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On Maximum Degree and Maximum Reverse Degree Energies of Splitting and Shadow graph of Complete graph

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Abstract: In this paper, the relations of maximum degree energy and maximum reserve degree energy of a complete graph after removing a vertex have been shown to be proportional to the energy of the complete graph. The results of splitting the graph and shadow graphs are also presented for the complete graph after removing a vertex.

Keywords: Splitting graph, Shadow graph, Maximum degree energy, Maximum reverse degree energy, Graph energy, Matrices, Eigenvalues

Mathematics Subject Classification: 05C10, 05C12, 05C69

1. Introduction

The study of spectral graph theory explores connections between the algebraic characteristics of the spectra of specific matrices associated with a graph. Various matrices, including the Laplacian matrix, incidence matrix, adjacency matrix, and distance matrix, are connected to a graph.

The idea of graph energy originated in quantum chemistry in 1930 when E. Huckel introduced the chemical applications of graph theory in molecular orbital theory for the π -electron network of conjugated hydrocarbons, leading to the discovery of eigenvalues.

In [1], the distance spectrum and distance energy of some families of graphs are calculated. Hampiholi in [2] presents results for different energies for the shadow graph. In [3], the author defines maximum reverse degree energy, provides some properties, and generates results for this energy for some families. In [4], the maximum degree energy of a graph is defined, and bounds for its results are given. [5] establishes relations between the maximum energy and minimum energy of the shadow and splitting graphs of a graph. In [6], the characteristic polynomial of the minimum degree matrix of graphs obtained by certain graph operations is discussed, along with bounds for the largest minimum degree eigenvalue and minimum degree energy of graphs. [7] provides the energy of a graph class in terms of another graph class after removing a vertex. In [8], the corresponding energy of a given graph G is expressed as a multiple of various graph energies of the regular splitting graph $S(G)$. Numerous applications of graph theory in computer science and engineering are detailed in [9]. [10] proves that the splitting graph $S'(G)$ is a Zagreb hyper-energetic graph. In [11], various lower and upper bounds for graph energy $E(G)$ are derived. [12] presents bounds and characterizations on the largest eigenvalue of the Sombor matrix $S(G)$ and Sombor energy of graphs. Statistical information on the research of graph energies and their applications is provided in [13]. [14] introduces the Sombor characteristic polynomial and the Sombor energy for some graph classes. In [15], the author establishes

the relationship between energy and Sombor energy of the m -splitting graph and m -shadow graph of a k -regular graph. [16] calculates average degree eigenvalues and average degree energy for some families of graphs. In [17], Kousar and Nazeer present results for numerous graph energies of the regular subdivision graph and complete graph. The study of the energies of some classes of non-regular graphs is shown in [18].

This paper is organized as follows: Section 2 presents the results of the maximum degree energy of the splitting graph and shadow graph of a complete graph after removing a vertex. Section 3 discusses the results of the maximum reverse degree energy of the splitting graph and shadow graph of a complete graph after removing a vertex.

2. Preliminaries

Let G be a simple, undirected and finite graph and let its vertex set and edge set be denoted by $V(G) = \{v_1, v_2, v_3, \dots, v_p\}$ and $E(G) = \{e_1, e_2, e_3, \dots, e_q\}$ respectively. The number of edges associated with a vertex v of a graph G is called the degree of vertex v and is denoted by d_v or $d(v)$ [19].

Let $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k$ be the eigenvalues of G . Gutman in 1978 [20] defined the energy of the graph G as the sum of the absolute values of all eigenvalues of G . Therefore,

$$E(G) = \sum_{i=1}^k |\lambda_i|.$$

The maximum degree matrix $M_e(G) = [M_{ij}]$ [4] of G is defined as

$$[M_{ij}] = \begin{cases} \max(d_i, d_j), & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases}$$

Adiga defined the maximum degree energy $M(G)$ of a simple connected graph G as the sum of the absolute values of eigenvalues of the maximum degree matrix $M_{[ij]}$ of G .

Let $\Delta(G)$ denote the maximum degree among the vertices of G . The reverse vertex degree of a vertex v_i in G is defined as $c_{v_i} = \Delta(G) - d(v_i) + 1$ where $d(v_i)$ is the degree of vertex v_i . Then the maximum reverse degree matrix is defined as $M_R(G) = (r_{ij})$ [3], where

$$[r_{ij}] = \begin{cases} \max(c_i, c_j), & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases}$$

The maximum reverse degree energy $M_R(G)$ of a simple connected graph G is the sum of the absolute values of eigenvalues of the maximum reverse degree matrix $r_{[ij]}$ of G .

3. Maximum Degree Energy of Complete Graph

Theorem 1. For the complete graph K_n with $n \geq 4$, the maximum degree energy $M_e(K_n)$ is given by $M_e(K_n) = 2 \cdot \deg(v)^2$, where v is any vertex.

Proof. The conclusion is derived from the fundamental properties of complete graphs K_n , where each vertex shares an edge with every other vertex, resulting in a degree of $(n - 1)$ for each vertex.

Vertices ($n \geq 4$)	Degree $\deg(K_n)$	Maximum Degree Energy $M_e(K_n)$
$n = 4$	$\deg(K_4) = 3$	$M_e(K_4) = 18$
$n = 5$	$\deg(K_5) = 4$	$M_e(K_5) = 32$
$n = 6$	$\deg(K_6) = 5$	$M_e(K_6) = 50$
\vdots	\vdots	\vdots
n th	$\deg(K_n) = n - 1$	$M_e(K_n) = 2 \cdot \deg(v)^2$

□

Theorem 2. For a complete graph K_n with $n \geq 4$, the maximum degree energy $M_e(K_n)$ satisfies the relation

$$M_e[K_n] = \left(\frac{\text{deg}(v)}{\text{deg}(v) - 1} \right)^2 M_e[K_n - v].$$

Proof. Consider a complete graph K_n where each vertex shares exactly one edge with every other vertex, resulting in a degree of $(n - 1)$ for each vertex. We will prove the given relation using mathematical induction.

Basic step: Let $n = 5$, then

$$\begin{aligned} M_e[K_5] &= \left(\frac{\text{deg}(v)}{\text{deg}(v) - 1} \right)^2 M_e[K_5 - v]. \\ &= \left(\frac{\text{deg}(v)}{\text{deg}(v) - 1} \right)^2 M_e[K_4]. \\ &= \left(\frac{4}{3} \right)^2 (18) = 32 = M_e[K_5]. \end{aligned}$$

Hence, the result holds for $n = 5$.

Induction hypothesis: Assume that the result holds for $n = m$, i.e.,

$$\begin{aligned} M_e[K_m] &= \left(\frac{\text{deg}(v)}{\text{deg}(v) - 1} \right)^2 M_e[K_m - v] \\ &= \left(\frac{(m - 1)}{(m - 2)} \right)^2 M_e[K_{m-1}]. \end{aligned}$$

Induction step: Now, we prove the result for $n = (m + 1)$.

$$\begin{aligned} M_e[K_{m+1}] &= \left(\frac{m}{m - 1} \right)^2 M_e[K_m] \\ &= \left(\frac{m}{m - 1} \right)^2 \left(\frac{\text{deg}(v)}{\text{deg}(v) - 1} \right)^2 M_e[K_m - v] \\ &= \left(\frac{\text{deg}(v)}{\text{deg}(v) - 1} \right)^2 \left(\frac{m}{m - 1} \right)^2 M_e[K_{m-1}] \\ &= \left(\frac{m - 1}{m - 2} \right)^2 \left(\frac{m}{m - 1} \right)^2 M_e[K_{m-1}] \\ &= \left(\frac{m}{m - 1} \right)^2 \left(\frac{m - 1}{m - 2} \right)^2 M_e[K_{m-1}] \\ &= \left(\frac{\text{deg}(v)}{\text{deg}(v) - 1} \right)^2 M_e[K_m]. \end{aligned} \tag{1}$$

Eq. (1) demonstrates that the result holds for $n = (m + 1)$.

Since mathematical induction performs the basis and induction steps, the result holds for all $n \geq 5$.

Vertices ($n \geq 5$)	$M_e(K_n)$	$M_e(K_n - v)$
$n = 5$	$M_e(K_5) = 32$	$M_e(K_5 - v) = 18$
$n = 6$	$M_e(K_6) = 50$	$M_e(K_6 - v) = 32$
$n = 7$	$M_e(K_7) = 72$	$M_e(K_7 - v) = 50$
$n = 8$	$M_e(K_8) = 98$	$M_e(K_8 - v) = 72$
\vdots	\vdots	\vdots

□

Corollary 1. For $n \geq 5$, the difference in energy between a complete graph K_n and its vertex deletion $K_n - v$ is given by

$$M_e(K_n) - M_e(K_n - v) = 4n + 14.$$

Vertices ($n \geq 5$)	$M_e(K_n)$	$M_e(K_n - v)$	Difference
$n = 5$	$M_e(K_5) = 32$	$M_e(K_5 - v) = 18$	14
$n = 6$	$M_e(K_6) = 50$	$M_e(K_6 - v) = 32$	18
$n = 7$	$M_e(K_7) = 72$	$M_e(K_7 - v) = 50$	22
$n = 8$	$M_e(K_8) = 98$	$M_e(K_8 - v) = 72$	26
\vdots	\vdots	\vdots	\vdots

4. Maximum Degree Energy of Splitting Graph

Theorem 3. For the splitting graph $Sp(K_n - v)$, where $n \geq 4$, we have

$$M_e(Sp(K_n - v)) = 2\sqrt{5}M_e(K_{n-1}).$$

Proof. Consider a complete graph K_6 , where $K_6 - v$ denotes the deletion of a vertex from K_6 and $Sp(K_6 - v)$ represents the splitting graph. To demonstrate that

$$M_e(Sp(K_6 - v)) = 2\sqrt{5}M_e(K_{6-1}),$$

we calculate the energy of $Sp(K_6 - v)$.

$$M_e(Sp(K_6 - v)) = \begin{pmatrix} 0 & 8 & 8 & 8 & 8 & 0 & 8 & 8 & 8 & 8 \\ 8 & 0 & 8 & 8 & 8 & 8 & 0 & 8 & 8 & 8 \\ 8 & 8 & 0 & 8 & 8 & 8 & 8 & 0 & 8 & 8 \\ 8 & 8 & 8 & 0 & 8 & 8 & 8 & 8 & 0 & 8 \\ 8 & 8 & 8 & 8 & 0 & 8 & 8 & 8 & 8 & 0 \\ 0 & 8 & 8 & 8 & 8 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 8 & 8 & 8 & 0 & 0 & 0 & 0 & 0 \\ 8 & 8 & 0 & 8 & 8 & 0 & 0 & 0 & 0 & 0 \\ 8 & 8 & 8 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\ 8 & 8 & 8 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial of $M_e(Sp(K_6 - v))$ is:

$$\lambda^{10} - 1920\lambda^8 - 40960\lambda^7 - 20480\lambda^6 + 5111808\lambda^5 + 13107200\lambda^4 - 293601280\lambda^3 - 25168240\lambda^2 + 8053063680\lambda - 17179869184 = 0.$$

The eigenvalues are:

$$\lambda_1 = 51.7771, \quad \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 4.9443, \quad \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = -12.9443, \quad \lambda_{10} = -19.7771.$$

Thus, the spectral matrix $\text{Spec}(M_e(Sp(K_6 - v)))$ is:

$$\begin{pmatrix} -19.7771 & -12.9443 & 4.9443 & 51.7771 \\ 1 & 4 & 4 & 1 \end{pmatrix}$$

Hence, $M_e(Sp(K_6 - v)) = 143.1084 = 2\sqrt{5}M_e(Sp(K_5)) = 2\sqrt{5}M_e(K_6 - v)$.

$M_e(K_n)$	$M_e(\text{Sp}(K_n - v))$	$M_e(K_n - v) = 2\sqrt{5}M_e(K_{n-1})$
$M_e(K_6) = 50$	$M_e(\text{Sp}(K_6 - v)) = 143.1084$	$M_e(\text{Sp}(K_6 - v)) = 2\sqrt{5}(32)$
$M_e(K_7) = 72$	$M_e(\text{Sp}(K_7 - v)) = 223.6068$	$M_e(\text{Sp}(K_7 - v)) = 2\sqrt{5}(50)$
$M_e(K_8) = 98$	$M_e(\text{Sp}(K_8 - v)) = 321.9938$	$M_e(\text{Sp}(K_8 - v)) = 2\sqrt{5}(72)$
$M_e(K_9) = 128$	$M_e(\text{Sp}(K_9 - v)) = 438.2698$	$M_e(\text{Sp}(K_9 - v)) = 2\sqrt{5}(98)$
\vdots	\vdots	\vdots

□

5. Maximum Degree Energy of Shadow Graph

Theorem 4. For a complete graph K_n with $n \geq 6$, the maximum degree energy of its shadow graph is given by

$$M_e(D(K_n - v)) = 4M_e(K_{n-1}).$$

Proof. Consider a complete graph K_6 . Let $K_6 - v$ denote the deletion of a vertex from K_6 , and let $D(K_6 - v)$ represent its shadow graph. We aim to demonstrate that

$$M_e(D(K_6 - v)) = 4M_e(K_6 - v).$$

The maximum degree energy matrix $M_e(D(K_6 - v))$ for $K_6 - v$ is as follows:

$$\begin{bmatrix} 0 & 8 & 8 & 8 & 8 & 0 & 8 & 8 & 8 & 8 \\ 8 & 0 & 8 & 8 & 8 & 8 & 0 & 8 & 8 & 8 \\ 8 & 8 & 0 & 8 & 8 & 8 & 8 & 0 & 8 & 8 \\ 8 & 8 & 8 & 0 & 8 & 8 & 8 & 8 & 0 & 8 \\ 8 & 8 & 8 & 8 & 0 & 8 & 8 & 8 & 8 & 0 \\ 0 & 8 & 8 & 8 & 8 & 0 & 8 & 8 & 8 & 8 \\ 8 & 0 & 8 & 8 & 8 & 8 & 0 & 8 & 8 & 8 \\ 8 & 8 & 0 & 8 & 8 & 8 & 8 & 0 & 8 & 8 \\ 8 & 8 & 8 & 0 & 8 & 8 & 8 & 8 & 0 & 8 \\ 8 & 8 & 8 & 8 & 0 & 8 & 8 & 8 & 8 & 0 \end{bmatrix}$$

The characteristic polynomial of $M_e(D(K_6 - v))$ is

$$\lambda^{10} - 2560\lambda^8 - 8190\lambda^7 - 983040\lambda^6 - 4194304\lambda^5 = 0.$$

Its eigenvalues are $\lambda_1 = 64$, $\lambda_2 = 8.5850e^{-15}$, $\lambda_3 = 5.3785e^{-15}$, $\lambda_4 = 3.8317e^{-15}$, $\lambda_5 = 1.1914e^{-15}$, $\lambda_6 = -4.3013e^{-15}$, $\lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = -16$. Thus,

$$\text{Spec}(M_e(D(K_6 - v))) = \begin{pmatrix} -16 & -4.3013e^{-15} & 1.1914e^{-15} & 3.8317e^{-15} & 5.3785e^{-15} & 8.5850e^{-15} & 64 \\ 4 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Therefore, $M_e(D(K_6 - v)) = 128 = 4M_e(K_5) = 4M_e(K_6 - v)$.

The Table below demonstrates the relationship between the maximum degree energies of the complete graph and its shadow graph after vertex deletion:

$M_e(K_n)$	$M_e(D(K_n - v))$	$M_e(D(K_n - v)) = 4M_e(K_{n-1})$
$M_e(K_6) = 50$	$M_e(D(K_6 - v)) = 128$	$M_e(D(K_6 - v)) = 4(32)$
$M_e(K_7) = 72$	$M_e(D(K_7 - v)) = 200$	$M_e(D(K_7 - v)) = 4(50)$
$M_e(K_8) = 98$	$M_e(D(K_8 - v)) = 288$	$M_e(D(K_8 - v)) = 4(72)$
$M_e(K_9) = 128$	$M_e(D(K_9 - v)) = 392$	$M_e(D(K_9 - v)) = 4(98)$
\vdots	\vdots	\vdots

□

6. Maximum Reverse Degree Energy of Complete Graph

Theorem 5. For the complete graph K_n with $n \geq 4$, we have $M_R[K_n] = 2(n-1)$, where v is any vertex.

Proof. The conclusion follows from the definition of a complete graph K_n , where each vertex has the same degree $(n-1)$.

Vertices ($n \geq 4$)	$\deg(K_n)$	$M_R(K_n)$
$n = 4$	$\deg(K_4) = 3$	$M_R(K_4) = 6$
$n = 5$	$\deg(K_5) = 4$	$M_R(K_5) = 8$
$n = 6$	$\deg(K_6) = 5$	$M_R(K_6) = 10$
\vdots	\vdots	\vdots
n th	$\deg(K_n) = n - 1$	$M_R(K_n) = 2(n - 1)$

□

7. Maximum Reverse Degree Energy of Complete Graph

Theorem 6. For a complete graph K_n where $n \geq 4$,

$$M_R[K_n] = \frac{\deg(v)}{\deg(v) - 1} M_R[K_n - v].$$

Proof. Consider a complete graph K_n where each vertex has degree $(n-1)$, and exactly one edge is shared by every pair of vertices. We utilize the principle of mathematical induction to prove this conclusion.

Base step: Let $n = 5$. Then,

$$\begin{aligned} M_R[K_5] &= \frac{\deg(v)}{\deg(v) - 1} M_R[K_5 - v] \\ &= \frac{\deg(v)}{\deg(v) - 1} M_R[K_5 - v] \\ &= \frac{\deg(v)}{\deg(v) - 1} M_R[K_4] \\ &= \frac{4}{3}(6) = 8 = M_R[K_5]. \end{aligned}$$

Hence, the result is true for $n = 5$.

Induction hypothesis: Assume that the result holds for $n = m$. Then,

$$\begin{aligned} M_R[K_m] &= \frac{\deg(v)}{\deg(v) - 1} M_R[K_m - v] \\ &= \frac{\deg(v)}{\deg(v) - 1} M_R[K_{m-1}] \\ &= \frac{(m-1)}{(m-2)} M_R[K_{m-1}]. \end{aligned}$$

Induction step: Now, we prove the result for $n = (m+1)$. That is,

$$M_R[K_{m+1}] = \frac{\deg(v)}{\deg(v) - 1} M_R[K_{m+1} - v].$$

Consider

$$M_R[K_{m+1}] = \frac{m}{m-1} M_R[K_m]$$

$$\begin{aligned}
 &= \left(\frac{m}{m-1}\right) \frac{\text{deg}(v)}{\text{deg}(v)-1} M_R[K_m - v] \\
 &= \frac{\text{deg}(v)}{\text{deg}(v)-1} \left(\frac{m}{m-1}\right) M_R[K_{m-1}] \\
 &= \left(\frac{m-1}{m-2}\right) \left(\frac{m}{m-1}\right) M_R[K_{m-1}] \\
 &= \left(\frac{m}{m-1}\right) \left(\frac{m-1}{m-2}\right) M_R[K_{m-1}] \\
 &= \frac{\text{deg}(v)}{\text{deg}(v)-1} M_R[K_m].
 \end{aligned} \tag{2}$$

The Eq. (2) demonstrates that the result is accurate for $n = (m + 1)$.

Eq. (2) is true because mathematical induction performs the basis and induction stages. Hence, the result holds for all $n \geq 5$.

Vertices ($n \geq 5$)	$M_R(K_n)$	$M_R(K_n - v)$
$n = 5$	$M_R(K_5) = 8$	$M_R(K_5 - v) = 6$
$n = 6$	$M_R(K_6) = 10$	$M_R(K_6 - v) = 8$
$n = 7$	$M_R(K_7) = 12$	$M_R(K_7 - v) = 10$
$n = 8$	$M_R(K_8) = 14$	$M_R(K_8 - v) = 12$
\vdots	\vdots	\vdots

□

Corollary 2. *The energy difference between a complete graph K_n and its vertex deletion $K_n - v$ is equal to 2 for $n \geq 5$:*

$$M_R(K_n) - M_R(K_n - v) = 2.$$

Vertices ($n \geq 5$)	$M_R(K_n)$	$M_R(K_n - v)$	Difference
$n = 5$	$M_R(K_5) = 8$	$M_R(K_5 - v) = 6$	2
$n = 6$	$M_R(K_6) = 10$	$M_R(K_6 - v) = 8$	2
$n = 7$	$M_R(K_7) = 12$	$M_R(K_7 - v) = 10$	2
$n = 8$	$M_R(K_8) = 14$	$M_R(K_8 - v) = 12$	2
\vdots	\vdots	\vdots	\vdots

8. Maximum Reverse Degree Energy of Splitting Graph

Theorem 7. *For a complete graph K_n where $n \geq 4$,*

$$M_R(\text{Sp}(K_n - v)) = \sqrt{4n^2 + 1} \cdot M_R(K_{n-1}).$$

Proof. Consider a complete graph K_6 , where $K_6 - v$ is the deletion of a vertex from K_6 , and $\text{Sp}(K_6 - v)$ is a splitting graph. We aim to prove that

$$M_R(\text{Sp}(K_6 - v)) = \sqrt{4n^2 + 1} \cdot M_R(K_{6-1}).$$

$$M_R(\text{Sp}(K_6 - v)) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 5 & 5 & 5 & 5 \\ 1 & 0 & 1 & 1 & 1 & 5 & 0 & 5 & 5 & 5 \\ 1 & 1 & 0 & 1 & 1 & 5 & 5 & 0 & 5 & 5 \\ 1 & 1 & 1 & 0 & 1 & 5 & 5 & 5 & 0 & 5 \\ 1 & 1 & 1 & 1 & 0 & 5 & 5 & 5 & 5 & 0 \\ 0 & 5 & 5 & 5 & 5 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 5 & 5 & 5 & 0 & 0 & 0 & 0 & 0 \\ 5 & 5 & 0 & 5 & 5 & 0 & 0 & 0 & 0 & 0 \\ 5 & 5 & 5 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 5 & 5 & 5 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial of $M_R(\text{Sp}(K_6 - v))$ is

$$\lambda^{10} - 510\lambda^8 - 1520\lambda^7 + 42235\lambda^6 + 111996\lambda^5 - 1468750\lambda^4 - 2787500\lambda^3 + 24140625\lambda^2 + 23437500\lambda - 15620000 = 0.$$

Its eigenvalues are

$$\begin{aligned} \lambda_1 &= 22.0998, \\ \lambda_2 &= \lambda_3 = \lambda_4 = \lambda_5 = 4.5249, \\ \lambda_6 &= \lambda_7 = \lambda_8 = \lambda_9 = -5.5249, \\ \lambda_{10} &= -18.0998. \end{aligned}$$

$$\text{Spec}(M_e(\text{Sp}(K_6 - v))) = \begin{pmatrix} -18.0998 & -5.5249 & 4.5249 & 22.0998 \\ 1 & 4 & 4 & 1 \end{pmatrix}$$

Thus,

$$M_R(\text{Sp}(K_6 - v)) = 80.3990 = \sqrt{4n^2 + 1} \cdot M_R(K_5) = \sqrt{4n^2 + 1} \cdot M_R(K_6 - v).$$

$M_R(K_n)$	$M_R(\text{Sp}(K_n - v))$	$M_R(K_n - v) = \sqrt{4n^2 + 1} \cdot M_R(K_{n-1})$
$M_R(K_6) = 10$	$M_R(\text{Sp}(K_6 - v)) = 80.3990$	$M_R(\text{Sp}(K_6 - v)) = \sqrt{4n^2 + 1} \cdot (8)$
$M_R(K_7) = 12$	$M_R(\text{Sp}(K_7 - v)) = 120.4159$	$M_R(\text{Sp}(K_7 - v)) = \sqrt{4n^2 + 1} \cdot (10)$
$M_R(K_8) = 14$	$M_R(\text{Sp}(K_8 - v)) = 168.4280$	$M_R(\text{Sp}(K_8 - v)) = \sqrt{4n^2 + 1} \cdot (12)$
$M_R(K_9) = 16$	$M_R(\text{Sp}(K_9 - v)) = 224.4371$	$M_R(\text{Sp}(K_9 - v)) = \sqrt{4n^2 + 1} \cdot (14)$
\vdots	\vdots	\vdots

□

9. Maximum Reverse Degree Energy of Shadow Graph

Theorem 8. For a complete graph K_n where $n \geq 4$,

$$M_R(D(K_n - v)) = 2n \cdot M_R(K_{n-1}).$$

Proof. Consider a complete graph K_6 , where $K_6 - v$ is the deletion of a vertex from K_6 , and $D(K_6 - v)$ is a shadow graph. We aim to prove that

$$M_R(D(K_6 - v)) = 2n \cdot M_R(K_6 - v).$$

$$M_R(D(K_6 - v)) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 5 & 5 & 5 & 5 \\ 1 & 0 & 1 & 1 & 1 & 5 & 0 & 5 & 5 & 5 \\ 1 & 1 & 0 & 1 & 1 & 5 & 5 & 0 & 5 & 5 \\ 1 & 1 & 1 & 0 & 1 & 5 & 5 & 5 & 0 & 5 \\ 1 & 1 & 1 & 1 & 0 & 5 & 5 & 5 & 5 & 0 \\ 0 & 5 & 5 & 5 & 5 & 0 & 1 & 1 & 1 & 1 \\ 5 & 0 & 5 & 5 & 5 & 1 & 0 & 1 & 1 & 1 \\ 5 & 5 & 0 & 5 & 5 & 1 & 1 & 0 & 1 & 1 \\ 5 & 5 & 5 & 0 & 5 & 1 & 1 & 1 & 0 & 1 \\ 5 & 5 & 5 & 5 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of $M_R(D(K_6 - v))$ is

$$\lambda^{10} - 520\lambda^8 - 3040\lambda^7 + 34320\lambda^6 + 203392\lambda^5 - 1036800\lambda^4 - 4792320\lambda^3 + 17141760\lambda^2 + 39813120\lambda - 127401984 = 0.$$

Its eigenvalues are

$$\lambda_1 = 24, \quad \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 4, \quad \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = -6, \quad \lambda_{10} = -16.$$

$$\text{Spec}(M_R(D(K_6 - v))) = \begin{pmatrix} -16 & -6 & 4 & 24 \\ 1 & 4 & 4 & 1 \end{pmatrix}$$

Thus,

$$M_R(D(K_6 - v)) = 80 = 2n \cdot M_R(K_5) = 2n \cdot M_R(K_6 - v).$$

$M_R(K_n)$	$M_R(D(K_n - v))$	$M_R(D(K_n - v)) = 2n \cdot M_R(K_{n-1})$
$M_R(K_6) = 10$	$M_R(D(K_6 - v)) = 80$	$M_R(D(K_6 - v)) = 2n \cdot (8)$
$M_R(K_7) = 12$	$M_R(D(K_7 - v)) = 120$	$M_R(D(K_7 - v)) = 2n \cdot (10)$
$M_R(K_8) = 14$	$M_R(D(K_8 - v)) = 168$	$M_R(D(K_8 - v)) = 2n \cdot (12)$
$M_R(K_9) = 16$	$M_R(D(K_9 - v)) = 224$	$M_R(D(K_9 - v)) = 2n \cdot (14)$
\vdots	\vdots	\vdots

□

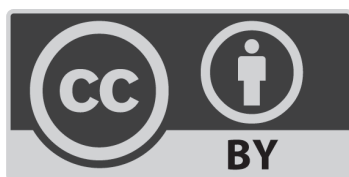
10. Conclusion

In this study, we have examined the family of complete graphs and their associated splitting and shadow graphs to investigate their energies after the deletion of a vertex. Specifically, we have established the relationship between the maximum degree energy and the maximum reverse degree energy of complete graphs and their splitting and shadow graphs for vertex deletions.

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