Utilitas Mathematica, 118: 63–67
DOI:10.61091/um118-06
http://www.combinatorialpress.com/um
Received 19 November 2023, Accepted 29 December 2023, Published 8 January 2024

Article

On $SCDF$-Modules

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Abstract: A module M over a commutative ring is termed an $SCDF$-module if every Dedekind finite object in $\sigma[M]$ is finitely cogenerated. Utilizing this concept, we explore several properties and characterize various types of $SCDF$-modules. These include local $SCDF$-modules, finitely generated $SCDF$-modules, and hollow $SCDF$-modules with $Rad(M) = 0 \neq M$. Additionally, we examine QF $SCDF$-modules in the context of duo-ring.

Keywords: Dedekind finite, Finitely cogenerated, $SCDF$-modules, Hollow modules, QF-modules
Mathematics Subject Classification: 13C05, 13C60.

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1. Introduction

Throughout this paper, all rings are assumed to be either commutative or duo-rings with $1 \neq 0$. A module M over a commutative ring is termed Dedekind finite (respectively, finitely cogenerated) if every monomorphism $f : M \longrightarrow M$ is an automorphism (respectively, if $Soc(M)$ is essential and finitely generated). Although any finitely cogenerated module is Dedekind finite, the converse is not generally true. In light of this fact, we utilize the $\sigma[M]$ category to introduce the concept of an $SCDF$-module, which is a generalization of the $SCDF$-ring. We define a non-zero R-module M as hollow if every proper submodule of it is superfluous. The socle of a module M, denoted as $Soc(M)$, is defined as the sum of its minimal non-zero submodules. Conversely, the radical of M is the intersection of all its maximal submodules. Let C be a subcategory of R-Mod. A module N in C is called finitely presented (for shot f.p) in C if:

1. N is finitely generated and
2. Every exact sequence $0 \to K \to L \to N \to 0$ in C, with L finitely generated, K is also finitely generated.

Let M be an R-module. A module $N \in \sigma[M]$ is called coherent in $\sigma[M]$ if

1. N is finitely generated and
2. any finitely generated submodule of $N$is finitely presented in $\sigma[M]$.

A non-zero module M is called a hollow module if every proper submodule of M is a small submodule of M. Let M be a faithful R-module. We say that M is a Quasi-Frobenius (in short QF) module if $Hom_R(P, M)$ is either zero or a simple R-module for each simple R-module P.

2. Some properties of $SCDF$-modules

Proposition 1. *1. Epimorphic image of $SCDF$-module is a $SCDF$-module;*

2. *If M is a product of modules M_i, $1 \leq i \leq n$ is a $SCDF$-module. Then so is every M_i;*
3. *Moreover if $Hom(M_i, M_j) = 0$ for all $1 \leq i \neq j \leq n$, then the converse of (2) is true;*
4. *Every factor module of $SCDF$-module is a $SCDF$-module.*

Proof. 1. Let M be a $SCDF$-module and $M^{'} = f(M)$ a homomorphic image of M, then $Gen(M^{'})$ is in $Gen(M)$ [4]. This implies that $\sigma[M^{'}]$ is a full subcategory of $\sigma[M]$. Hence $M^{'}$ is a $SCDF$-module of finite.

2. Results from (1).
3. Suppose that every M_i for $1 \leq i \leq n$ is a $SCDF$-module. As $Hom(M_i, M_j) = 0$ for M_i for $1 \leq i \neq j \leq n$, then by Proposition 2.2 of [1], for every $N \in \sigma[\prod_{i=1}^{n} M_i]$ there is a unique $N_i \in \sigma[M_i]$ $1 \leq i \leq n$ such that $N = \prod_{i=1}^{n} N_i$. If N is a Dedekind finite, N_i is also a Dedekind finite for all $1 \leq i \leq n$ because if a module is a Dedekind finite, then so is any direct summand of that module. Since M_i is a $SCDF$-module, then N_i is finitely cogenerated for all $1 \leq i \leq n$. Hence $N = \prod_{i=1}^{n} N_i$ is finitely cogenerated. Thus, M is a $SCDF$-module.
4. Let M a $SCDF$-module and N a submodule of M. Then by proposition (15.1) of [2], $M/N \in \sigma[M]$. Let $K \in \sigma[M/N]$, K is also belongs $\sigma[M]$. Wich implies K is finitely cogenerated. Therefore M/N is a $SCDF$-module.

□

Lemma 1. *(15.4 of [2])*

1. *If a R-module M is finitely generated as a module over $S = End_R(M)$ then $\sigma[M] = R/Ann(M)$-MOD.*
2. *If R is commutative, then for every finitely generated R-module we have $\sigma[M] = R/Ann(M)$-MOD.*

Lemma 2. *(Theorem 2.1 of [3])*
Let R be a commutative ring. Then the following statements are equivalent:

1. *R is an artinian principal ideal ring*
2. *R is a $SCDF$-ring.*

Proposition 2. *Let M a local R-module over $S = End(M)$. If M is $SCDF$-module, then M is coherent.*

Proof. If M is local over S, then M is finitely generated over $S = End(M)$. Referring to the first point of Lemma 1 and Lemma 2 $M \cong R/Ann(M)$ and $R/Ann(M)$ is artinian. It is well know that any artinian ring is noetherian. Therefore M coherent because every noetherian module is coherent.

□

Recall an R-module M is called locally artinian if every finitely generated module in $\sigma[M]$ is artinian.

Lemma 3. *(41.4 of [2])*
Let M be a non-zero R-module. The following are equivalent:

1. *M is hollow module and $Rad(M) \neq M$*
2. *M is local.*

Proposition 3. *Let R commutative ring and M a hollow module with $Rad(M) = 0 \neq M$. If M is a $SCDF$-module, then M is locally artinian.*

Proof. Let N be a finitely generated module in $\sigma[M]$. Successively by Lemmas 3, 1 and 2, M is artinian. In addition, $Rad(M) = 0$ implies by referring to [4] proposition 10.15 M is semisimple and noetherian. If M is semisimple, every module of $\sigma[M]$ is also semisimple. By [4] Corollary 10.16 finitely generated and artinian are equivalent. Therefore M is locally artinian.

□

3. Characterization theorems

Lemma 4. *Suppose a ring R has zero nilradical. R is Dedekind finite if and only if R is artinian.*

Definition 1. *A non empty set S of a ring R is said to be a mutiplicatively closed subset (briefly m.c.s) if $1 \in S$ and $ab \in S$ for each $a, b \in S$.*

Remark 1. *We denote the set of all prime and maximal ideals of R by $Spec(R)$ and $Max(R)$ respectively.*

A ring R is called a S-artinian if for each descending chain of ideals $\{I_i\}_{i\in\mathbf{N}}$ of R these exist $s \in S$ and $k \in \mathbf{N}$ such that $sI_k \subseteq I_n$ for all $n \geq k$.

Lemma 5 (By example 3 of [5])**.** *Every artinian R-module M is a S-artinian module where $S \subseteq R$ is m.c.s.*

Definition 2. *Let M be a R-module. A propoer submodule P of M is said to be prime if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$.*
M is said to be reduced if intersection of all prime submodules of M is equal to zero.

If N is a submodule of a R-module, then $cl(N) = \{m \in M, mI \subseteq N$for some large left ideal of $R\}$. If $cl(N) = N$, then N is said to be closed.

Definition 3. *A submodule N of a R-module M is termed closed prime provided the following two conditions are satisfied:*

1. *if N' is a submodule such that $N \subset N' \subseteq M$, then $(N : N') \subset (N : M)$.*
2. *$cl(N) = M$*

Theorem 1. *Let M be a finitely generated module over commutative ring. The following properties are equivalent.*

1. *M is a $SCDF$-module*
2. *M is artinian*
3. *M is P-artinian for each $P \in Spec(R)$*
4. *M is μ-artinian for each $\mu \in Max(R)$.*

Proof. (1) ⇒ (2) Result from Proposition 1
(2) ⇔ (3) ⇔ (4) these equivalences follow the theorem 2 [5]
(2) ⇒ (1) Let N be a Dedekind finite object of $\sigma[M]$. There is an epimorphism $\varphi : M^{(\Lambda)} \longrightarrow K$ such that N is a submodule of K. Since M is finitely generated then $card(\Lambda)$ is finite. The first theorem of isomorphism implies $M^{(\Lambda)}/ker\varphi \simeq K$.
M artinian, $card(\Lambda)$ finite implies $M^{(\Lambda)}$ is also artinian. Since artinian modules are stable of submodules and factor modules, then K and N are artinian. Since N is artinian, has a simple submodule, in fact $Soc(N)$ is an essential submodule.
Now let's prouve that $Soc(N)$ is finitely generated.
M finitely generated implies $\sigma[M] \simeq R/Ann(M)$-MOD. Hence every object of $\sigma[M]$ is a $R/Ann(M)$-module therefore $R/Ann(M)$ is an artinian ring. In addition for an artinian ring, finitely generated is equivalent to artinian. Then N is finitely generated. $Soc(N)$ is also finitely generated because every submodule of artinian finitely generated module is finitely generated. □

Theorem 2. *Let M be an hollow module and $Rad(M) \neq M$. The following are equivalent:*

1. *M is a $SCDF$-module*
2. *M is locally artinian*
3. *Every cyclic module in $\sigma[M]$is a direct sum of a self-projective M-injective module and a finitely cogenerated module.*

Proof. (1) ⇒ (2) Result of Proposition 3
(2) ⇒ (1) M locally artinian and M finitely generated implies M artinian. In addition, $Rad(M) = 0$ implies M is semisimple. Then $M = \bigoplus_i M_i$ is a direct sum of a finite set of simple modules. $N \in \sigma[M]$, then N is also semisimple and finite lenght. Therefore N is artinian and noetherian. For a semisimple module, artinian, noetherian and finitely cogenerated coincide.
(2) ⇔ (3) Results from 3.Theorem of [6]. □

Recall a R-module M is uniform if every non-zero submodule of M is an essential submodule.

Definition 4. *Let M be a module such that every module in $\sigma[M]$ is a direct sum of uniform modules. Then we will say that M is an SU-module.*

Definition 5. *A module M is said to be pure-semisimple if every module in $\sigma[M]$ is a direct sum of finitely presented modules.*

Theorem 3. *Let R be a commutative ring and M a local R-module. Then the following are equivalent:*

1. *M is a $SCDF$-module;*
2. *M is a SU-module;*
3. *M is pure-semisimple;*

Proof. Referring Lemmas 1 and 2 $M \simeq R/Ann(M)$ is an artinian ideal principal ring. Basing on [7], artinian with principal ideal and uniserial coincide. Therefore M is uniserial. In addition, it is well known that every uniserial module is serial. Basing on Theorem 5.2.1 from [8] we have the equivalences. □

Definition 6. *A uniserial module M is said to be homo-uniserial if whenever A, B, C and D are submodules of M such that A and C are maximal submodules of B and D respectively, then $B/A \simeq D/C$.*

Corollary 1. *Let M be a module over a commutative ring R. Then the following are equivalent:*

1. *M is a $SCDF$-module;*
2. *Every module in $\sigma[M]$ is a direct sum of homo-uniserial modules.*
3. *Every module in $\sigma[M]$ is a direct sum of homo-uniserial modules of finite length.*

Proof. Results from Theorem 3 and Theorem 5.2.4 of [8] □

Now, we suppose that R is a duo-ring. It is a ring such that every one-sided ideal is two-sided ideal.We have the following theorem.

Theorem 4. *Let R be a duo-ring and M faithfully balanced left finite generated R-module. Assume $soc(M)$ is square-free. then the following statements are equivalent*

1. *M is a $SCDF$-module;*
2. *M is a QF-module;*
3. *$M \simeq M_1 \times M_2 \times ...... \times M_s$ where each M_i is a local artinian module with a simple socle.*

Proof. Assume $S = End(M)$ the ring of endomorphism of M. If M is an R-module, then M is an S-module. Thus it follows from the first point of lemma 1 $\sigma[M] = R/Ann(M)$-Mod. As $R/Ann(M)$ is a duo ring, and M is isomorphic to $R/Ann(M)$ and from Theorem 9 of [9] M is artinian .Since M is also square-free,it results from Theorem 15.27 of [10] that the statements are equivalent. □

Corollary 2. *Let R be a duo-ring and M faithfully balanced left finite generated R-module. Assume soc(M) is square-free. then the following statements are equivalent*

1. *M is a $SCDF$-module;*
2. *M is a QF-module;*
3. *M is self injective;*
4. *M is a cogenerator;*

Proof. The corollary results from Theorem 4 and theorem 30.7 from [4] □

Conflict of Interest

The authors declare no conflict of interest.

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