



Article

Odd Graceful Labeling of W -Tree $WT(n, k)$ and its Disjoint Union

Abaid ur Rehman Virk^{1,*} and A. Riasat²

¹ University of Management and Technology, Lahore, Pakistan.

² University of Engineering and Technology, Lahore, KSK campus, Pakistan.

* **Correspondence:** abaidrehman@umt.edu.pk

Abstract: Let $G = (V(G), E(G))$ be a graph with p vertices and q edges. A graph G of size q is said to be odd graceful if there exists an injection $\lambda : V(G) \rightarrow 0, 1, 2, \dots, 2q - 1$ such that assigning each edge xy the label or weight $|\lambda(x) - \lambda(y)|$ results in the set of edge labels being $1, 3, 5, \dots, 2q - 1$. This concept was introduced in 1991 by Gananajothi. In this paper, we examine the odd graceful labeling of the W -tree, denoted as $WT(n, k)$.

Keywords: Graph, Odd graceful labeling, W -tree $WT(n, k)$.

Mathematics Subject Classification: Primary 11B39, Secondary 05A10.

1. Introduction

The graph considered in this study is finite, connected, and simple. The symbols $V(G)$ and $E(G)$ denote the vertex set and edge set of the graph G , respectively. Additionally, p and q represent the number of vertices and edges in G , respectively.

Definition 1. [1] Rosa defined a β -valuation of a graph G with q edges as an injection from the vertices of G to the set $\{1, 2, \dots, q\}$. This valuation is such that when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct.

Definition 2. A graceful labeling f possesses the property that there exists an integer λ such that for each edge xy , either

$$f(x) \leq \lambda < f(y) \quad \text{or} \quad f(y) \leq \lambda < f(x).$$

If f satisfies this condition, it is referred to as an α -labeling.

Definition 3. [2] A graph G of size q is odd graceful if there exists an injection f from $V(G)$ to $\{0, 1, 2, \dots, 2q - 1\}$. When each edge is assigned the label or weight $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, \dots, 2q - 1\}$.

Many types of labeling exist, such as magic, total, antimagic, harmonious, graceful, and odd-graceful. This work focuses on formulating odd-graceful labeling for some new graphs. The study of graceful graphs and graceful labeling was introduced by Rosa. β -valuations are functions that produce graceful labeling. The term 'graceful labeling' became widely used after Golomb's study of such labeling several years later [3]. Graceful labeling and α -labeling are among the most popular classes of graph labeling [4]. It is known that not every graph is graceful. The Ringel-Kotzig conjecture [5], which posits that all trees are graceful, remains a well-known open problem.

The concept of odd graceful labeling was introduced by Gnanajothi [2]. Gnanajothi proposed the conjecture that all trees are odd-graceful and provided evidence for this conjecture in trees of order up to 10. Barrientos extended this conjecture to trees of order up to 12, proving that certain graphs are odd graceful, such as:

- Subdividing each edge exactly once,
- Every forest whose components are caterpillars,
- Every tree with a diameter of at most five,
- All disjoint unions of caterpillars.

Barrientos also conjectured that every bipartite graph is odd graceful.

Eldergill generalized Gnanajothi’s results on stars, demonstrating that a graph obtained by joining one endpoint from each of an odd number of paths of equal length is odd graceful [6]. Gao worked on the union of graphs and proved that unions of paths, stars, cycles $C_m \cup P_n$, $C_m \cup C_n$, and unions of cycles of order divisible by 4 are odd or both types of graceful.

S.K. Vaidya and L.Bijukumar investigated odd graceful labeling, focusing on combining two copies of C_n with a path P_n and two copies of a cycle C_n sharing a common edge, leading to various results.

Graph labeling has become an increasingly useful mathematical model for a wide range of applications, including ambiguities in X-Ray crystallography, communication network labeling, and circuit layout optimization. Most graph labeling problems involve three elements:

1. A set of numbers S from which the labels are chosen,
2. A rule assigning a value to each edge,
3. A condition that these values must satisfy.

Motivated by Gnanajothi’s research, many other graphs have been studied since then, which are odd graceful, leading to many new results. However, there still exist many interesting open problems. Inspired by the research in [2, 7], we began investigating the odd graceful labeling of the W -tree and its disjoint union with other graph families.

In this work, we demonstrate that a subclass, namely the W -tree $WT(n, k)$, admits odd graceful labeling. We also prove that the graph obtained by the disjoint union of a W -tree $WT(n, k)$ with a path P_m , a star $K_{1,m}$, a bistar $B_{m,l}$, and a ladder L_h are odd graceful graphs.

Firstly, we define some terms that will aid in understanding the subsequent work.

Definition 4. [4] A path is a tree P_n with vertex set $V(P_n) = \{x_i : 1 \leq i \leq n\}$ and edge set $E(G) = \{x_i x_{i+1} : 1 \leq i \leq n - 1\}$.

Definition 5. [4] Let x_0 define the central vertex of star S_n , $1 \leq n$ and let x_i , $1 \leq i \leq n$ be its leaves.

Definition 6. A bistar is a graph obtain by joining the central vertices of two star graphs.

Definition 7. [4] Let $L_n = P_n \times P_2$ be a ladder with $V(L_n) = \{u_i v_i : 1 \leq i \leq n\}$ and $E(L_n) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n\} \cup \{u_i v_i : 1 \leq i \leq n\}$

Definition 8. Let G be a graph with the set of vertices and edges as

$$V(G) = \{(c_1, c_2, b, w, d) \cup (x^1, x^2, \dots, x^n) \cup (y^1, y^2, \dots, y^n)\}$$

$$E(G) = \{(c_1 x^1, c_1 x^2, \dots, c_1 x^n) \cup (c_2 y^1, c_2 y^2, \dots, c_2 y^n) \cup (c_1 b, c_1 w) \cup (c_2 w, c_2 d)\}$$

we call it w -graph and it shall be denoted by $W(n)$.

Definition 9. Suppose that we have k isomorphic copies of w -graphs $W(n)$. A w -tree $WT(n, k)$ is a tree obtain by taking a new vertex a and joining it with $\{d_i : 1 \leq i \leq k\}$, where $n \geq 2$ and $k \geq 3$.

2. Odd-graceful labeling of w -tree

In this section, we will define the odd graceful labeling of W -tree.

Theorem 1. For $k \geq 3$, $G \cong WT(n, k)$ admits odd graceful labeling if $n \geq 2k - 2$.

Proof. Let G be a graph, we denote the vertex and edge set of G as follows:

$$\begin{aligned} \mathbf{V(G)} &= \{a\} \cup \{b_i : 1 \leq i \leq n\} \cup \{W_i : 1 \leq i \leq n\} \\ &\cup \{d_i : 1 \leq i \leq n\} \cup \{C_{i1} : 1 \leq i \leq n\} \cup \{C_{i2} : 1 \leq i \leq n\} \\ &\cup \{x_i^p : 1 \leq i \leq n, 1 \leq p \leq k\} \cup \{y_i^p : 1 \leq i \leq n, 1 \leq p \leq k\} \\ \mathbf{E(G)} &= \{b_i c_{i1} : 1 \leq i \leq n\} \cup \{w_i c_{i1} : 1 \leq i \leq n\} \\ &\cup \{w_i c_{i2} : 1 \leq i \leq n\} \cup \{d_i c_{i2} : 1 \leq i \leq n\} \\ &\cup \{c_{i1} x_i^p : 1 \leq i \leq n, 1 \leq p \leq k\} \cup \{c_{i2} y_i^p : 1 \leq i \leq n, 1 \leq p \leq k\} \end{aligned}$$

$$|V(G)| = (2n + 3)k + 2k + 1$$

and

$$|E(G)| = (2n + 5)k$$

where $v = |V(G)|$ and $q = |E(G)|$.

Now, we define the labeling as follows

$$\lambda : V(G) \longrightarrow \{0, 1, 2, \dots, 2q - 1\}$$

where $1 \leq i \leq k$

$$\lambda(c_{i1}) = \{2q - 1 - 4(i - 1)\}$$

$$\lambda(c_{i2}) = \{2q - 3 - 4(i - 1)\}$$

$$\lambda(b_i) = \{(2n + 3)(2i - 2) \text{ for } 1 \leq i \leq k\}$$

$$\lambda(w_i) = \{(2n + 2)(2i - 1) + 2(i - 1) \text{ for } 1 \leq i \leq k\}$$

$$\lambda(d_i) = \{\lambda(w_i) + 4i - 2 \text{ for } 1 \leq i \leq k\}$$

For $1 \leq i \leq k$ and $1 \leq p \leq n$

$$\lambda(x_i^p) = \{2p + (4n + 6)(i - 1)\}$$

$$\lambda(y_i^p) = \begin{cases} \lambda(w_i) + 2p & \text{if } 1 \leq p \leq 2i - 2 \\ \lambda(w_i) + 2(p + 1) & \text{if } 2i - 2 \leq p \leq n \end{cases}$$

$$\lambda(a) = \{\lambda(d_k) + 1\}$$

□

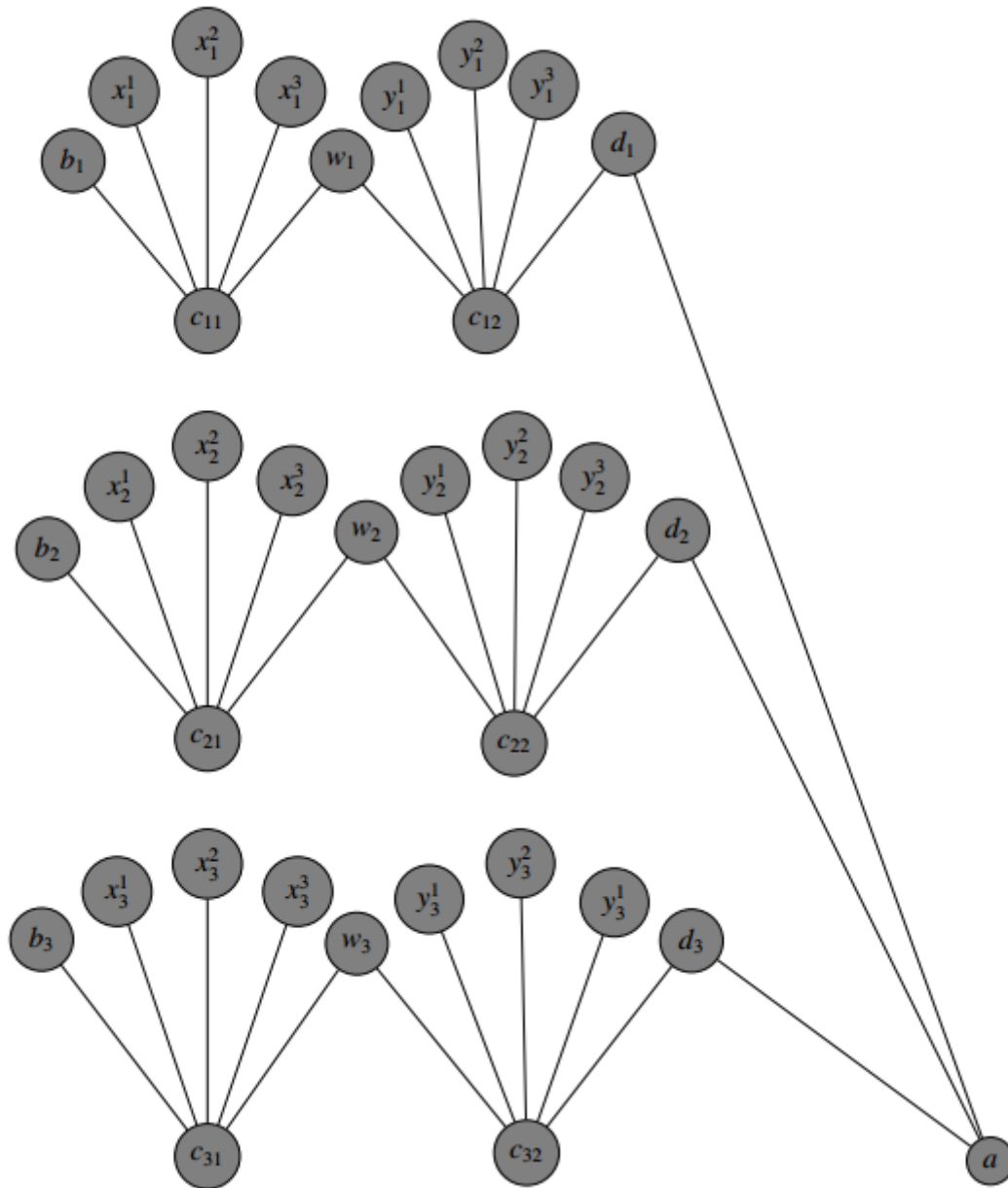


Figure 1. $WT(3, 3)$

3. Disjoint union of two isomorphic copies of w -tree $WT(n, k)$

The odd graceful labeling of disjoint union of two isomorphic copies of w -tree will presented in this section.

Theorem 2. *Let $G \cong 2WT(n, k)$ be an odd graceful graph for $n \geq 4k - 2$.*

Proof. Let G be a graph, we denote the vertex and edge set of G as follows:

$$\begin{aligned} \mathbf{V(G)} = & \{a\} \cup \{b_i : 1 \leq i \leq k\} \cup \{w_i : 1 \leq i \leq k\} \\ & \cup \{d_i : 1 \leq i \leq k\} \cup \{c_{i1} : 1 \leq i \leq k\} \cup \{c_{i2} : 1 \leq i \leq k\} \\ & \cup \{x_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \cup \{y_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \\ & \cup \{a\} \cup \{b'_i : 1 \leq i \leq k\} \cup \{w'_i : 1 \leq i \leq k\} \\ & \cup \{d'_i : 1 \leq i \leq k\} \cup \{c'_{i1} : 1 \leq i \leq k\} \cup \{c'_{i2} : 1 \leq i \leq k\} \\ & \cup \{x_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \cup \{y'_i : 1 \leq i \leq k, 1 \leq p \leq n\} \end{aligned}$$

$$\begin{aligned} \mathbf{E(G)} = & \{b_i c_{i1} : 1 \leq i \leq k\} \cup \{w_i c_{i1} : 1 \leq i \leq k\} \\ & \cup \{w_i c_{i2} : 1 \leq i \leq k\} \cup \{d_i c_{i2} : 1 \leq i \leq k\} \\ & \cup \{c_{i1} x_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \cup \{c_{i2} y_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \\ & \cup \{a d_i : 1 \leq i \leq k\} \cup \{a' d'_i : 1 \leq i \leq k\} \\ & \cup \{b'_i c'_{i1} : 1 \leq i \leq k\} \cup \{w'_i c'_{i1} : 1 \leq i \leq k\} \\ & \cup \{w'_i c'_{i2} : 1 \leq i \leq k\} \cup \{d'_i c'_{i2} : 1 \leq i \leq k\} \\ & \cup \{c'_{i1} x_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \cup \{c'_{i2} y_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \end{aligned}$$

then

$$|V(G)| = 2(2n + 3)k + 2k + 1$$

and

$$|E(G)| = 2(2n + 5)k$$

Now we define the labeling as follows.

$$\lambda : V(G) \longrightarrow \{0, 1, 2, \dots, 2q - 1\}$$

where $1 \leq i \leq k/2$

$$\lambda(c_{i1}) = \{2q - 1 - 4(i - 1)$$

$$\lambda(c_{i2}) = \{2q - 3 - 4(i - 1)$$

$$\lambda(c'_{i1}) = \{2q - 1 - 4(i - 1) - 2k$$

$$\lambda(c'_{i2}) = \{2q - 3 - 4(i - 1) - 2k$$

$$\lambda(b_i) = \{(2n + 3)(2i - 2) \text{ for } 1 \leq i \leq k/2$$

$$\lambda(b'_i) = \{(2n + 3)(2i - 2 + 2k) \text{ for } 1 \leq i \leq k/2$$

$$\lambda(w_i) = \{(2n + 2)(2i - 1) + 2(i - 1) \text{ for } 1 \leq i \leq k/2$$

$$\lambda(w'_i) = \{(2n + 2)(k + 2i - 1) + 2(k/2 + i - 1) \text{ for } 1 \leq i \leq k/2$$

$$\lambda(d_i) = \{\lambda(w_i) + 4i - 2 \text{ for } 1 \leq i \leq k/2$$

$$\lambda(d'_i) = \{\lambda(w'_i) + 2k + 4(i - 1) \text{ for } 1 \leq i \leq k/2$$

For $1 \leq i \leq k/2$ and $1 \leq p \leq n$

$$\lambda(x_i^p) = \{2p + (4n + 6)(i - 1)$$

$$\lambda(x_i^p) = \{\lambda(b'_i) + 2p + (4n + 6)(i - 1)$$

$$\lambda(y_i^p) = \begin{cases} \lambda(w_i) + 2p & \text{if } 1 \leq p \leq 2i - 2 \\ \lambda(w_i) + 2(p + 1) & \text{if } 2i - 2 \leq p \leq n \end{cases}$$

$$\lambda(y_i^p) = \begin{cases} \lambda(w'_i) + 2p & \text{if } 1 \leq p \leq 2i + 3 \\ \lambda(w'_i) + 2(p + 1) & \text{if } 2i + 4 \leq p \leq n \end{cases}$$

Now for the apex vertex a and a' , we have:

$$\lambda(a) = \{\lambda(c'_{(k/2)2} - 2(n - 9)$$

$$\lambda(a) = \{\lambda(d'_{k/2}) + 1$$

□

4. Disjoint union of W -tree $WT(n, k)$ with other families of Graphs

The odd graceful labeling of w -tree with some other known families of graphs like path, star, bistar, lader has been formulated in this section.

4.1. Disjoint union of W -tree $WT(n, k)$ with path P_m

Theorem 3. A graph $G \cong WT(n, k) \cup P_m$ admits the odd graceful labeling if $n \geq 2k - 2$.

Proof. Let G be a graph, where q represents the total number of edges of path P_m and $WT(n, k)$ and q' represents the total number of edges of path p_m and m represents the total number of vertices of path P_m . The vertex and the edge set of G can be represented as follows:

$$V(G) = V(P_m) \cup V(WT(n, k))$$

$$V(G) = \{a\} \cup \{v_{i'} : 1 \leq i' \leq m\} \cup \{b_i : 1 \leq i \leq k\} \cup \{w_i : 1 \leq i \leq k\}$$

$$\cup \{d_i : 1 \leq i \leq k\} \cup \{c_{i1} : 1 \leq i \leq k\} \cup \{c_{i2} : 1 \leq i \leq k\}$$

$$\cup \{x_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \cup \{y_i^p : 1 \leq i \leq k, 1 \leq p \leq n\}$$

and

$$E(G) = E(P_m) \cup E(WT(n, k))$$

$$E(G) = \{v_{i'}v_{i'+1} : 1 \leq i' \leq m\} \cup \{ad_i : 1 \leq i \leq k\} \cup \{w_ic_{i1} : 1 \leq i \leq k\}$$

$$\cup \{d_ic_{i2} : 1 \leq i \leq k\} \cup \{w_ic_{i2} : 1 \leq i \leq k\} \cup \{b_ic_{i1} : 1 \leq i \leq k\}$$

$$\cup \{c_{i1}x_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \cup \{c_{i2}y_i^p : 1 \leq i \leq k, 1 \leq p \leq n\}$$

then

$$|V(G)| = (2n + 3)k + 2k + m + 1$$

and

$$|E(G)| = (2n + 5)k + m - 1$$

Now we define the labeling as follows.

$$\lambda : V(G) \longrightarrow \{0, 1, 2, \dots, 2q - 1\}$$

We have different schemes for both path and w-tree,

where $1 \leq i' \leq m$

$$\lambda(v_{i'}) = \begin{cases} 2q - i' & \text{if } i' \equiv 1 \pmod{2} \\ i' - 2 & \text{if } i' \equiv 0 \pmod{2} \end{cases}$$

Now for the w-tree, we have the following scheme where $1 \leq i \leq k$

$$\lambda(c_{i1}) = \{2q - q' - 1 - 4(i - 1)$$

$$\lambda(c_{i2}) = \{2q - q' - 3 - 4(i - 1)$$

$$\lambda(b_i) = \{q' + (4n + 6)(i - 1) \text{ for } 1 \leq i \leq k$$

$$\lambda(w_i) = \{\lambda(b_i) + 2n + 2 \text{ for } 1 \leq i \leq k$$

$$\lambda(d_i) = \{\lambda(w_i) + 4i - 2 \text{ for } 1 \leq i \leq k$$

For $1 \leq i \leq k$ and $1 \leq p \leq n$

$$\lambda(x_i^p) = \{\lambda(b_i) + 2p$$

$$\lambda(y_i^p) = \begin{cases} \lambda(w_i) + 2p & \text{if } 1 \leq p \leq 2i - 2 \\ \lambda(w_i) + 2(p + 1) & \text{if } 2i - 2 \leq p \leq n \end{cases}$$

$$\lambda(a) = \{\lambda(d_k) + 1$$

□

4.2. Disjoint union of W -tree $WT(n, k)$ with star $K_{(1,m)}$

Theorem 4. Let $G \cong WT(n, k) \cup K_{(1,m)}$ admits the odd graceful labeling if $n \geq 2k - 2$.

Proof. Let G be a graph and q is the total number of edges of star $K_{(1,m)}$ and w-tree $WT(n, k)$ and q' represents the total number of edges of star $K_{(1,m)}$ and m represents the total number of vertices. The vertex and edge set of graph G is follows:

$$V(G) = V(K_{(1,m)}) \cup V(WT(n, k))$$

$$\begin{aligned} \mathbf{V(G)} = & \{v_o\} \cup \{v_{i'} : 1 \leq i' \leq m\} \cup \{a\} \cup \{b_i : 1 \leq i \leq k\} \cup \{w_i : 1 \leq i \leq k\} \\ & \cup \{d_i : 1 \leq i \leq k\} \cup \{c_{i1} : 1 \leq i \leq k\} \cup \{c_{i2} : 1 \leq i \leq k\} \\ & \cup \{x_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \cup \{y_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \end{aligned}$$

and

$$E(G) = E(K_{(1,m)}) \cup E(WT(n, k))$$

$$\mathbf{E(G)} = \{v_o v_{i'} : 1 \leq i' \leq m\} \cup \{a d_i : 1 \leq i \leq k\} \cup \{w_i c_{i1} : 1 \leq i \leq k\}$$

$$\cup \{d_i c_{i2} : 1 \leq i \leq k\} \cup \{w_i c_{i2} : 1 \leq i \leq k\} \cup \{b_i c_{i1} : 1 \leq i \leq k\}$$

$$\cup \{c_{i1} x_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \cup \{c_{i2} y_i^p : 1 \leq i \leq k, 1 \leq p \leq n\}$$

then

$$|V(G)| = (2n + 3)k + 2k + m + 1$$

and

$$|E(G)| = (2n + 5)k + m - 1$$

Now we define the labeling as follows.

$$\lambda : V(G) \longrightarrow \{0, 1, 2, \dots, 2q - 1\}$$

Now we have different schemes for both star and w-tree.

For star $K_{(1,m)}$, we have the following scheme:

where $1 \leq i' \leq m$,

$$\lambda(v_i) = \begin{cases} 0 & \text{if } i = 0 \\ 2q - 1 - 2(i' - 1) & \text{if } i \equiv 1 \pmod{2} \\ 2q - 3 - 2(i' - 1) & \text{if } i \equiv 0 \pmod{2} \end{cases}$$

Now for the w-tree, we have the following scheme where $1 \leq i \leq k$

$$\lambda(c_{i1}) = \{2q - q' - 1 - 4(i - 1)$$

$$\lambda(c_{i2}) = \{2q - 2q' - 2 - 4(i - 1)$$

$$\lambda(b_i) = \{(4n + 6)(i - 1) + 1 \text{ for } 1 \leq i \leq k$$

$$\lambda(w_i) = \{\lambda(b_i) + 2(n + 1) \text{ for } 1 \leq i \leq k$$

$$\lambda(d_i) = \{\lambda(w_i) + 4i - 2 \text{ for } 1 \leq i \leq k$$

For $1 \leq i \leq k$ and $1 \leq p \leq n$

$$\lambda(x_i^p) = \{\lambda(b_i) + 2p$$

$$\lambda(y_i^p) = \begin{cases} \lambda(w_i) + 2p & \text{if } 1 \leq p \leq 2i - 2 \\ \lambda(w_i) + 2(p + 1) & \text{if } 2i - 2 \leq p \leq n \end{cases}$$

$$\lambda(a) = \{\lambda(d_k) + 1$$

□

4.3. Disjoint union of W-tree $WT(n, k)$ with bistar $B_{(m,l)}$

Theorem 5. Let $G \cong WT(n, k) \cup B_{(m,l)}$ admits odd graceful labeling if $n \geq 2k - 2$.

Proof. Let G be a graph and q is the total number of edges of bistar $B_{(m,l)}$ and w-tree $WT(n, k)$ and q' represents the total number of edges of bistar $B_{(m,n)}$ and m represents the total number of vertices. The vertex and edge set of graph G is follows:

$$V(G) = V(B_{(m,l)}) \cup V(WT(n, k))$$

$$V(G) = \{a\} \cup \{u, v, u_{i'} v_{i'} : 1 \leq i' \leq m\} \cup \{b_i : 1 \leq i \leq k\} \cup \{w_i : 1 \leq i \leq k\}$$

$$\cup \{d_i : 1 \leq i \leq k\} \cup \{c_{i1} : 1 \leq i \leq k\} \cup \{c_{i2} : 1 \leq i \leq k\}$$

$$\cup \{x_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \cup \{y_i^p : 1 \leq i \leq k, 1 \leq p \leq n\}$$

and

$$E(G) = E(B_{(m,l)}) \cup E(WT(n, k))$$

$$\begin{aligned} \mathbf{E}(G) = & \{vv_{i'} : 1 \leq i' \leq m\} \cup \{uv\} \cup \{uu_{i'} : 1 \leq i \leq k\} \cup \{w_i c_{i1} : 1 \leq i \leq k\} \\ & \cup \{d_i c_{i2} : 1 \leq i \leq k\} \cup \{w_i c_{i2} : 1 \leq i \leq k\} \cup \{b_i c_{i1} : 1 \leq i \leq k\} \\ & \cup \{c_{i1} x_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \cup \{c_{i2} y_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \end{aligned}$$

then

$$|V(G)| = (2n + 3)k + 2k + 2m + 1$$

and

$$|E(G)| = (2n + 5)k + 2m$$

Now we define the labeling as follows.

$$\lambda : V(G) \longrightarrow \{0, 1, 2, \dots, 2q - 1\}$$

Now we have different schemes for both bistar and w -tree. As we know that a bistar consist of two isomorphic copies of a star, so we discuss each star separately.

Now for the first star. we have the following scheme:

where $1 \leq i' \leq m$,

$$\lambda(v_i) = \begin{cases} 0 & \text{if } i' = 0 \\ 2q - 1 - 2(i' - 1) & \text{if } i' \equiv 1(\text{mod}2) \\ 2q - 3 - 2(i' - 1) & \text{if } i' \equiv 0(\text{mod}2) \end{cases}$$

Now for the second star. we have the following scheme:

where $1 \leq i' \leq l$,

$$\lambda(v_i) = \begin{cases} 1 & \text{if } i' = 0 \\ \lambda(v_i) - 1 - 2(i' - 1) & \text{if } i' \equiv 1(\text{mod}2) \\ \lambda(v_i) - 3 - 2(i' - 1) & \text{if } i' \equiv 0(\text{mod}2) \end{cases}$$

Now for the w -tree, we have the following scheme where $1 \leq i \leq k$

$$\lambda(c_{i1}) = \{2q - q' - 1 - 4(i - 1)$$

$$\lambda(c_{i2}) = \{2q - 2q' - 2 - 4(i - 1)$$

$$\lambda(b_i) = \{q' - 1 + (4n + 6)(i - 1) \text{ for } 1 \leq i \leq k$$

$$\lambda(w_i) = \{\lambda(b_i) + 2(n + 1) \text{ for } 1 \leq i \leq k$$

$$\lambda(d_i) = \{\lambda(w_i) + 4i - 2 \text{ for } 1 \leq i \leq k$$

For $1 \leq i \leq k$ and $1 \leq p \leq n$

$$\lambda(x_i^p) = \{\lambda(b_i) + 2p$$

$$\lambda(y_i^p) = \begin{cases} \lambda(w_i) + 2p & \text{if } 1 \leq p \leq 2i - 2 \\ \lambda(w_i) + 2(p + 1) & \text{if } 2i - 2 \leq p \leq n \end{cases}$$

$$\lambda(a) = \{\lambda(d_k) + 3$$

□

4.4. Disjoint union of w-tree $WT(n, k)$ with ladder L_h

Theorem 6. Let $G \cong WT(n, k) \cup L_h$ admits odd graceful labeling if $n \geq 2k - 2$.

Proof. Let G be a graph and q is the total number of edges of ladder L_h and w-tree $WT(n, k)$ and q' represents the total number of edges of ladder L_h and m represents the total number of vertices of path P_m . The vertex and edge set of graph G is follows:

$$V(G) = V(L_h) \cup V(WT(n, k))$$

$$\begin{aligned} \mathbf{V(G)} = & \{a\} \cup \{u_{i'}v_{i'} : 1 \leq i' \leq m\} \cup \{b_i : 1 \leq i \leq k\} \cup \{w_i : 1 \leq i \leq k\} \\ & \cup \{d_i : 1 \leq i \leq k\} \cup \{c_{i1} : 1 \leq i \leq k\} \cup \{c_{i2} : 1 \leq i \leq k\} \\ & \cup \{x_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \cup \{y_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \end{aligned}$$

and

$$E(G) = E(L_h) \cup E(WT(n, k))$$

$$\begin{aligned} \mathbf{E(G)} = & \{u_{i'}u_{i'+1}v_{i'}v_{i'+1} : 1 \leq i' \leq m\} \cup \{u_{i'}v_{i'} : 1 \leq i \leq k\} \cup \{w_i c_{i1} : 1 \leq i \leq k\} \\ & \cup \{d_i c_{i2} : 1 \leq i \leq k\} \cup \{w_i c_{i2} : 1 \leq i \leq k\} \cup \{b_i c_{i1} : 1 \leq i \leq k\} \\ & \cup \{c_{i1}x_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \cup \{c_{i2}y_i^p : 1 \leq i \leq k, 1 \leq p \leq n\} \cup \{ad_i : 1 \leq i \leq k\} \end{aligned}$$

then

$$|V(G)| = (2n + 3)k + 2k + 3m + 2$$

and

$$|E(G)| = (2n + 5)k + 3m$$

Now we define the labeling as follows.

$$\lambda : V(G) \longrightarrow \{0, 1, 2, \dots, 2q - 1\}$$

Now we have different schemes for both ladder and w-tree.

where $1 \leq i' \leq m$,

$$\lambda(x_{1i'}) = \begin{cases} 2q' - 1 & \text{if } i' = 1 \\ 2q' - 2i' & \text{if } i' \equiv 1 \pmod{2} \quad i' \neq 1 \\ i' + [6(i' - 2)/2] & \text{if } i' \equiv 0 \pmod{2} \end{cases}$$

$$\lambda(x_{2i'}) = \begin{cases} 0 & \text{if } i' = 1 \\ 2q' - 2i' + 1 & \text{if } i' \equiv 1 \pmod{2} \quad i' \neq 1 \\ 4i' - 2 & \text{if } i' \equiv 0 \pmod{2} \end{cases}$$

Now for w-tree we have following scheme, where s is the least edge weight in the ladder graph. where $1 \leq i \leq k$

$$\lambda(c_{i1}) = \{s - 1 - 4(i - 1)$$

$$\lambda(c_{i2}) = \{s - 3 - 4(i - 1)$$

$$\lambda(b_i) = \{1 + (4n + 6)(i - 1) \text{ for } 1 \leq i \leq k$$

$$\lambda(w_i) = \{\lambda(b_i) + 2(n + 1) \text{ for } 1 \leq i \leq k$$

$$\lambda(d_i) = \begin{cases} \lambda(w_i) + 4i - 2 & \text{for } 1 \leq i \leq k \end{cases}$$

For $1 \leq i \leq k$ and $1 \leq p \leq n$

$$\begin{aligned} \lambda(x_i^p) &= \lambda(b_i) + 2p \\ \lambda(y_i^p) &= \begin{cases} \lambda(w_i) + 2p & \text{if } 1 \leq p \leq 2i - 2 \\ \lambda(w_i) + 2(p + 1) & \text{if } 2i - 2 \leq p \leq n \end{cases} \\ \lambda(a) &= \lambda(d_k) + 1 \end{aligned}$$

□

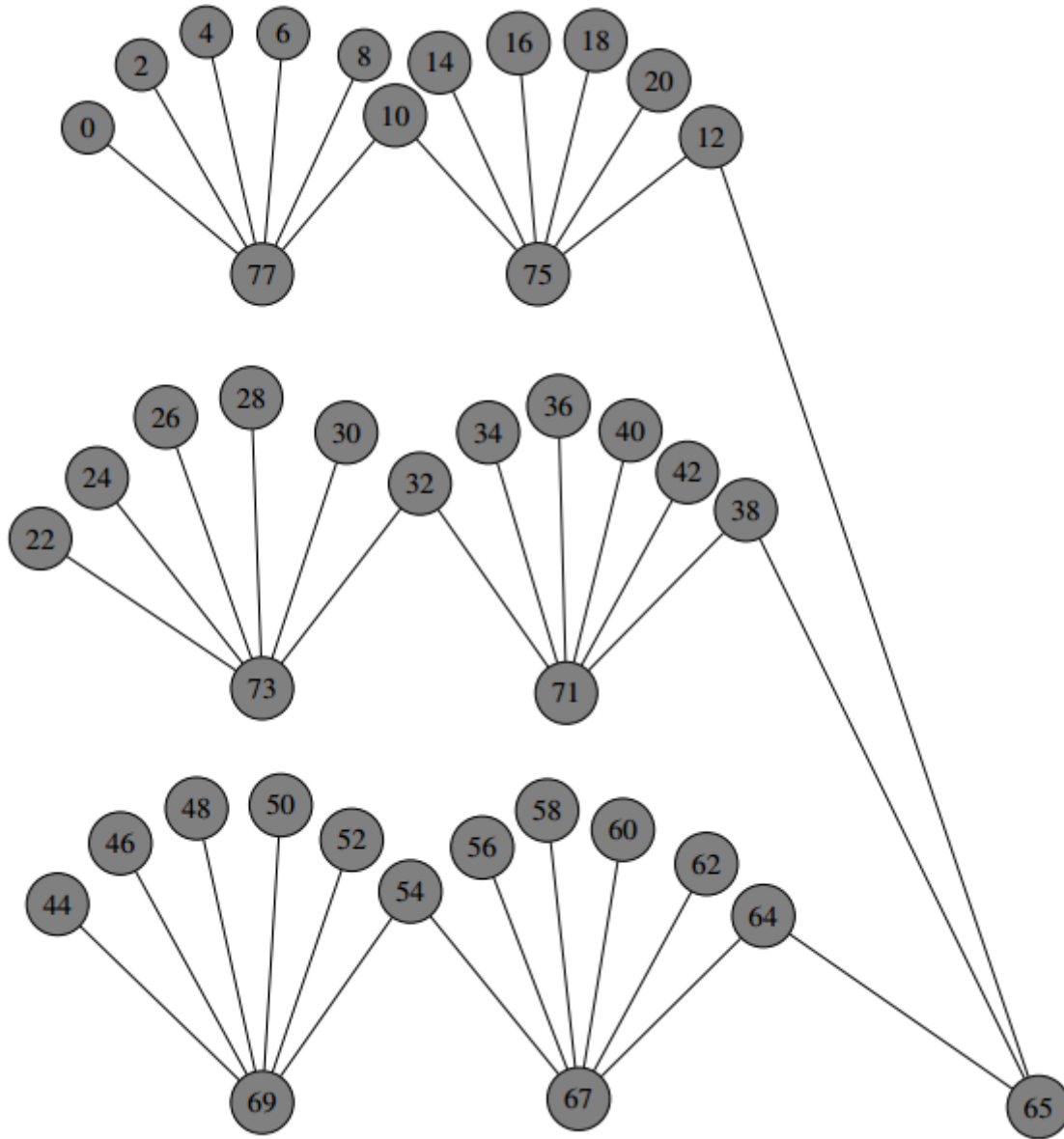


Figure 2. $WT(4, 3)$

5. Conclusion

In this paper, we have demonstrated that a specific subclass, the W -tree denoted by $WT(n, k)$, admits an odd graceful labeling. Additionally, we prove that graphs obtained through the union of a W -tree $WT(n, k)$ with various other graphs—namely, a path P_m , a star $K_{1,m}$, a bistar $B_{m,m}$, and a ladder $L_n = P_n \times P_n$ —are also odd graceful graphs. However, a more comprehensive understanding of the odd graceful labeling of W -trees requires further investigation.

Conflict of Interest

The authors declare no conflict of interest.

References

1. Rosa, A., 1966, July. On certain valuations of the vertices of a graph. In *Theory of Graphs (Internat. Symposium, Rome)* (pp. 349-355).
2. Gnanajothi, R.B., 1991. *Topics in graph theory* (Doctoral dissertation, Ph. D. Thesis, Madurai Kamaraj University).
3. Golomb, S.W., 1972. How to number a graph, *Graph Theory and Computing. Academic Press, New York*, 23, p.37.
4. Baca, M. and Miller, M., 2008. *Super edge-antimagic graphs: A wealth of problems and some solutions*. Universal-Publishers.
5. Wang, T.M., Yang, C.C., Hsu, L.H. and Cheng, E., 2015. Infinitely many equivalent versions of the graceful tree conjecture. *Applicable Analysis and Discrete Mathematics*, pp.1-12.
6. Daoud, S.N., 2017. Edge odd graceful labeling of some path and cycle related graphs. *AKCE international journal of graphs and combinatorics*, 14(2), pp.178-203.
7. Gallian, J.A., 2018. A dynamic survey of graph labeling. *Electronic Journal of combinatorics*, 1(DynamicSurveys), p.DS6.