



Article

On Constructions of GMGDs

Dinesh G Sarvate¹, Somnuek Worawiset² and Li Zhang^{3,*}

¹ Department of Mathematics, College of Charleston, Charleston, SC USA

² Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen, Thailand

³ Department of Mathematical Sciences, The Citadel Charleston, SC USA

* **Correspondence:** li.zhang@citadel.edu

Abstract: Modified group divisible designs $MGD(k, \lambda, m, n)$ are extensively studied because of an intriguing combinatorial structure that they possess and their applications. In this paper, we present a generalization of MGDs called $GMGD(k, \lambda_1, \lambda_2, m, n)$, and we provide some elementary results and constructions of some special cases of GMGDs. In addition, we show that the necessary conditions are sufficient for the existence of a $GMGD(3, \lambda, 2\lambda, m, n)$ for any positive integer λ , and a $GMGD(3, 2, 3, m, n)$. Though not a general result, the construction of a $GMGD(3, 3, 2, 2, 6)$ given in the paper is worth mentioning in the abstract. Along with another example of a $GMGD(3, 3, 2, 2, 4)$, and n to tn construction, we have families of $GMGD(3, 3\lambda, 2\lambda, 2, n)$ s for $n = 4t$ or $6t$ when $t \equiv 0, 1 \pmod{3}$, for any positive integer λ .

Keywords: Triple systems, Group divisible designs, Modified group divisible designs, BIBD

Mathematics Subject Classification: Primary 0505; Secondary 0530 0507

1. Introduction

One of the most studied designs in combinatorial design theory is a balanced incomplete block design:

Definition 1. A balanced incomplete block designs $BIBD(v, k, \lambda)$ is a pair (V, B) where V is a v -set of points and B is a collection of k -subsets of V called blocks such that every pair of distinct elements of V occurs in exactly λ blocks.

As given in [1],

Theorem 1. The necessary and sufficient conditions for the existence of a $BIBD(v, 3, \lambda)$ are given in Table 1.

λ	v
0 (mod 6)	all $v \neq 2$
1, 5 (mod 6)	1, 3 (mod 6)
2, 4 (mod 6)	0, 1 (mod 3)
3 (mod 6)	odd

Table 1. Necessary and Sufficient Conditions for a $BIBD(v, 3, \lambda)$

BIBDs can be applied to construct another well studied designs called group divisible designs (GDDs). GDDs have many applications, one of which is to construct other designs including BIBDs.

Definition 2. [2] A group divisible design $GDD(n, m, k; \lambda_1, \lambda_2)$, is a collection of k -element subsets of a v -set V called blocks which satisfies the following properties: the $v = nm$ elements of V are partitioned into m subsets (called groups) of size n each; distinct points within the same group are called first associates of each other and appear together in λ_1 blocks; any two points not in the same group are called second associates of each other and appear together in λ_2 blocks, where λ_1 and λ_2 are called indices of the GDD.

Note that in a $GDD(n, m, k; \lambda_1, \lambda_2)$, each point of V appears in $r = \frac{\lambda_1(n-1) + \lambda_2n(m-1)}{k-1}$ (called the replication number) of the $b = \frac{nmr}{k}$ blocks.

Theorem 2. [3] The necessary and sufficient conditions for the existence of a $GDD(n, m, 3; \lambda_1 = 0, \lambda_2 = \lambda)$ are

- $m \geq 3$,
- $\lambda(m - 1)n \equiv 0 \pmod{2}$, and
- $\lambda m(m - 1)n^2 \equiv 0 \pmod{6}$.

As a consequence of the above result, we have

Corollary 1. A $GDD(n, m, 3; \lambda_1 = 0, \lambda_2 = 1)$ exists when

m	n
0 (mod 6)	Even
1 (mod 6)	Any
2 (mod 6)	0 (mod 6)
3 (mod 6)	Any
4 (mod 6)	Even
5 (mod 6)	0 (mod 3)

Table 2. Necessary and Sufficient Conditions for a $GDD(n, m, 3; 0, 1)$

Though the definition of GDDs include nonzero λ_1 , the above theorem only gives the existence of GDDs with $\lambda_1 = 0$ and $k = 3$. The general existence of GDDs for $k = 3$ was obtained in [2] and [4].

Assaf [5] generalized the concept of GDDs and defined the modified group divisible designs as follows.

Definition 3. A modified group divisible design $MGD(k, \lambda, m, n)$ is a pair (V, B) where $V = \{(x_i, y_j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ is a set of order mn , and B is a collection of k -subsets of V satisfying the following conditions:

- every pair of distinct points (x_{i_1}, y_{j_1}) and (x_{i_2}, y_{j_2}) is contained in exactly λ blocks when $i_1 \neq i_2$ and $j_1 \neq j_2$;
- any pair of distinct points (x_{i_1}, y_{j_1}) and (x_{i_2}, y_{j_2}) where $i_1 = i_2$ or $j_1 = j_2$ is not contained in any block.

MGDs possess an intriguing combinatorial structure and are studied extensively. Also, MGDs have many applications, for example, see Assaf [6], Assaf and Wei [7], Abel and Assaf [8], Danziger and Wang [9], Ling and Colbourn [10], and Ge, Wang and Wei [11]. Similar to the more general definition of GDDs, MGDs can be further generalized and studied. With this motivation, we define generalized modified group divisible designs, GMGDs, as follows.

Definition 4. A generalized modified group divisible design $GMGD(k, \lambda_1, \lambda_2, m, n)$ is a pair (V, B) where $V = \{(x_i, y_j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ is a set of order mn , and B is a collection of k -subsets of V satisfying the following conditions:

- every pair of distinct points (x_{i_1}, y_{j_1}) and (x_{i_2}, y_{j_2}) is contained in exactly λ_2 blocks when $i_1 \neq i_2$ and $j_1 \neq j_2$;
- every pair of distinct points (x_{i_1}, y_{j_1}) and (x_{i_2}, y_{j_2}) where $i_1 = i_2$ or $j_1 = j_2$ is contained in λ_1 blocks.

The subsets $\{(x_i, y_j), 1 \leq i \leq m\}$ where $j = 1, 2, \dots, n$ are called the first set of groups (or columns), and the subsets $\{(x_i, y_j), 1 \leq j \leq n\}$ where $i = 1, 2, \dots, m$ are called the second set of groups (or rows).

In other words, the points of the mn -set are partitioned into m groups R_1, R_2, \dots, R_m (for rows) and n groups C_1, C_2, \dots, C_n (for columns) with $|R_i \cap C_j| = 1$ and if two elements are in the same row or column they occur together in λ_1 blocks, and otherwise they occur together in λ_2 blocks. Notice that a $MGD(k, \lambda, m, n)$ is a special case of a $GMGD(k, \lambda_1, \lambda_2, m, n)$, i.e., a $GMGD(k, \lambda_1 = 0, \lambda_2 = \lambda, m, n)$.

Remark 1 (Case $k = 2$). $GMGDs$ for block size $k = 2$ are trivial. One can use λ_1 pairs of each of the groups to form subsets of size 2 and if a pair is not from the same row group or column group, repeat it λ_2 times, to get a $GMGD(2, \lambda_1, \lambda_2, m, n)$ for any positive integer λ_1, λ_2, m and n .

Example 1. A $GMGD(3, 2, 1, 2, 3)$ on $V = \{1, 2, \dots, 6\}$ and

	C_1	C_2	C_3	
R_1 :	1	3	5	.
R_2 :	2	4	6	

The blocks are written in columns below

1	1	3	3	5	5	1	2
2	2	4	4	6	6	3	4
3	4	5	6	1	2	5	6

As $m = 2 < 3$, a $MGD(3, 1, 2, 3)$ does not exist.

A very useful construction of a $GMGD$ is given below, for any block size if a MGD exists.

Theorem 3. If a $MGD(k, \lambda_2, m, n)$, a $BIBD(m, k, \lambda_1)$ and a $BIBD(n, k, \lambda_1)$ exist, then a $GMGD(k, \lambda_1, \lambda_2, m, n)$ exists.

Proof. The blocks of the $MGD(k, \lambda_2, m, n)$ together with the blocks of the $BIBD(m, k, \lambda_1)$ on each column group, and the blocks of the $BIBD(n, k, \lambda_1)$ on each row group give the required $GMGD$. \square

In particular, Assaf [5] proved that the necessary conditions given below were sufficient for block size $k = 3$.

Theorem 4. The necessary conditions for the existence of a $MGD(k, \lambda, m, n)$ are that $m, n \geq k$, $\lambda(mn + 1 - m - n) \equiv 0 \pmod{k - 1}$ and $\lambda mn(mn + 1 - m - n) \equiv 0 \pmod{k(k - 1)}$.

Theorem 5. A $MGD(3, \lambda, m, n)$ exists whenever a $BIBD(m, 3, \lambda)$ or a $BIBD(n, 3, \lambda)$ exists.

Proof. The necessary conditions given in Theorem 4 are satisfied by any (v, λ) pair from Table 1 given in Theorem 1 for $v = m$ (or $v = n$) when $k = 3$. Instead of showing the proof for each pair (m, λ) and similarly for any pair (n, λ) , we show the calculations just for one pair when $m \equiv 1 \pmod{6}$ and $\lambda = 5$, say $(m = 6t + 1, \lambda = 5)$ where t is any positive integer.

We want to show that

$$\lambda(mn + 1 - m - n) \equiv 0 \pmod{2}$$

and

$$\lambda mn(mn + 1 - m - n) \equiv 0 \pmod{6}.$$

First, note $mn + 1 - m - n = (6t + 1)n + 1 - (6t + 1) - n = 6nt + n + 1 - 6t - 1 - n = 6(n - 1)t$, which is even, and hence

$$\lambda(mn + 1 - m - n) \equiv 0 \pmod{2}$$

as required.

Second, note $mn(mn + 1 - m - n) = (6t + 1)n(6(n - 1)t)$, which is divisible by 6, and hence

$$\lambda mn(mn + 1 - m - n) \equiv 0 \pmod{6}.$$

As the necessary conditions are sufficient for the existence of the required MGD, we have the result. \square

From Theorem 3, and Theorem 5 we get,

Corollary 2. *A GMGD(3, $\lambda_1, \lambda_2, m, n$) exists, if a BIBD($m, 3, \lambda_2$) or a BIBD($n, 3, \lambda_2$) and a BIBD($m, 3, \lambda_1$) and a BIBD($n, 3, \lambda_1$) exist.*

For example, if $\lambda_1 \equiv 1 \pmod{6}$, and $\lambda_2 \equiv 5 \pmod{6}$, and $m, n \equiv 1$ or $3 \pmod{6}$, a GMGD(3, $\lambda_1, \lambda_2, m, n$) exists. Another example, $\lambda_1 = 3$, $\lambda_2 = 2$, if $m \equiv 3 \pmod{6}$ and $n \equiv 1 \pmod{6}$, a GMGD(3, 3, 2, m, n) exists.

2. Necessary Conditions For GMGD($k, \lambda_1, \lambda_2, m, n$)

Counting the number of pairs with a fixed point, we have $\lambda_1(m+n-2) + \lambda_2(mn - (m+n-1)) = r(k-1)$. Hence

$$r = \frac{\lambda_1(m+n-2) + \lambda_2(mn - (m+n-1))}{k-1}. \quad (1)$$

From $vr = bk$, we have $vr = (mn) \left(\frac{\lambda_1(m+n-2) + \lambda_2(mn - (m+n-1))}{k-1} \right) = bk$, we have

$$b = \frac{mn(\lambda_1(m+n-2) + \lambda_2(mn - (m+n-1)))}{k(k-1)}. \quad (2)$$

Since r and b must be positive integers, we have the following result.

Theorem 6. *The necessary conditions for the existence of a GMGD($k, \lambda_1, \lambda_2, m, n$) are $\lambda_1(m+n-2) + \lambda_2(mn - (m+n-1)) \equiv 0 \pmod{(k-1)}$ and $mn(\lambda_1(m+n-2) + \lambda_2(mn - (m+n-1))) \equiv 0 \pmod{k(k-1)}$.*

3. GMGD($k = 3, \lambda_1, \lambda_2, m = 2, n$)

Examining different types of blocks in a GMGD(3, $\lambda_1, \lambda_2, m, n$), there are six possible types of blocks:

- Type I: Blocks are subsets of the row groups. These blocks have 3 first associate pairs. Assume there are x such blocks which provide $3x$ first associate pairs and 0 second associate pairs.
- Type II: Blocks are subsets of the column groups. These blocks have 3 first associate pairs. Assume there are y such blocks which provide $3y$ first associate pairs and 0 second associate pairs.
- Type III: Blocks contain two elements from a row R_i and two elements from a column C_j where one element is from both R_i and C_j . These blocks have 2 first associate pairs and 1 second associate pair. Assume there are z such blocks which provide $2z$ first associate pairs and z second associate pairs.

Type IV: Blocks contain only one elements from a row R_i and two elements from a column C_j for $i \neq j$ where no two elements are from the same row. These blocks have 2 second associate pairs and 1 first associate pair. Assume there are w such blocks which provide w first associate pairs and $2w$ second associate pairs.

Type V: Blocks contain only one elements from a column C_j and two elements from a row R_i for $i \neq j$ where no two elements are from the same column. These blocks have 2 second associate pairs and 1 first associate pair. Assume there are u such blocks which provide u first associate pairs and $2u$ second associate pairs.

Type VI: Blocks contain three elements where no two elements are from the same row or column. These blocks have 3 second associate pairs. Assume there are t such blocks which provide 0 first associate pairs and $3t$ second associate pairs.

Considering the total number of blocks b , the total number of first associate pairs, and the total number of the second associate pairs in a $\text{GMGD}(3, \lambda_1, \lambda_2, m, n)$, we have the following three equations.

$$x + y + z + w + u + t = b = \frac{mn(\lambda_1(m + n - 2) + \lambda_2(mn - (m + n - 1)))}{k(k - 1)}, \tag{3}$$

$$3x + 3y + 2z + w + u = \frac{\lambda_1 mn(m + n - 2)}{2}, \tag{4}$$

and

$$z + 2w + 2u + 3t = \frac{\lambda_2 mn^2(mn - m - n + 1)}{2}. \tag{5}$$

For the $m = 2$ case, we first consider an example where a design does not exist, although the necessary conditions in Theorem 6 are satisfied. One can check that the necessary conditions in Theorem 6 are satisfied for a $\text{GMGD}(3, \lambda_1 = 1, \lambda_2 = 2, m = 2, n = 6)$, but we show that a design does not exist by counting different types of blocks. Let $R_1 = \{1, 2, 3, 4, 5, 6\}$ and $R_2 = \{7, 8, 9, 10, 11, 12\}$ and $C_i = \{i, i + 6\}, i = 1, 2, 3, 4, 5, 6$. If a $\text{GMGD}(3, 1, 2, 2, 6)$ exists, then $y = 0, w = 0, t = 0$ and $z = 6$, and there should be 36 first associate pairs and 60 second associate pairs packed into blocks of size 3. That is,

$$3x + 12 + u = 36,$$

and

$$6 + 2u = 60.$$

The solution to the equations is $(x, u) = (-1, 27)$, a contradiction to x and u must be non-negative integers. Therefore, a design does not exist. This non-existence shown by counting different types of blocks gives us an additional necessary condition for the existence of a $\text{GMGD}(3, \lambda_1, \lambda_2, m = 2, n)$. From Eqs (3), (4) and (5), respectively, we have the following (in this case, $y = 0, w = 0, t = 0$ and $z = n$):

$$n + x + u = \frac{2n(\lambda_1 n + \lambda_2(n + 1))}{6},$$

$$3x + 2n + u = \lambda_1 n^2,$$

$$n + 2u = \lambda_2 n(n - 1).$$

We have $u = \frac{\lambda_2 n(n-1)-n}{2}$ and $x = \frac{2\lambda_1 n^2 - \lambda_2 n(n-1) - 3n}{6}$. Since x must be a non-negative integer, we have

$$2\lambda_1 n^2 - \lambda_2 n(n - 1) - 3n \geq 0, \tag{6}$$

and

$$2\lambda_1 n^2 - \lambda_2 n(n - 1) - 3n \equiv 0 \pmod{6}. \tag{7}$$

From Eq. (6), we should include the following as additional necessary condition where $m = 2$.

Theorem 7. *An additional necessary condition for the existence of a GMGD(3, λ₁, λ₂, 2, n) is λ₁ ≥ $\frac{\lambda_2(n-1)+3}{2n}$.*

3.1. GMGD(k = 3, λ₁ = 1, λ₂ = 1, m = 2, n)

If λ₂ = 1, then λ₁ ≥ 1 by Theorem 7. First, we start with λ₁ = 1. Notice that a GMGD(3, 1, 1, 2, n) is just a BIBD(2n, 3, 1). From Eq. (7), we have n ≡ 0, 2 (mod 6). Since 2n ≡ 0, 4 (mod 6) and a BIBD(2n, 3, 1) does not exist, a GMGD(3, 1, 1, 2, n) does not exist.

Next, we examine the case GMGD(3, 2, 1, 2, n). From Eq. (7), we have n ≡ 0 (mod 6). If n = 6p, from Eq. (1), we have r = $\frac{12t+\lambda_2(12t-6t-1)}{2} = \frac{18t-1}{2}$ which is not an integer. Therefore, a GMGD(3, 2, 1, 2, n) does not exist. In addition, if n ≡ 0 (mod 6) and m = 2, λ₂ must be an even number.

3.2. GMGD(k = 3, λ₁, λ₂ = 2, m = 2, n)

If λ₂ = 2, then λ₁ ≥ 2 by Theorem 7. If λ₁ = 2, we have n ≡ 0, 2 (mod 6) from Eq. (7). Since 2n ≡ 0, 4 (mod 6), a BIBD(2n, 3, 2) exists by Theorem 1. Therefore, a GMGD(3, 2, 2, 2, n) (which is just a BIBD(2n, 3, 2)) exists (unlike the case where λ₁ = λ₂ = 1).

For the case where λ₁ = 3 and λ₂ = 2, we have n ≡ 0, 4 (mod 6) from Eq. (7). The blocks of a GMGD(3, 3, 2, 2, 4) are provided below.

Example 2. A GMGD(3, 3, 2, 2, 4) on V = {1, 2, ..., 8} and

	C ₁	C ₂	C ₃	C ₄
R ₁ :	1	2	3	4
R ₂ :	5	6	7	8

The blocks are written in columns below

1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	4	5	5	6	6
2	2	2	3	3	4	4	5	5	6	6	6	3	4	7	4	4	4	7	5	6	6	7	7
3	4	5	6	7	6	8	7	8	3	4	5	8	7	8	8	7	5	5	8	7	8	8	8

The following example is the one noted in the abstract.

Example 3. A GMGD(3, 3, 2, 2, 6) on V = {1, 2, ..., 12} and

	C ₁	C ₂	C ₃	C ₄	C ₅	C ₆
R ₁ :	1	2	3	4	5	6
R ₂ :	7	8	9	10	11	12

The following blocks (where the blocks are written in columns) combined with blocks from a BIBD(12, 3, 2) where the three blocks {4, 5, 9}, {4, 6, 7}, and {5, 6, 8} are removed from the 44 blocks of the BIBD(12, 3, 2), and together with the block {4, 5, 6} give the required 56 blocks of a GMGD(3, 3, 2, 2, 6). Note that using appropriate relabeling, the blocks {1, 5, 0}, {1, 6, 2}, and {5, 6, 10} in the blocks of the BIBD(12, 3, 2) given in [12] give us a BIBD(12, 3, 2) with the required blocks {4, 5, 9}, {4, 6, 7}, and {5, 6, 8}.

1	1	2	3	7	7	8	9	1	2	3	4	5	6
2	4	4	5	8	11	10	10	6	5	4	7	9	8
3	5	6	6	9	12	11	12	7	8	9	10	11	12

3.3. n to tn Construction of a GMGD

To better understand the construction presented in the Theorem 8, first let us see Example 4 below where t = 3.

Example 4. Get a GMGD(3, $\lambda_1, \lambda_2, 2, 3n$) from a GMGD(3, $\lambda_1, \lambda_2, 2, n$) where $\lambda_1 \geq \lambda_2$.

Let R_1, \dots, R_6 be 6 disjoint sets of size n . Construct a GMGD(3, $\lambda_1, \lambda_2, 2, n$) on groups $R_1, R_2; R_3, R_4;$ and R_5, R_6 , respectively. Then construct λ_2 copies of eight GDD($n, 3, 3; 0, 1$)s where the groups are from each of these blocks $\{R_1, R_3, R_6\}, \{R_1, R_5, R_4\}, \{R_1, R_4, R_6\}, \{R_1, R_3, R_5\}, \{R_2, R_4, R_6\}, \{R_2, R_3, R_6\}, \{R_2, R_4, R_5\}, \{R_2, R_3, R_5\}$, respectively. Also, construct $\lambda_1 - \lambda_2$ copies of GDD($n, 3, 3; 0, 1$)s on groups R_1, R_3, R_5 and on groups R_2, R_4, R_6 . The blocks of these GDDs together with the blocks of the GMGDs provide the blocks of a GMGD(3, $\lambda_1, \lambda_2, 2, 3n$).

Theorem 8. If a GMGD(3, $\lambda_1, \lambda_2, 2, n$) exists with $\lambda_1 \geq \lambda_2$ and $\lambda_2 t(t - 1) \equiv 0 \pmod{3}$, then a GMGD(3, $\lambda_1, \lambda_2, 2, tn$) exists.

Proof. Let $R_1, R_2, \dots, R_{2t-1}, R_{2t}$ be $2t$ disjoint sets of size n . First, we form a set X of blocks by constructing t GMGD(3, $\lambda_1, \lambda_2, 2, n$)s on groups R_{2i-1}, R_{2i} with column groups, say $C_{1i}, C_{2i}, \dots, C_{ni}, i = 1, \dots, t$. Second, we construct GDD($2, t, 3; 0, \lambda_2$)s on groups $\{R_{2i-1}, R_{2i}\}, i = 1, \dots, t$. Next, we form a set Y of blocks by constructing GDD($n, 3, 3; 0, 1$)s on groups of each block of the GDD($2, t, 3; 0, \lambda_2$)s as follows. Each block of the GDD($2, t, 3; 0, \lambda_2$) has three elements, say R_i, R_j, R_k . So we construct a GDD($n, 3, 3; 0, 1$) with groups R_i, R_j, R_k . Note that when $\lambda_2 t(t - 1) \equiv 0 \pmod{3}$, a GDD($2, t, 3; 0, \lambda_2$) exists. Lastly, we form a set Z of blocks by constructing $\lambda_1 - \lambda_2$ copies of a GDD($n, t, 3; 0, 1$) on groups $R_1, R_3, \dots, R_{2t-1}$, and also $\lambda_1 - \lambda_2$ copies of a GDD($n, t, 3; 0, 1$) on groups R_2, R_4, \dots, R_{2t} . The blocks in X, Y and Z together give us the blocks for a GMGD(3, $\lambda_1, \lambda_2, 2, tn$).

Now we do the checking for λ_1 and λ_2 . If two elements are from R_{2i} (or from R_{2i-1}), they occur together λ_1 times in the GMGD(3, $\lambda_1, \lambda_2, 2, n$) on groups R_{2i-1}, R_{2i} with column groups, say $C_{1i}, C_{2i}, \dots, C_{ni}$.

If two elements are in the same column group, say C_{ji} , they occur together λ_1 times in the GMGD on R_{2i-1}, R_{2i} .

If two elements $x \in R_{2i-1}$ and $y \in R_{2i}$, and x and y are not in the same column group, then (x, y) occur λ_2 times in the GMGD on R_{2i-1}, R_{2i} .

For $x \in R_{2i-1}$ and $y \in R_{2j}$ where $i \neq j$, since R_{2i-1} and R_{2j} occur in λ_2 blocks of GDD($2, t, 3; 0, \lambda_2$) on groups $\{R_{2i-1}, R_{2i}\}, i = 1, \dots, t$. Each of these blocks is used to construct a GDD($n, 3, 3; 0, 1$). Therefore, (x, y) occur together in λ_2 blocks in the set Y of blocks.

For $x \in R_{2i-1}$ and $y \in R_{2j-1}$ ($i \neq j$), (x, y) occur together in λ_2 blocks in the set Y of blocks using the same argument. Also, they occur in exactly $\lambda_1 - \lambda_2$ blocks in the set Z of blocks. Therefore, all together (x, y) occur $\lambda_1 - \lambda_2 + \lambda_2 = \lambda_1$ times. \square

Note that it is not necessary to use GDD($2, t, 3; 0, \lambda_2$) as in Example 4. We may use GDD($2, t, 4; 0, \lambda_2$) as shown in Example 5 below for $t = 4$.

Example 5. Get a GMGD(3, $\lambda_1, \lambda_2, 2, 4n$) from a GMGD(3, $\lambda_1, \lambda_2, 2, n$) where $\lambda_1 \geq \lambda_2$ and n is even.

Let R_1, \dots, R_8 be 8 disjoint sets of size n . Construct a GMGD(3, $\lambda_1, \lambda_2, 2, n$) on groups $R_1, R_2; R_3, R_4;$ $R_5, R_6;$ and R_7, R_8 , respectively. Then construct λ_2 copies of six GDD($n, 4, 3; 0, 1$)s where groups are from each of these blocks $\{R_1, R_3, R_6, R_8\}, \{R_1, R_5, R_4, R_8\}, \{R_1, R_7, R_4, R_6\}, \{R_3, R_5, R_2, R_8\}, \{R_3, R_7, R_2, R_6\}, \{R_5, R_7, R_2, R_4\}$, respectively. Also, construct $\lambda_1 - \frac{\lambda_2}{2}$ copies of GDD($n, 4, 3; 0, 1$)s on groups R_1, R_3, R_5, R_7 and on groups R_2, R_4, R_6, R_8 . Note that from the necessary conditions for the GDDs, if $m = 4$ and n is even, then λ_2 is even. The blocks of these GDDs together with the blocks of the GMGDs provide the blocks of a GMGD(3, $\lambda_1, \lambda_2, 2, 4n$).

Examples 2 and 3 and Theorem 8 imply Corollary 4 below. Note that when $\lambda_2 = 2$, the condition $\lambda_2 t(t - 1) \equiv 0 \pmod{3}$ in Theorem 8 becomes $t \equiv 0, 1 \pmod{3}$.

Corollary 3. A GMGD(3, 3, 2, 2, 4t) and a GMGD(3, 3, 2, 2, 6t) exist for $t \equiv 0, 1 \pmod{3}$.

Taking λ copies of a GMGD(3, 3, 2, 2, n) for $n = 4t$ or $6t$, we have the following corollary.

Corollary 4. A GMGD(3, 3 $\lambda, 2\lambda, 2, 4t$) and a GMGD(3, 3 $\lambda, 2\lambda, 2, 6t$) exist for $t \equiv 0, 1 \pmod{3}$, for any positive integer λ .

4. GMGD($k = 3, \lambda_1 = \lambda, \lambda_2 = 2\lambda, m, n$)

In this section, we study the case where $\lambda_2 = 2\lambda_1$ and $k = 3$ and $m, n \geq 3$. We start with $\lambda_1 = 1$. If $\lambda_1 = 1$ and $\lambda_2 = 2$, from the necessary conditions in Theorem 6, we need first

$$(m + n - 2) + 2(mn - (m + n - 1)) \equiv 0 \pmod{2},$$

i.e.,

$$2mn - m - n \equiv 0 \pmod{2},$$

which implies m and n are of the same parity. Second, we need

$$mn((m + n - 2) + 2(mn - (m + n - 1))) \equiv 0 \pmod{6},$$

i.e.,

$$mn(2mn - m - n) \equiv 0 \pmod{6}.$$

From this and keeping in mind that m and n are of the same parity, we have the necessary conditions given in Table 3.

m	n
0 (mod 6)	0, 2, 4 (mod 6)
1 (mod 6)	1, 3 (mod 6)
2 (mod 6)	0 (mod 6)
3 (mod 6)	1, 3, 5 (mod 6)
4 (mod 6)	0, 4 (mod 6)
5 (mod 6)	3 (mod 6)

Table 3. Necessary Conditions for a GMGD(3, 1, 2, m, n)

Here is an important observation.

Theorem 9. *If a GDD($n, m, k; 0, \lambda$) and a GDD($m, n, k; 0, \lambda$) exists, then a GMGD($k, \lambda_1 = \lambda, \lambda_2 = 2\lambda, m, n$) exists.*

Proof. From the definition of a GMGD, we know that the required GMGD has m row groups and n column groups. The blocks of a GDD($n, m, k; 0, \lambda$) on these m row groups together with the blocks of a GDD($m, n, k; 0, \lambda$) on these n column groups together give the required GMGD($k, \lambda_1 = \lambda, \lambda_2 = 2\lambda, m, n$). □

The above theorem along with Corollary 1 gives us

Corollary 5. *Necessary conditions are sufficient for the existence of a GMGD(3, $\lambda_1 = 1, \lambda_2 = 2, m, n$).*

Proof. A GMGD(3, $\lambda_1 = 1, \lambda_2 = 2, m, n$) exists for the values of m and n which satisfies both tables given below which are respectively necessary and sufficient conditions for the existence of a GDD($n, m, 3; 0, \lambda = 1$) and a GDD($m, n, 3; 0, \lambda = 1$) are given in Tables 4 and 5.

m	n
0 (mod 6)	Even
1 (mod 6)	Any
2 (mod 6)	0 (mod 6)
3 (mod 6)	Any
4 (mod 6)	Even
5 (mod 6)	0 (mod 3)

Table 4. Necessary and Sufficient Conditions for a GDD($n, m, 3; 0, \lambda = 1$)

n	m
0 (mod 6)	Even
1 (mod 6)	Any
2 (mod 6)	0 (mod 6)
3 (mod 6)	Any
4 (mod 6)	Even
5 (mod 6)	0 (mod 3)

Table 5. Necessary and Sufficient Conditions for a $GDD(m, n, 3; 0, \lambda = 1)$

Combining Tables 4 and 5, we get the following values of m and n in Table 6 where both a $GDD(n, m, 3; 0, 1)$ and a $GDD(m, n, 3; 0, 1)$ exist. Hence a $GMGD(3, \lambda_1 = 1, \lambda_2 = 2, m, n)$ exists.

n	m
0 (mod 6)	0, 2, 4 (mod 6)
1 (mod 6)	1, 3 (mod 6)
2 (mod 6)	0 (mod 6)
3 (mod 6)	1, 3, 5 (mod 6)
4 (mod 6)	0, 4 (mod 6)
5 (mod 6)	3 (mod 6)

Table 6. Necessary and Sufficient Conditions for a $GMGD(3, \lambda_1 = 1, \lambda_2 = 2, m, n)$

These are precisely the necessary conditions for the existence of a $GMGD(3, \lambda_1 = 1, \lambda_2 = 2, m, n)$. Hence the necessary conditions are sufficient for the existence of a $GMGD(3, \lambda_1 = 1, \lambda_2 = 2, m, n)$. □

Taking λ_1 copies of a $GMGD(3, \lambda_1 = 1, \lambda_2 = 2, m, n)$ we have;

Corollary 6. A $GMGD(3, \lambda_1, \lambda_2 = 2\lambda_1, m, n)$ exists for the values of m and n given in the Table 6.

In fact, we can generalize this as follows.

Theorem 10. Necessary conditions for the existence of a $GMGD(k = 3, \lambda, 2\lambda, m, n)$ in Theorem 6 are equivalent to the necessary and sufficient conditions for the existence of a $GDD(m, n, 3; 0, \lambda)$ and a $GDD(n, m, 3; 0, \lambda)$ in Theorem 2, i.e., $\lambda(m - 1)n \equiv 0 \pmod{2}$, and $\lambda m(m - 1)n^2 \equiv 0 \pmod{6}$, and $\lambda(n - 1)m \equiv 0 \pmod{2}$, and $\lambda n(n - 1)m^2 \equiv 0 \pmod{6}$.

Proof. Necessary conditions for the existence of GDDs imply necessary conditions for the existence of a GMGD from Theorem 9. To prove the other way around, we observe that the first necessary condition for the existence of a $GMGD(k = 3, \lambda, 2\lambda, m, n)$ in Theorem 6 ($\lambda(m+n-2) + 2\lambda(mn - m - n + 1) \equiv 0 \pmod{2}$) imply $\lambda m + \lambda n \equiv 0 \pmod{2}$, which implies $\lambda \equiv 0 \pmod{2}$ or $m \equiv n \pmod{2}$. These imply the first necessary condition for both GDDs, i.e., $\lambda(m - 1)n \equiv 0 \pmod{2}$ and $\lambda(n - 1)m \equiv 0 \pmod{2}$. From the second necessary condition for the existence of a $GMGD(k = 3, \lambda, 2\lambda, m, n)$ in Theorem 6 ($mn(\lambda(m + n - 2) + 2\lambda(mn - m - n + 1)) \equiv 0 \pmod{6}$), we have $\lambda m(m - 1)n^2 + \lambda n(n - 1)m^2 \equiv 0 \pmod{6}$. Our aim is to demonstrate that given $\lambda \equiv 0 \pmod{2}$ or $m \equiv n \pmod{2}$, if $\lambda m(m - 1)n^2 + \lambda n(n - 1)m^2 \equiv 0 \pmod{6}$ is valid, then $\lambda m(m - 1)n^2 \equiv 0 \pmod{6}$ and $\lambda n(n - 1)m^2 \equiv 0 \pmod{6}$. To this end, we first notice that if any one of $\lambda n(n - 1)m^2$ or $\lambda m(m - 1)n^2$ is 0 (mod 6), then the second expression is also 0 (mod 6). Now it is an easy exercise to show that for all possible (m, n) pairs, one of the terms is 0 (mod 6). □

Corollary 7. As the necessary conditions are sufficient for a $GDD(m, n, 3; 0, \lambda)$, necessary conditions are sufficient for the existence of a $GMGD(3, \lambda, 2\lambda, m, n)$.

Corollary 8. A $GMGD(3, 6t + \lambda, 6t + 2\lambda, m, n)$ exists when the necessary conditions for a $GMGD(3, \lambda, 2\lambda, m, n)$ are satisfied.

Proof. Use the blocks of a $BIBD(mn, 3, 6t)$ and the blocks of a $GMGD(3, \lambda, 2\lambda, m, n)$ to get the blocks of a $GMGD(3, 6t + \lambda, 6t + 2\lambda, m, n)$. □

5. $GMGD(k = 3, \lambda_1 = 2, \lambda_2 = 3, m, n)$

We start with an example of a $GMGD(3, 2, 3, 3, 4)$.

Example 6. $V = \{1, 2, \dots, 12\}$ and

	C_1	C_2	C_3	C_4
$R_1 :$	1	2	3	4
$R_2 :$	5	6	7	8
$R_3 :$	9	10	11	12

The blocks of a $GDD(3, 4, 3; 0, 2)$ on the column groups, the blocks of a $GDD(4, 3, 3; 0, 1)$ on the row groups, together with the column groups as blocks, give us a $GMGD(3, 2, 3, 3, 4)$.

This example can be generalized to give a construction, which also generalizes Theorem 9.

Theorem 11. If a $GDD(m, n, 3; 0, \lambda_2)$, a $GDD(n, m, 3; 0, \mu_2)$ and a $BIBD(m, 3, \lambda_2 - \mu_2)$ for $\lambda_2 > \mu_2$ (or a $BIBD(n, 3, \mu_2 - \lambda_2)$ for $\lambda_2 < \mu_2$) exist, then a $GMGD(3, \lambda_2, \lambda_2 + \mu_2, m, n)$ exists for $m, n \geq 3$.

Proof. Assume $\lambda_2 > \mu_2$, let $\{a_i, a_j\}$ and $\{b_i, b_j\}$ be the elements from two different column groups, so that $\{a_i, b_i\}$ and $\{a_j, b_j\}$ are elements from two different row groups. Except the pairs $\{a_i, a_j\}$ and $\{b_i, b_j\}$ all other four pairs occur together λ_2 times from the blocks of $GDD(m, n, 3; 0, \lambda_2)$. Similarly, except the pairs $\{a_i, b_i\}$ and $\{a_j, b_j\}$ all other four pairs occur μ_2 times in the blocks of $GDD(n, m, 3; 0, \mu_2)$. The blocks of $BIBD(m, 3, \lambda_2 - \mu_2)$ on each of the column groups, contain the pairs $\{a_i, a_j\}$ and $\{b_i, b_j\}$ exactly $\lambda_2 - \mu_2$ times. Hence we see that the pairs from the same row group or column group occur λ_2 times and all other pairs occur $\lambda_2 + \mu_2$ times as required. Similarly, if $\lambda_2 < \mu_2$, use the blocks of a $BIBD(n, 3, \mu_2 - \lambda_2)$ instead. □

Remark 2. If $\lambda_2 = \mu_2$, we have Theorem 9.

Corollary 9. If $m \equiv 1, 3 \pmod{6}$ (resp $n \equiv 1, 3 \pmod{6}$) and $n \equiv 0, 4 \pmod{6}$ (resp $m \equiv 0, 4 \pmod{6}$), then a $GMGD(3, 2, 3, m, n)$ exists.

Next, we find necessary conditions the existence of a $GMGD(k = 3, \lambda_1 = 2, \lambda_2 = 3, m, n)$. We have

$$r = \frac{2(m + n - 2) + 3(mn - (m + n - 1))}{2}$$

and

$$b = \frac{mn[2(m + n - 2) + 3(mn - (m + n - 1))]}{6}.$$

As b and r must be integers, following table gives possible values for m and n in modulo 6.

m	n
0	1, 3, 5
1	0, 1, 3, 4
2	3
3	any n
4	1, 3
5	0, 3

Corollary 10. *If $m \equiv 3 \pmod{6}$ and any n , or $m \equiv 1 \pmod{6}$ and $n \equiv 0, 1, 3, 4 \pmod{6}$, then a GMGD(3, 2, 3, m, n) exists.*

Proof. From Theorem 2, a GDD($m, n, 3; \lambda_1 = 0, \lambda_2 = 2$), and a GDD($n, m, 3; \mu_1 = 0, \mu_2 = 1$) exist, respectively, for each of the two cases. From Theorem 1, a BIBD($m, 3, 1$) exists for $m \equiv 1, 3 \pmod{6}$. Therefore, a GMGD(3, 2, 3, m, n) exists from Theorem 11. \square

We can also apply Theorem 4 to show the existence of a GMGD(3, 2, 3, m, n) as follows.

Theorem 12. *A GMGD(3, 2, 3, m, n) exists if the values of m and n satisfy Table 6.*

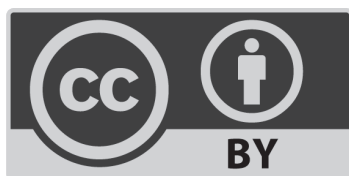
Proof. The necessary and sufficient conditions for the existence of a MGD(3, λ, m, n) for $m, n \geq 3$ in Theorem 4 are $\lambda(mn + 1 - m - n) \equiv 0 \pmod{2}$ and $\lambda mn(mn + 1 - m - n) \equiv 0 \pmod{6}$. For each of the six cases in the table above, a MGD(3, 1, m, n) exists as the necessary conditions are satisfied. We also note that a BIBD($mn, 3, 2$) exists from Theorem 1. The blocks of a BIBD($mn, 3, 2$) on mn elements together with the blocks of a MGD(3, 1, m, n) give the required GMGD. \square

Conflict of Interest

The authors declare no conflict of interest.

References

1. Lindner, C. C., and Rodger, C. A., 1997. *Design Theory*. CRC Press.
2. Fu, H. L., Rodgers, C. A., and Sarvate, D. G., 2000. The existence of group divisible designs with first and second associate having block size 3. *Ars Combinatoria*, 54, pp.33–50.
3. Ge, G., 2007. Group divisible designs. In C. J. Colbourn and J. H. Dinitz (Eds.), *Handbook of Combinatorial Design Theory* (2nd ed., pp. 255–260). CRC Press.
4. Fu, H. L., and Rodgers, C. A., 1998. Group divisible designs with two associate classes: $m = 2$ or $n = 2$. *Journal of Combinatorial Theory, Series A*, 83, pp.94–117.
5. Assaf, A. M., 1990. Modified group divisible designs. *Ars Combinatoria*, 29, pp.13–20.
6. Assaf, A. M., 1997. Modified group divisible designs with block size 4 and $\lambda > 1$. *Australian Journal of Mathematics*, 16, pp.229–238.
7. Assaf, A., and Wei, R., 1999. Modified group divisible designs with block size 4 and $\lambda = 1$. *Discrete Math.*, 195, pp.15–25.
8. Abel, J., and Assaf, A. M., 2002. Modified group divisible designs with block size 5 and $\lambda = 1$. *Discrete Mathematics*, 256, pp.1–22.
9. Danziger, P., and Wang, C., 2008. Resolvable modified group divisible designs with higher index. *Australian Journal of Mathematics*, 41, pp.37–44.
10. Ling, A. C. H., and Colbourn, C. J., 2000. Modified group divisible designs with block size 4. *Discrete Mathematics*, 219(1–3), pp.207–221.
11. Ge, G., Wang, J., and Wei, R., 2003. MGDDs with block size 4 and its application to sampling designs. *Discrete Mathematics*, 272, pp.277–283.
12. Colbourn, C.J and Dinitz, J. H., 2007. *Handbook of Combinatorial Design Theory*, Second edition, CRC Press.



©2024 the Author(s), licensee Combinatorial Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)