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Article

On the Laplacian Energy of an Orbit Graph of Finite Groups

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Abstract: Let β_H denote the orbit graph of a finite group H. Let ζ be the set of commuting elements in H with order two. An orbit graph is a simple undirected graph where non-central orbits are represented as vertices in ζ, and two vertices in ζ are connected by an edge if they are conjugate. In this article, we explore the Laplacian energy and signless Laplacian energy of orbit graphs associated with dihedral groups of order $2w$ and quaternion groups of order 2^w.

Keywords: Laplacian energy, Signless laplacian energy, Orbit graph, Finite group
Mathematics Subject Classification: 05C10, 05C12, 05C69

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1. Introduction

In this manuscript, we are concerned with the orbit graph of some finite group H. Let β be a graph, possessing a subset of commuting elements of order two in H, denoted by ζ. The vertices of the graph represent non-central orbits in ζ, and two vertices have an edge between them if they are conjugate.

Let $D(\beta)$ and $A(\beta)$ be the degree matrix and adjacency matrix of the graph β respectively. Then $L(\beta) = D(\beta) - A(\beta)$ and $Q(\beta) = D(\beta) + A(\beta)$ are the Laplacian matrix and signless Laplacian matrix of β respectively. Let d_r be the degree of the r-th vertex of β, where $r = 1, 2, \ldots, n$.

The spectrum of the graph β is a multiset given by $\mathrm{Spec}(\beta) = \{\alpha_1^{j_1}, \alpha_2^{j_2}, \ldots, \alpha_l^{j_l}\}$, where $\alpha_1, \alpha_2, \ldots, \alpha_l$ are the eigenvalues of $A(\beta)$ having multiplicities $j_1, j_2, \ldots, j_l$ respectively. Similarly, the Laplacian and signless Laplacian spectrum of β is given by $L - \mathrm{Spec}(\beta) = \{\eta_1^{t_1}, \eta_2^{t_2}, \ldots, \eta_m^{t_m}\}$, and $Q - \mathrm{Spec}(\beta) = \{\lambda_1^{s_1}, \lambda_2^{s_2}, \ldots, \lambda_n^{s_n}\}$ respectively, where $\eta_1, \eta_2, \ldots, \eta_m$ are representing the eigenvalues of $L(\beta)$ having multiplicities $t_1, t_2, \ldots, t_m$ where as $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $Q(\beta)$ having multiplicities $s_1, s_2, \ldots, s_n$ respectively.

A graph's energy can be expressed as energy $\xi(\beta)$, laplacian energy $L.E(\beta)$, or signless laplacian energy $L.E^+(\beta)$, depending on its different spectra. In 1978, Gutman [1] defined the energy of a graph as

$$\xi(\beta) = \sum_{\alpha \in \mathrm{Spec}(\beta)} |\alpha| \,.$$

The concept of laplacian energy of a graph was initially coined by Gutman and Zhou [2] and is defined as

$$L.E(\beta) = \sum_{\mu \in L-\text{Spec}(\beta)} \left| \mu - \frac{2|E(\beta)|}{|V(\beta)|} \right|.$$

Similarly, Cvetkovic et al. [3], introduced the signless laplacian energy as

$$L.E^{+}(\beta) = \sum_{\lambda \in \text{Spec}(\beta)} \left| \lambda - \frac{2|E(\beta)|}{|V(\beta)|} \right|,$$

where $V(\beta)$ and $E(\beta)$ represents the set of vertices of β and set of edges of β respectively.

Zhou [4] studied about an upper bound for the graph's energy in terms the number of edges, the number of vertices, and the number of zero eigenvalues. Some results about cospectral graphs and equienergetic graphs were examined by Balakrishnan [5]. In [6], Dutta et al., introduced the concept on laplacian energy for non-commuting graphs of groups having finite number of elements. Then Mahmoud et al. in [7], introduced the concept of laplacian energy on conjugacy class graph of certain finite groups. Sharma and Bhat [8] studied the orbit and conjugacy class graphs for some dihedral groups. Pirzada et al. [9] determine the Laplacian as well as signless Laplacian eigenvalues for graphs like unitary cayley graph of a commutative ring. Sharma and Bhat [10] also examined topological indices for the orbit graphs of dihedral groups. Recently, Nath et al. in [11], introduced the concept on various energies for commuting graphs of finite non-abelian groups. Das et al. [12] introduced the concepts of distance Laplacian energy as well as distance signless Laplacian energy for a connected graph Orbital regular graphs using a regular action for some solvable groups with finite number of elements has been studied by Sharma et al. [13].

The main purpose of this work is to compute the formulas for the laplacian energy and signless laplacian energy of orbit graphs of some finite groups which are dihedral groups and quaternian groups. Based on the orbits of the group's elements, the results presented in this study revealed more group characteristics and classifications. Here, we also observe that the laplacian energy is equal to signless laplacian energy.

2. Preliminaries

Some fundamental concepts of an orbit graph that are used in this manuscript are presented in this section.

Definition 1. *(The set ζ). The set ζ consists of all pairs of commuting elements of H that are of the form (g, h) where g and h are elements of the metabelian groups having a finite number of elements and the least common multiple of the order of the elements is two, i.e.,*

$$\zeta = \{(g, h) \in H \times H \mid gh = hg, g \neq h, lcm(|g|, |h|) = 2\}.$$

Definition 2. [14]*(Orbit). Consider a group H having a finite number of elements which acts on set ζ, and $u \in \zeta$. The orbit of u, denoted by $O(u)$, is the subset defined as $O(u) = \{hu \mid h \in H, u \in \zeta\}$. In this paper, the group action is considered as a conjugation action. Hence, the orbit is defined as*

$$O(u) = \{huh^{-1} \mid h \in H, u \in \zeta\}.$$

Definition 3. [15]*(Orbit Graph, β_H^{ζ}). Let H be a metabelian group and ζ be a set. An orbit graph is denoted by β_H^{ζ} and can be defined as a graph where non-central orbits under the group action on the set ζ represent vertices, i.e., $|V(\beta_H^{\zeta})| = |\zeta| - |J|$, where ζ is a disjoint union of distinct orbits and $J = \{v \in H | vh = hv,\ h \in H\}$. In the orbit graph, two vertices x_1 and x_2 are adjacent if x_1 and x_2 are conjugate, i.e., $x_1 = h^{x_2}$.*

Theorem 1. [15] *If H is a dihedral group having $2w$ number of elements and H acts on ζ by conjugation, then*

$$\beta_H^{\zeta} = \begin{cases} \bigcup_{i=1}^{5} K_{\frac{w}{2}} i, & \text{if } k \text{ is even and } \frac{w}{2} \text{ is odd,} \\ \left(\bigcup_{i=1}^{4} K_{\frac{w}{2}} i\right) \cup \left(\bigcup_{i=1}^{2} K_{\frac{w}{4}} i\right), & \text{if } w \text{ and } \frac{w}{2} \text{ are odd,} \\ K_w, & \text{if } w \text{ is odd.} \end{cases}$$

Theorem 2. [15] *Let H be a quaternion group having 2^w number of elements. If H acts on ζ by conjugation, then β_H^{ζ} is an empty or null graph.*

Proposition 1. [16] *The number of connected components in the graph is the multiplicity of* 0 *as the eigenvalue of $L(\beta)$.*

Proposition 2. [17] *The eigenvalues of the Laplacian matrix of the complete graph, K_n, are n with multiplicity $n-1$ and* 0 *with multiplicity* 1.

3. Main Results

The Laplacian energy as well as signless Laplacian energy of an orbit graph of some dihedral groups having order $2w$ and quaternion group of order 2^w have been studied in this particular section.

Theorem 3. *Let $D_{2w} = \langle s, t \mid s^w = t^2 = 1, tst^{-1} = s^{-1}\rangle$ be a dihedral group having order $2w$, where w is even and $\frac{w}{2}$ is odd, and let $\beta_{D_{2w}}^{\zeta}$ be the orbit graph. Then $L.E(\beta_{D_{2w}}^{\zeta}) = L.E^{+}(\beta_{D_{2w}}^{\zeta}) = 5(w-2)$.*

Proof. Let D_{2w} be a dihedral group having $2w$ elements, where $\frac{w}{2}$ is odd and w is even, and $\beta_{D_{2w}}^{\zeta} = \bigcup_{i=1}^{5} K_{\frac{w}{2}}^{i}$. Then, we have:

$$\left|V(\beta_{D_{2w}}^{\zeta})\right| = \frac{5w}{2}, \qquad \left|E(\beta_{D_{2w}}^{\zeta})\right| = \frac{\frac{5w}{2}\left(\frac{w}{2}-1\right)}{2} = \frac{5w(w-2)}{8}$$

and therefore,

$$\frac{2\left|E\left(\beta_{D_{2w}}^{\zeta}\right)\right|}{\left|V\left(\beta_{D_{2w}}^{\zeta}\right)\right|} = \frac{w-2}{2}.$$

Case I (Laplacian Energy): Here, we find out the Laplacian energy of an orbit graph $\beta_{D_{2w}}^{\zeta} = \bigcup_{i=1}^{5} K_{\frac{w}{2}}^{i}$ of the dihedral group D_{2w} having order $2w$. For this, we have:

$$L-\mathrm{Spec}\left(\beta_{D_{2w}}^{\zeta}\right) = \left\{0^5, \left(\frac{w}{2}\right)^{5\left(\frac{w}{2}-1\right)}\right\}.$$

Now, the Laplacian eigenvalues of $\beta_{D_{2w}}^{\zeta}$ are $\mu_1 = 0$ having multiplicity 5 and $\mu_2 = \frac{w}{2}$ having multiplicity $5(\frac{w}{2}-1)$. Using the definition of Laplacian energy, we get:

$$\begin{aligned} L.E\left(\beta_{D_{2w}}^{\zeta}\right) &= \sum_{i=1}^{w}\left|\mu_i - \frac{2m}{w}\right| \\ &= 5\left|0-\left(\frac{w-2}{2}\right)\right| + 5\left(\frac{w}{2}-1\right)\left|\frac{w}{2}-\frac{w}{2}+1\right| \\ &= 5\left|-\left(\frac{w-2}{2}\right)\right| + 5\left(\frac{w-2}{2}\right) \\ &= 2\left(5\left(\frac{w-2}{2}\right)\right) \end{aligned}$$

$$= 10\left(\frac{w-2}{2}\right)$$
$$= 5(w-2).$$

Case II (Signless Laplacian Energy): In this case, we find out the signless Laplacian energy of an orbit graph $\beta^{\zeta}_{D_{2w}} = \bigcup_{i=1}^{5} K^{i}_{\frac{w}{2}}$ of the dihedral group D_{2w} having order $2w$. For this, we have:

$$Q - \operatorname{Spec}\left(\beta^{\zeta}_{D_{2w}}\right) = \left\{(w-2)^5, \left(\frac{w}{2}-2\right)^{5\left(\frac{w}{2}-1\right)}\right\}.$$

Now, the signless Laplacian eigenvalues of $\beta^{\zeta}_{D_{2w}}$ are $\lambda_1 = (w-2)$ having multiplicity 5 and $\lambda_2 = \left(\frac{w}{2}-2\right)$ with multiplicity $5\left(\frac{w}{2}-1\right)$. Using the definition of signless Laplacian energy, we get:

$$\begin{aligned}
L.E^{+}\left(\beta^{\zeta}_{D_{2w}}\right) &= \sum_{i=1}^{w}\left|\lambda_i - \frac{2m}{w}\right| \\
&= 5\left|(w-2) - \left(\frac{w-2}{2}\right)\right| + 5\left(\frac{w}{2}-1\right)\left|(\frac{w}{2}-2) - \left(\frac{w-2}{2}\right)\right| \\
&= 5\left|w-2-\frac{w}{2}+1\right| + 5\left(\frac{w}{2}-1\right)\left|\frac{w}{2}-2-\frac{w}{2}+1\right| \\
&= 5\left(\frac{w}{2}-1\right) + 5\left(\frac{w}{2}-1\right) \\
&= 2\left(5\left(\frac{w}{2}-1\right)\right) \\
&= 10\left(\frac{w}{2}-1\right) \\
&= 5(w-2).
\end{aligned}$$

□

Theorem 4. *If $D_{2w} = \langle s, t \mid s^w = t^2 = 1, tst^{-1} = s^{-1}\rangle$ be a dihedral group having order $2w$, where w and $\frac{w}{2}$ are even, and let $\beta^{\zeta}_{D_{2w}}$ be the orbit graph. Then $L.E(\beta^{\zeta}_{D_{2w}}) = L.E^{+}(\beta^{\zeta}_{D_{2w}}) = 4(w-2) \cup (w-4)$.*

Proof. Consider a dihedral group D_{2w} having $2w$ elements, where w and $\frac{w}{2}$ are even, and $\beta^{\zeta}_{D_{2w}} = \left(\bigcup_{i=1}^{4} K^{i}_{\frac{w}{2}}\right) \cup \left(\bigcup_{i=1}^{2} K^{i}_{\frac{w}{4}}\right)$, then, we have:

$$\left|V\left(\beta^{\zeta}_{D_{2w}}\right)\right| = \frac{4w}{2} = 2w, \qquad \left|E\left(\beta^{\zeta}_{D_{2w}}\right)\right| = \frac{\frac{4w}{2}(\frac{w}{2}-1)}{2} = \frac{2w(w-2)}{4} = \frac{w(w-2)}{2}$$

and therefore,

$$\frac{2\left|E\left(\beta^{\zeta}_{D_{2w}}\right)\right|}{\left|V\left(\beta^{\zeta}_{D_{2w}}\right)\right|} = \frac{w-2}{2}.$$

Case I (a) (Laplacian Energy): Here, we find out the Laplacian energy of an orbit graph $\beta^{\zeta}_{D_{2w}} = \left(\bigcup_{i=1}^{4} K^{i}_{\frac{w}{2}}\right)$. For this, we have:

$$L - \operatorname{Spec}\left(\beta^{\zeta}_{D_{2w}}\right) = \left\{0^4, \left(\frac{w}{2}\right)^{4\left(\frac{w}{2}-1\right)}\right\}.$$

Now, the Laplacian eigenvalues of $\beta^{\zeta}_{D_{2w}}$ are $\mu_3 = 0$ having multiplicity 4 and $\mu_4 = \frac{w}{2}$ having multiplicity $4\left(\frac{w}{2}-1\right)$. Using the definition of Laplacian energy, we get:

$$\begin{aligned}
L.E\left(\beta^{\zeta}_{D_{2w}}\right) &= \sum_{i=1}^{w}\left|\mu_i - \frac{2m}{w}\right| \\
&= 4\left|0-\left(\frac{w-2}{2}\right)\right| + 4\left(\frac{w}{2}-1\right)\left|\frac{w}{2}-\left(\frac{w}{2}-1\right)\right| \\
&= 4\left|-\left(\frac{w-2}{2}\right)\right| + 4\left(\frac{w-2}{2}\right) \\
&= 2\left(4\left(\frac{w-2}{2}\right)\right) \\
&= 8\left(\frac{w-2}{2}\right) \\
&= 4(w-2).
\end{aligned}$$

Case I (b): Here, we find out the Laplacian energy of an orbit graph $\beta^{\zeta}_{D_{2w}} = \left(\bigcup_{i=1}^{2} K^i_{\frac{w}{4}}\right)$. For this, we have:

$$L-\text{Spec}\left(\beta^{\zeta}_{D_{2w}}\right) = \left\{0^2, \left(\frac{w}{4}\right)^{2\left(\frac{w}{4}-1\right)}\right\}.$$

Now, the Laplacian eigenvalues of $\beta^{\zeta}_{D_{2w}}$ are $\mu_5 = 0$ having multiplicity 2 and $\mu_6 = \frac{w}{4}$ having multiplicity $2\left(\frac{w}{4}-1\right)$. Using the definition of Laplacian energy, we have:

$$\begin{aligned}
L.E\left(\beta^{\zeta}_{D_{2w}}\right) &= \sum_{i=1}^{w}\left|\mu_i - \frac{2m}{w}\right| \\
&= 2\left|0-\left(\frac{w-4}{4}\right)\right| + 2\left(\frac{w}{4}-1\right)\left|\frac{w}{4}-\left(\frac{w}{4}-1\right)\right| \\
&= 2\left|-\left(\frac{w-4}{4}\right)\right| + 2\left(\frac{w-4}{4}\right) \\
&= 2\left(2\left(\frac{w-4}{4}\right)\right) \\
&= 4\left(\frac{w-4}{4}\right) \\
&= (w-4).
\end{aligned}$$

Case II (a) (Signless Laplacian Energy): In this case, we find out the signless Laplacian energy of an orbit graph of $\beta^{\zeta}_{D_{2w}} = \left(\bigcup_{i=1}^{4} K^i_{\frac{w}{2}}\right)$. For this, we have:

$$Q-\text{Spec}\left(\beta^{\zeta}_{D_{2w}}\right) = \left\{(w-2)^4, \left(\frac{w}{2}-2\right)^{4\left(\frac{w}{2}-1\right)}\right\}.$$

Now, the signless Laplacian eigenvalues of $\beta^{\zeta}_{D_{2w}}$ are $\lambda_2 = (w-2)$ having multiplicity 4 and $\lambda_3 = \left(\frac{w}{2}-2\right)$ having multiplicity $4\left(\frac{w}{2}-1\right)$. Using the definition of signless Laplacian energy, we get:

$$L.E^{+}\left(\beta^{\zeta}_{D_{2w}}\right) = \sum_{i=1}^{t}\left|\lambda_i - \frac{2m}{w}\right|$$

$$
\begin{aligned}
&= 4\left|(w-2)-\left(\frac{w-2}{2}\right)\right| + 4\left(\frac{w}{2}-1\right)\left|\left(\frac{w}{2}-2\right)-\left(\frac{w-2}{2}\right)\right| \\
&= 4\left|w-2-\frac{w}{2}+1\right| + 4\left(\frac{w}{2}-1\right)\left|\frac{w}{2}-2-\frac{w}{2}+1\right| \\
&= 4\left(\frac{w}{2}-1\right)+4\left(\frac{w}{2}-1\right) \\
&= 2\left(4\left(\frac{w}{2}-1\right)\right) \\
&= 8\left(\frac{w}{2}-1\right) \\
&= 4(w-2).
\end{aligned}
$$

Case II (b): Here we find out the signless Laplacian energy of an orbit graph of $\beta^{\zeta}_{D_{2w}} = \left(\bigcup_{i=1}^{2} K^{i}_{\frac{w}{4}}\right)$. For this, we have:

$$
Q-\text{Spec}\left(\beta^{\zeta}_{D_{2w}}\right) = \left\{\left(\frac{w}{2}-2\right)^{2}, \left(\frac{w}{4}-2\right)^{2\left(\frac{w}{4}-1\right)}\right\}.
$$

Now, the signless Laplacian eigenvalues of $\beta^{\zeta}_{D_{2w}}$ are $\lambda_5 = \left(\frac{w}{2}-2\right)$ having multiplicity 2 and $\lambda_6 = \left(\frac{w}{4}-2\right)$ having multiplicity $2\left(\frac{w}{4}-1\right)$. Using the definition of signless Laplacian energy, we get:

$$
\begin{aligned}
L.E^{+}\left(\beta^{\zeta}_{D_{2w}}\right) &= \sum_{i=1}^{w}\left|\lambda_i - \frac{2m}{w}\right| \\
&= 2\left|\left(\frac{w}{2}-2\right)-\left(\frac{w-4}{4}\right)\right| + 2\left(\frac{w}{4}-1\right)\left|\left(\frac{w}{4}-2\right)-\left(\frac{w-4}{2}\right)\right| \\
&= 2\left|\frac{w}{2}-2-\frac{w}{4}+1\right| + 2\left(\frac{w}{4}-1\right)\left|\frac{w}{4}-2-\frac{w}{4}+1\right| \\
&= 2\left(2\left(\frac{w}{4}-1\right)\right) \\
&= 4\left(\frac{w}{4}-1\right) \\
&= (w-4).
\end{aligned}
$$

□

Theorem 5. *Let $D_{2w} = \langle s, t \mid s^{w} = t^{2} = 1, tst^{-1} = s^{-1}\rangle$ be a dihedral group having order $2w$, where w is odd, and let $\beta^{\zeta}_{D_{2w}}$ be the orbit graph. Then $L.E\left(\beta^{\zeta}_{D_{2w}}\right) = L.E^{+}\left(\beta^{\zeta}_{D_{2w}}\right) = 2(w-1)$.*

Proof. Consider a dihedral group D_{2w} having order $2w$; w is odd and $\beta^{\zeta}_{D_{2w}} = K_w$, then, we have:

$$
\left|V\left(\beta^{\zeta}_{D_{2w}}\right)\right| = w, \qquad \left|E\left(\beta^{\zeta}_{D_{2w}}\right)\right| = \frac{w(w-1)}{2}
$$

and therefore,

$$
\frac{2\left|E\left(\beta^{\zeta}_{D_{2w}}\right)\right|}{\left|V\left(\beta^{\zeta}_{D_{2w}}\right)\right|} = w-1.
$$

Case I (Laplacian Energy): Here, we find out the Laplacian energy of an orbit graph $\beta^{\zeta}_{D_{2w}} = K_w$ of the dihedral group D_{2w} of order $2w$. For this, we have:

$$L - \text{Spec}\left(\beta^{\zeta}_{D_{2w}}\right) = \left\{0^1, w^{(w-1)}\right\}.$$

Now, the Laplacian eigenvalues of $\beta^{\zeta}_{D_{2w}}$ are $\mu_7 = 0$ having multiplicity 1 and $\mu_8 = w$ have multiplicity $(w-1)$. Using the definition of Laplacian energy, we get:

$$\begin{aligned}
L.E\left(\beta^{\zeta}_{D_{2w}}\right) &= \sum_{i=1}^{t} \left|\mu_i - \frac{2m}{w}\right| \\
&= 1\,|0-(w-1)| + (w-1)\,|w-(w-1)| \\
&= |-(w-1)| + (w-1)\,|w-w+1| \\
&= (w-1) + (w-1) \\
&= 2(w-1).
\end{aligned}$$

Case II (Signless Laplacian Energy): Here, we find out the signless Laplacian energy of an orbit graph $\beta^{\zeta}_{D_{2w}} = K_w$ of the dihedral group D_{2w} having order $2w$. For this, we have:

$$Q - \text{Spec}\left(\beta^{\zeta}_{D_{2w}}\right) = \left\{(2w-2)^1, (w-2)^{(w-1)}\right\}.$$

Now, the signless Laplacian eigenvalues of $\beta^{\zeta}_{D_{2w}}$ are $\lambda_7 = (2w-2)$ with multiplicity 1 and $\lambda_8 = (w-2)$ with multiplicity $(w-1)$. Using the definition of signless Laplacian energy, we get:

$$\begin{aligned}
L.E^{+}\left(\beta^{\zeta}_{D_{2w}}\right) &= \sum_{i=1}^{w} \left|\lambda_i - \frac{2m}{w}\right| \\
&= |(2w-2)-(w-1)| + (w-1)\,|(w-2)-(w-1)| \\
&= |(2w-2-w+1)+(w-1)| + (w-1)|-1| \\
&= (w-1) + (w-1) \\
&= 2(w-1).
\end{aligned}$$

□

Proposition 3. *Let Q_{2^w} be a quaternion group of order 2^w, and let $\beta^{\zeta}_{Q_{2w}}$ be the orbit graph. Then $L.E\left(\beta^{\zeta}_{Q_{2w}}\right) = L.E^{+}\left(\beta^{\zeta}_{Q_{2w}}\right) = 0$.*

Proof. Since the orbit graph $\beta^{\zeta}_{Q_{2w}}$ of the quaternion group Q_{2^w} is a null graph, its vertices and edges are zero. Therefore, $L.E\left(\beta^{\zeta}_{Q_{2w}}\right) = L.E^{+}\left(\beta^{\zeta}_{Q_{2w}}\right) = 0$. □

4. Conclusion

In this manuscript, the general formula for the Laplacian energy and the general formula for signless Laplacian energy for the orbit graph of dihedral groups having order $2w$ are found. For w even and $\frac{w}{2}$ odd, $L.E\left(\beta^{\zeta}_{D_{2w}}\right) = L.E^{+}\left(\beta^{\zeta}_{D_{2w}}\right) = 5(w-2)$, while for w and $\frac{w}{2}$ even, $L.E\left(\beta^{\zeta}_{D_{2w}}\right) = L.E^{+}\left(\beta^{\zeta}_{D_{2w}}\right) = 4(w-2)\cup(w-4)$, and for w odd, $L.E\left(\beta^{\zeta}_{D_{2w}}\right) = L.E^{+}\left(\beta^{\zeta}_{D_{2w}}\right) = 2(w-1)$. We also find that the Laplacian energy as well as signless Laplacian energy of an orbit graph of the quaternion group having order 2^w are zero.

Conflict of Interest

The authors declare no conflict of interest.

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