserial.

Proof. (2) \( \Rightarrow \) (1) obvious

(1) \( \Rightarrow \) (2) \( M \) a finitely generated EKFN-module implies \( M \cong R/\text{Ann}(M) \) is an artinian principal ideal ring. Hence \( R/\text{Ann}(M) \) is a finite product of commutative artinian local rings. Suppose \( R/\text{Ann}(M) = R_1/\text{Ann}(M) \times R_2/\text{Ann}(M) \) with \( R_1 \) and \( R_2 \) are local rings.

Let \( K \) be a cu-uniserial object of \( \sigma \) and \( L \) a non-zero finitely generated submodule of \( K \). Then \( K \) is uniform and Bezout by Theorem 2.3 of [5]. This means that \( L \) is cyclic and hence there exist \( I_1 = K_1/\text{Ann}(M) \) with \( \text{Ann}(M) \subset K_1 \) and \( I_2 = K_2/\text{Ann}(M) \) with \( \text{Ann}(M) \subset K_2 \) such that \( L \cong (R_1/I_1) \times (R_2/I_2) \). Since \( L \) is uniform, either \( L \cong (R_1/I_1) \) or \( L \cong (R_2/I_2) \). Therefore \( L \) is a local submodule. Hence \( L/\text{Rad}(L) \) is simple submodule of \( M \) thus \( K \) is uniserial. □

Theorem 2. Let \( R \) be a ring and \( M \) a finitely generated \( R \)-module. We suppose \( M/\text{Rad}(M) \) is a subgenerator in \( \sigma \). Then the following statements are equivalent.

1. \( M \) is an EKFN-module.
2. \( M \) is a local cu-uniserial module.
3. \( M \) is a semilocal cu-uniserial module.
4. \( M \) is a virtually uniserial module.

Proof. (1) \( \Rightarrow \) (2) \( M \) finitely generated and EKFN-module implies \( M \cong R/\text{Ann}(M) \) is an artinian principal ideal ring. By [6], modules over commutative artinian principal ideal ring and uniserial modules coincide. And it is easy to see that uniserial modules are cu-uniserial. In addition, \( M \) artinian implies \( M \) is a finite product of artinian local submodules . Hence \( M \) is local.

(2) \( \Rightarrow \) (3) Obvious

(3) \( \Rightarrow \) (1) It follows from theorem 2.10 of [7]

(4) \( \Rightarrow \) (1) Let \( N \) be an endo-noetherian object of \( \sigma \). \( M \) semilocal, hence \( M/\text{Rad}(M) \) is a semisimple artinian module. In addition \( M/\text{Rad}(m) \) subgenerator in \( \sigma \) implies \( \sigma = \sigma[M/\text{Ann}(M)] \) meaning that every object of \( \sigma[M] \) is an object of \( \sigma[M/\text{Ann}(M)] \). Hence \( N \in \sigma[M/\text{Ann}(M)] \). As \( M/\text{Ann}(M) \) is
semisimple then every object of $\sigma[M/\text{Ann}(M)]$ is semisimple therefore $N$ is semisimple. It is well known that for a semisimple module, endo-noetherian and noetherian coincide. In conclusion $M$ is an EKFN-module.

Recall $M$ is locally noetherian if every finitely generated submodule of $M$ is noetherian.

Lemma 4. (From 27.3 of [2])

For an $R$-module $M$ the following assertions are equivalent:

1. $M$ is locally noetherian;
2. Every finitely generated module in $\sigma$ is noetherian;

Theorem 3. Let $M$ be a finitely generated module. The following conditions are equivalent:

1. $M$ is EKFN-module;
2. $M$ is locally noetherian;

Proof. 1) $\implies$ 2) Let $M$ be an EKFN-module and $N$ be a finitely generated module $\in \sigma$. $M$ finitely generated implies $\sigma = R/\text{Ann}(M)$-MOD i.e. every object of $\sigma$ is a $R/\text{Ann}(M)$-module. In addition $M$ EKFN-module implies by referring to Lemma 2 $M \simeq R/\text{Ann}(M)$ is an artinian and so noetherian. We know that any finitely generated module over noetherian ring is noetherian; therefore $N$ is noetherian. By Lemma 4 we can deduce that $M$ is locally noetherian.

2) $\implies$ 1) Let $N \in \sigma$ be an endo-noetherian module. Since $M$ is locally noetherian, by Corollary 2.3 of [8]; $R/\text{Ann}(M)$ is an noetherian ring and $\sigma = R/\text{Ann}(M)$-MOD. Hence $N$ is a $R/\text{Ann}(M)$-module. $M \simeq R/\text{Ann}(M)$ is finitely generated and noetherian. Moreover $N \in \sigma$ implies $N$ is an ideal of $R/\text{Ann}(M)$; hence a submodule of $M$. It’s well know over noetherian ring, every submodule of finitely generated module is finitely generated. So $N$ is noetherian because over noetherian ring, finitely generated and noetherian coincide. Therefore $M$ is an EKFN-module.

Corollary 2. For an $R$-module $M$ the following assertions are equivalent:

1. $M$ is EKFN-module;
2. every injective module in $\sigma$ is a direct sum of indecomposable. modules;

Proof. It results from Theorem 3 and 27.5 of [2].

Recall a $R$ is a CD-ring if every consigular $R$-module is discrete, and $M$ is a CD-module if every $M$-cosingular module in $\sigma$ is discrete.

Lemma 5. (Theorem 2.23 of [9]) The following are equivalent for a CD-module $M$.

1. $M$ has finite hollow dimension;
2. $M$ is semilocal and finitely generated

Lemma 6. (Proposition 2.26 of [9]) Let $R$ be a commutative domain. Then the following are equivalent.

1. $R$ is a CD-ring;
2. Every consigular $R$-module is projective;
3. $R$ is a field.

Theorem 4. Let $M$ be a $R$-module with finite hollow dimension then the class of EKFN-modules contains the class of CD-modules.

Proof. Suppose that $M$ is a CD-module. Let $N \in \sigma$ be an endo-noetherian module. Since $M$ is CD-module with finite hollow dimension referring to Lemma 5 $M$ is finitely generated and so $M \simeq R/\text{Ann}(M)$. As $M$ is CD-module, $R/\text{Ann}(M)$ is a CD-ring and by Lemma 6 $R/\text{Ann}(M)$ is a field.
Hence $R/\text{Ann}(M)$ is noetherian; $M \cong R/\text{Ann}(M)$ is finitely generated and noetherian. Moreover $N \in \sigma$ implies $N$ is an ideal of $R/\text{Ann}(M)$; hence a submodule of $M$. It’s well known over noetherian ring, every submodule of finitely generated module is finitely generated. So $N$ is noetherian because over noetherian ring, finitely generated and noetherian coincide. Therefore $M$ is an EKFN-module. □

Conflict of Interest

The authors declare no conflict of interest.

References