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Article

On the Orbital Regular Graph of Finite Solvable Groups

Karnika Sharma[1], **Vijay Kumar Bhat**[1,*] **and Pradeep Singh**[2]

[1] School of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, Jammu and Kashmir, India.

[2] Department of Mathematics, Maharishi Markandeshwar Deemed to be University, Mullana-133207, Haryana, India.

* **Correspondence:** vijaykumarbhat2000@yahoo.com

Abstract: Let G be a finite solvable group and Δ be the subset of $\Upsilon \times \Upsilon$, where Υ is the set of all pairs of size two commuting elements in G. If G operates on a transitive G - space by the action $(v_1, v_2)^g = (v_1^g, v_2^g)$; $v_1, v_2 \in \Upsilon$ and $g \in G$, then orbits of G are called orbitals. The subset $\Delta_o = \{(v, v); v \in \Upsilon, (v, v) \in \Upsilon \times \Upsilon\}$ represents $G's$ diagonal orbital. The orbital regular graph is a graph on which G acts regularly on the vertices and the edge set. In this paper, we obtain the orbital regular graphs for some finite solvable groups using a regular action. Furthermore, the number of edges for each of a group's orbitals is obtained.

Keywords: Solvable group, Orbital, Orbital graph, Orbital regular graph, Regular action
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1. Introduction

Let G be a group that acts on a finite set Υ. Then the orbit of v is the subset $O(v) = \{gv \mid g \in G, v \in \Upsilon\}$ [1]. Later on, Omer et al. [2] define orbit as the set of all conjugates of the elements, where G acts on itself by conjugation. Furthermore, by defining an orbit graph as a graph whose vertices are non-central orbits under group action on Υ, Omer et al. [2] extended the work on conjugate graphs. Using various group actions, they constructed orbit graphs for various groups, such as finite non-abelian groups, finite p-groups, and groups of order pq. They also used a regular action to introduce the orbit graph for some finite solvable groups.

If G is transitive on Υ then Fang et al. [3] define orbitals of G as the orbits of $G's$ transitive action on Υ and the subset $\Delta_o = \{(v, v); v \in \Upsilon, (v, v) \in \Upsilon \times \Upsilon\}$ form a diagonal orbital of G. Looking forward to the work, Smith [4] then constructed a new graph on Υ, which he called an orbital graph having vertex set Υ and an arc set of orbitals. He introduced the concepts of sub orbits and orbitals using transitive action on a set. He constructed an orbital graph from orbitals, which shows that the orbital graphs for each orbital are different [4].

In the recent past, several research articles based on orbital graphs have been studied related to groups. The primitive group with small suborbital of length 3 or 4 and their orbital graph were introduced by Li et al. [5]. They also constructed vertex primitive half arc-transitive graphs of valency $2k$ for an infinite number of integers k, with fourteen being the smallest valency. Smith [4] looked into the diameter of an orbital graph that was linked to a group.

Sheikh [6] proposed that the orbital diameter must be bounded by a constant c and that the actions must be bounded by 5. He also determined the infinite families of orbital graphs with a diameter of 2. The action of SL(2, C) on hyperbolic 3-space and orbital graphs was first observed by Besenk [7]. Recently, Nagnibeda et al. [8] published an article where they show interest in orbital graphs for the action of spinal groups on d- regular rooted trees and on their boundaries. Pogorelov just published a classification for distance transitive orbital graphs over groups of the Jevons group [9]. Rakvenyi [10] introduced the concept of the orbital diameter of groups of diagonal type.

Also, orbitals have a wide role in the field of sciences like physics, chemistry, biology and many more. Hoffmann [11] investigates the orbitals' interaction through space and bonds. King [12] found that to form hybrid orbitals of special symmetries, the combination of atomic orbitals can be related to the individual orbital polynomials. Using this approach, he found the system of atomic orbital hybridization of coordination polyhedra and the role of f orbitals. Next, Rahaman and Gagliardi [13] introduced a deep learning-based framework that combines large organic molecules' total energies and orbital energies using molecular fingerprints' hybridisation.

Using the concept of orbitals, Sole [14] constructed an orbital regular graph as a graph, if it is regular for some $G = Aut(\Gamma)$ and derived an edge-forwarding index formula for it. According to Fang et al., [3] almost all orbital regular graphs are Frobenius graphs. However, many groups, such as finite solvable groups for which the orbital regular graph is yet to be constructed, are regular in their orbits.

In this article, we use regular action on a finite set Δ. With this, we may now define orbitals of G as the orbits of the regular action of G on Δ. Note that Δ must be a subset of $\Upsilon \times \Upsilon$. In this work, we use the concept of [14] and [2], to determine the number of edges for each orbital of a group G as well as the orbital regular graphs of a finite solvable group whose vertices are adjacent if there are $\upsilon_1, \upsilon_2 \in \Upsilon$ and $g \in G$ such that $(\upsilon_1, \upsilon_2)^g = (\upsilon_1^g, \upsilon_2^g)$. The graphs examined in this study are undirected.

2. Preliminaries

The orbit graph, orbital regular graph, group actions, and solvable groups are all discussed in this section, along with some basic concepts, definitions, and current results.

Suppose $\Gamma^* = (V, E)$ is a non-trivial, simple, and finite graph with E as the edge set and V as the vertex set. Let G be a group that acts on a finite set of Υ on a regular basis. Then G takes action on Δ element-wise. When there is no room for ambiguity, we write Γ^* instead of Γ and examine Δ a subset of the manuscript.

Definition 1. *A group G is said to be solvable if it has a normal series such that each normal factor is abelian.*

Theorem 1. [2] *The symmetric group S_n is a solvable group if $n \leq 4$.*

Definition 2 (Orbit). [15] *Let G be a group that acts on a set Υ and $\upsilon \in \Upsilon$. The orbit of υ, denoted by $O(\upsilon)$ is the subset $O(\upsilon) = \{g\upsilon \mid g \in G, \upsilon \in \Upsilon\}$. In this study, the group action is considered as a conjugation action. Hence, the orbit is given as*

$$O(\upsilon) = \{g\upsilon g^{-1} \mid g \in G, \upsilon \in \Upsilon\}.$$

Definition 3 (Orbit Graph). [2] *Let G ba a group and Υ be a set. Then an orbit graph, Γ_G^Υ is defined as a graph whose vertices are non central orbits under group action on the set Υ that is $|V(\Gamma_G^\Upsilon)| = |\Upsilon| - |B|$, where Υ is a disjoint union of distinct orbits and $B = \{\upsilon \in \Upsilon | \upsilon g = g\upsilon, g \in G\}$. Two vertices υ_1, υ_2 are adjacent if υ_1, υ_2 are conjugate that is $\upsilon_1 = g^{\upsilon_2}$.*

Definition 4 (Orbital). [3] *Let G be transitive on $\Upsilon \times \Upsilon$ then the orbits of G on $\Upsilon \times \Upsilon$ are called as orbitals of G, denoted by O^*.*

Definition 5 (Orbital Graph). [4] *Let G be a group acting on Υ and O^* be its orbital. Then the orbital graph with respect to O^* is the graph having Υ as a vertex set and O^* as its arc set.*

Next, we define the finite set on which G acts and its group actions.

Definition 6. [15] *Let G be a group and S be a set. G acts on S if there is a function which maps $G \times S \to S$ such that it satisfies the following axioms:*

1. *Identity: $e.s = s.e, \forall s \in S$.*
2. *Compatibility: $(gh)s = g(hs), \forall s \in S, g, h \in G$.*

Definition 7. [2] *The set Υ is the set of all pairs of commuting elements of G which are in the form of (α, β) where α, β are the elements of the finite solvable groups and the least common multiple of the order of the elements is two. Symbolically, it is represented as*

$$\Upsilon = \{(a, b) \in G \times G | ab = ba, a \neq b, lcm(|a|, |b|) = 2\}.$$

Definition 8. [4] *The action of G on a non empty set Υ is transitive if for each pair υ_1, υ_2 in Υ there exist a g in G such that $g.\upsilon_1 = \upsilon_2$.*

We recall [3] that the group G acts regularly on Υ if it is both transitive and $|G| = |\Upsilon|$. In this paper, we defined regular action as follows:

Definition 9. *A group G acts regularly on a set Υ if for any pair $\upsilon_1, \upsilon_2 \in \Upsilon$ there exist exactly one $g \in G$ such that $(\upsilon_1, \upsilon_2)^g = (\upsilon_1^g, \upsilon_2^g)$.*

On the basis of group actions, we defined orbital regular graph of a group G.

Definition 10. *Let G be a group which act regularly on the set Δ. Then the orbital regular graph is the undirected graph if G is regular on each of its orbits in V and one of these orbits is exactly E.*

The following corollary shows that a finite solvable group acts regularly on a finite set Υ.

Corollary 1. [2] *Let G be a finite solvable groups on a set Υ. If G acts regularly on Υ. Then*

$|E(\Gamma)|$	G	Γ	$|\Omega|$	$|V(\Gamma)|$
$\frac{(2^{n-2})!}{2!(2^{n-2}-2)!} + 2^{n-1}$	$\langle a, b : a^{2n} = b^2 = e, ab = ba^{2^{n-1}-1} \rangle$	$K_2 \bigcup K_{2^{n-2}}$	$2^n + 2^{n-2} + 1$	$2^n + 2^{n-2}$
β	$\langle a, b : a^{2^{\beta}} = b^2 = e, ab = a^{-1}b \rangle$	$\bigcup_{i=1}^{\beta} K_{2_i}$	$2n + 2$	$2n + 1$
$|\Omega| - |\omega| + 3(|\Omega| - |Z|)$	$S_n, n > 2, n$ is even	$\bigcup_{i=1}^{|\Omega|-|\omega|} K_{2_i} \bigcup_{i=1}^{|\Omega|-|Z|} K_{3_i}$	-	$|\Omega|$
-	S_n, n is odd	*empty*	-	-

In Corollary 1, Omer et al. [2] found the orbit graph for some finite solvable groups and general formula for number of vertices and edges.

Using Corollary 1, we prove our main results.

3. Main result

In this section, we present some results on the calculation of orbitals concerning finite solvable groups. We use regular action to find the orbital regular graph of a finite solvable group based on the orbitals of the group. We also get the number of edges for each of a group's orbitals.

Theorem 2. *Let $G = \langle a, b : a^{2^{\beta}} = b^2 = e, ab = a^{-1}b \rangle$ be a finite solvable group, where β is even. Then each orbital of G has a disconnected orbital regular graph with connected components, except one orbital $O((e, a^{2^{\beta-1}})(e, a^n b)), 0 \leq n \leq 2^{\beta} - 1$.*

Proof. Consider a finite solvable group $G = \langle a, b : a^{2^\beta} = b^2 = e, ab = a^{-1}b \rangle$, where β is even. Let Δ be the subset of $\Upsilon \times \Upsilon$ and G acts on Δ by the action $(v_1, v_2)^g = (v_1^g, v_2^g)$, where $v_1, v_2 \in \Upsilon$ and $g \in G$. This implies, the number of elements in Δ is $2(2^\beta)^2 + 2^\beta$. Based on regular action, we found three different types of orbitals.

Case I For orbital of the form $O((e, a^m b), (e, a^n b))$, $0 \le m \le 2^\beta - 1$ and $1 \le n \le 2^\beta - 1$. We consider two subcases.

Subcase I For $m \le \beta + 3$, the vertices in the form of $(e, a^m b)$ and $(a^{2^{\beta-1}}, a^{2^{\beta-1}+i}b)$, $0 \le i \le 2^\beta - 1$ are adjacent to the vertices in the form of $(e, a^n b)$ and $(a^{2^{\beta-1}}, a^{2^{\beta-1}+i}b)$, $0 \le i \le 2^\beta - 1$. Thus, we have $2^\beta - (n+1)$, $1 \le n \le 2^{\beta-1} + 1$ components of $\bigcup_{i=1}^2 K_2^i$.

On the other hand, the vertices in the form of $(e, a^m b)$, $(e, a^n b)$, $(a^{2^{\beta-1}}, a^m b)$ and $(a^{2^{\beta-1}}, a^n b)$ are adjacent to one another to form $2^{\beta-1}$ connected component of four vertices.

Thus, it follows that $\Gamma_G^{*\Delta} = \bigcup_{i=1}^2 K_2^i \bigcup C_4$.

Subcase II For $m \ge \beta + 4$, we found that the $O((e, a^m b), (e, a^n b))$ consist $2^{\beta-1} - (n+1)$, $0 \le n \le 2^{\beta-1} - 2$ components of $\bigcup_{i=1}^2 K_2^i$.

This implies, $\Gamma_G^{*\Delta} = \bigcup_{i=1}^2 K_2^i$.

Case II For orbital of the type $O((e, a^m b), (a^{2^{\beta-1}}, a^n b))$, $0 \le m \le 2^\beta - 1$ and $1 \le n \le 2^\beta - 1$ there exist $v_1, v_2 \in \Upsilon$ such that $v_1 = (e, a^m b)$ is adjacent to $v_2 = (a^{2^{\beta-1}}, a^m b)$ to form one component, $\bigcup_{i=1}^2 K_2^i$ and one complete component, K_2. Hence it follows that $\Gamma_G^{*\Delta} = \bigcup_{i=1}^2 K_2^i \bigcup K_2$.

Case III (Shows exception) Based on regular action the orbital of the type $O((e, a^{2^{\beta-1}})(e, a^n b))$, $0 \le n \le 2^\beta - 1$ of size two, there is always one common vertex of the form $(e, a^{2^{\beta-1}})$ adjacent to the two vertices of the type $(e, a^n b)$ and $(a^{2^{\beta-1}}, a^n b)$. Hence, there are 2^β connected orbital regular graph of three vertices.

□

Example 1. *Consider a finite solvable group* $G = \langle a, b : a^{2^2} = b^2 = e, ab = a^{-1}b \rangle$ *of 8 elements* $\{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ *and the elements of order two in G are* $\{e, a^2, b, ab, a^2b, a^3b\}$. *This implies* $\Delta \subset \Upsilon \times \Upsilon$ *contains 55 elements.*

If we take the orbital $O((e, b)(a^2, a^n b))$, $1 \le n \le 3$. *By applying regular action on the orbital, we see that only the element* $a^2 \in g$ *acts on it.*

That is,

$$O((e, b)(a^2, a^n b))a^2 = ((e, a^m b)(a^2, a^2 b)), \qquad 0 \le m \le 3$$

and contains two components $\bigcup_{i=1}^2 K_2^i$ *and one complete component* K_2.

This implies that the orbital regular graph for the orbital is disconnected, $\Gamma_G^{*\Delta} = \bigcup_{i=1}^2 K_2^i \bigcup K_2$. *The orbital regular graph for* $O((e, b)(a^2, a^n b))$ *is presented in Figure 1*

Now, if we take a different orbital of the form $O((e, a^2)(e, a^n b))$, $0 \le n \le 3$ *of size two. we produce four such orbitals which have connected orbital regular graph of three vertices. The orbital regular graph* $O((e, a^2)(e, a^n b))$ *is presented in Figure 2.*

Again, if we take orbital $O((e.b)(e, a^n b))$, $1 \le n \le 3$ *we have two orbitals of size two and one orbital of size four.*
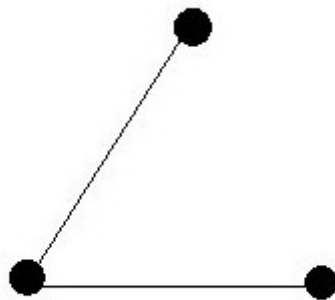
This implies that there is one component of the type $\bigcup_{i=1}^2 K_2^i$ *and one component is connected graph of four vertices.*

Thus, $\Gamma_G^{*\Delta} = \bigcup_{i=1}^2 K_2^i \bigcup C_4$. *The orbital regular graph* $O((e.b)(e, a^n b))$ *is presented in Figure 3.*
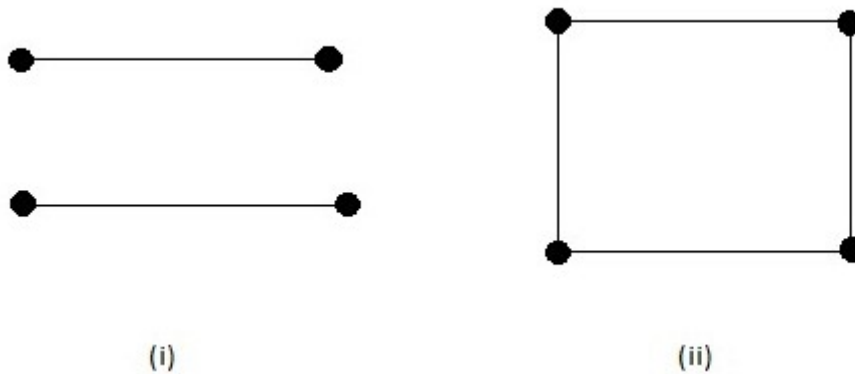
Here, we can find the graph for the other remaining orbitals and see that each orbital have disconnected graph except one orbital.
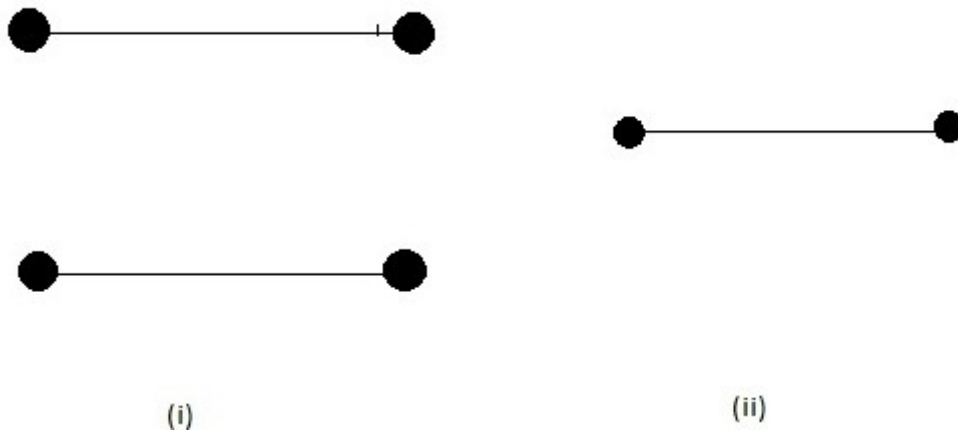
Figure 1. Orbital Regular Graph for $O((e, b)(a^2, a^n b))$

Figure 2. Orbital Regular Graph $O((e, a^2)(e, a^n b))$

Figure 3. Orbital Regular Graph $O((e.b)(e, a^n b))$

Theorem 3. *Let $G = \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} \rangle$ be a finite solvable group with n even. Then for each even i and $0 \leq i \leq 2^n - 1$ the orbital*

i. $O((e, a^{2^{n-1}})(e, a^i b))$ *have* $\Gamma_G^{*\Delta} = C_4$,
ii. $O((e, a^i b)(a^i b, a^{i+2^{n-1}} b))$ *have connected orbital regular graph of three vertices,*
iii. $O((e, a^{2^{n-1}})(a^i b, a^{i+2^{n-1}} b))$ *have* $\Gamma_G^{*\Delta} = K_2$,
iv. $O((a^{i'} b, a^{i+2^{n-1}} b)(a^{i'+2} b, a^{i+2^{n-1}+2} b))$, $0 \leq i' \leq 2^{n-1}$ *have* $\Gamma_G^{*\Delta} = K_2$,
v. $O((e, a^i b)(a^{2^{n-1}}, a^i b))$ *have disconnected orbital regular graph,*
vi. $O((e, a^i b)(e, a^{i+2} b))$ *have disconnected orbital regular graph.*

Proof. Consider a finite solvable group $G = \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} \rangle$, n is even. Let Δ be the subset of $\Upsilon \times \Upsilon$ and G acts regularly on Δ. If Γ^* is an orbital graph then each orbital have distinct orbital graph and two vertices of a graph are linked to each other if $v_1, v_2 \in \Upsilon$ there exist $g \in G$ such

that $(v_1, v_2)^g = (v_1^g, v_2^g)$. To show orbital regular graph of different orbitals and for i even, we have four cases:

Case I In the orbital $O((e, a^{2^{n-1}})(e, a^i b))$, where $0 \leq i \leq 2^n - 1$. If G acts regularly on Δ then the vertices of the form $(e, a^{2^{n-1}})$ and $(e, a^i b)$ are adjacent to the vertices of the form $(a^{2^{n-1}}, a^{2^{n-1}} b)$ and $(b, a^{2^{n-1}} b)$. Thus, we found that the orbital $O((e, a^{2^{n-1}})(e, a^i b))$ consist of 2^m, $m \geq 2$ connected orbital regular graph of four vertices.

This implies, $\Gamma_G^{*\Delta} = C_4$.

On the other hand, for orbital $O((e, a^i b)(a^i b, a^{i+2^{n-1}} b))$, $0 \leq i \leq 2^n - 1$, the vertex $v_1 = (a^i b, a^{i+2^{n-1}} b)$ is adjacent to the vertices $v_2 = (e, a^i b)$ and $v_3 = (a^{2^{n-1}}, a^i b)$, $0 \leq i \leq 2^n - 1$. Hence, we have 2^{n-2} connected orbital regular graph of three vertices.

Case II Consider the orbital of the form $O((e, a^{2^{n-1}}), (a^i b, a^{i+2^{n-1}} b))$ and $O((a^{i'} b, a^{i+2^{n-1}} b), (a^{i'+2} b, a^{i+2+2^{n-1}} b))$, $0 \leq i \leq 2^n - 1$, $0 \leq i' \leq 2^{n-1}$.

If G acts regularly on Δ, we found that both the orbitals are central orbitals and each having two adjacent vertices of the type $(e, a^{2^{n-1}})$, $(a^i b, a^{i+2^{n-1}} b)$ and $(a^i b, a^{i+2^{n-1}} b)$, $(a^{i+2} b, a^{i+2+2^{n-1}} b)$. Hence, for both the orbitals we have $\Gamma_G^{*\Delta} = K_2$.

Case III For orbital of the form $O((e, a^i b), (e, a^{i+2} b))$ where $0 \leq i \leq 2^n - 1$, the orbital regular graph of a particular orbitals are;

$(e, a^i b)$	$(e, a^{i+2} b)$	No of orbitals ($n \in 2l$),$l \in \mathbb{Z}$	$\Gamma_G^{*\Delta}$
$i = 0$	$i = 0, 2, ..., 2^{n-1}$	$2n - 1$	$(2n - 2)$ connected component of $\bigcup_{i=1}^2 K_2^i$ and one connected component of C_4
$i = 2$	$i = 4, 6, ..., 2^{n-1}$	$2n - 2$	$(2n - 3)$ connected component of $\bigcup_{i=1}^2 K_2^i$ and one connected component of C_4
.	.	.	.
.	.	.	.
.	.	.	.
$i = 2^n - 4$	$i = 2^n - 2$	1	one connected component of $\bigcup_{i=1}^2 K_2^i$

Hence, we found that each orbital have $\Gamma_G^{*\Delta} = \bigcup_{i=1}^2 K_2^i \bigcup C_4$ and also there exist one orbital which always have $\Gamma_G^{*\Delta} = \bigcup_{i=1}^2 K_2^i$.

Thus, we can say that the orbital $O((e, a^i b), (e, a^{i+2} b))$ have disconnected orbital regular graph.

Case IV For Orbital $O((e, a^i b), (a^{2^{n-1}}, a^i b))$ where $0 \leq i \leq 2^n - 1$, we have;

$(e, a^i b)$	$(a^{2^{n-1}}, a^i b)$	No of orbitals ($n \in 2l$),$l \in \mathbb{Z}$	$\Gamma_G^{*\Delta}$
$i = 0$	$i = 2, ..., 2^{n-1}$	$2n - 1$	$(2n - 2)$ connected component of $\bigcup_{i=1}^2 K_2^i$ and one connected component of K_2
$i = 2$	$i = 4, 6, ..., 2^{n-1}$	$2n - 2$	$(2n - 3)$ connected component of $\bigcup_{i=1}^2 K_2^i$ and one connected component of K_2
.	.	.	.
.	.	.	.
$i = 2^n - 4$	$i = 2^n - 2$	1	one connected component of K_2

From above table, we see that each orbital have disconnected orbital regular graph, $\Gamma_G^{*\Delta} = \bigcup_{i=1}^2 K_2^i \bigcup K_2$ except last orbital, $\Gamma_G^{*\Delta} = K_2$.

$\square$

Example 2. *Consider a finite solvable group $G = \langle a, b : a^{2^4} = b^2 = e, ab = ba^7 \rangle$ of 32 elements* $\{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, a^{13}, a^{14}, a^{15}, b, ab, a^2 b, a^3 b, a^4 b, a^5 b, a^6 b,$ $a^7 b, a^8 b, a^9 b, a^{10} b, a^{11} b, a^{12} b, a^{13} b, a^{14} b, a^{15} b\}$ *and order two elements are* $\{e, a^8, b, ab, a^2 b, a^3 b,$ $a^4 b, a^5 b, a^6 b, a^7 b, a^8 b, a^9 b, a^{10} b, a^{11} b, a^{12} b, a^{13} b, a^{14} b, a^{15} b\}$. *Hence, $\Delta \subset \Upsilon \times \Upsilon$ contains 222 elements.*

If we take the orbital $O((e, a^8)(e, a^i b))$, $0 \leq i \leq 15$ and i is even. By applying regular action on the orbital we see that only the element $a^8 \in g$ acts on it, that is

$$O((e, a^8)(e, a^i b))a^8 = ((e, a^8)(a^8, a^i b)), 0 \leq i \leq 15.$$

Therefore, the orbital of size four contains connected orbital regular graph of four vertices, C_4. Figure 4 shows the orbital regular graph for $O((e, a^8)(e, a^i b))$ and Figure 5 shows the orbital regular graph for $O((e, b)(b, a^8 b))$. Figure 6 shows the orbital regular graph for $O((e, b)(e, a^{i+2} b))$.
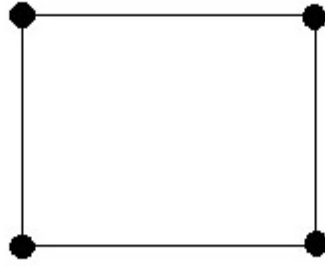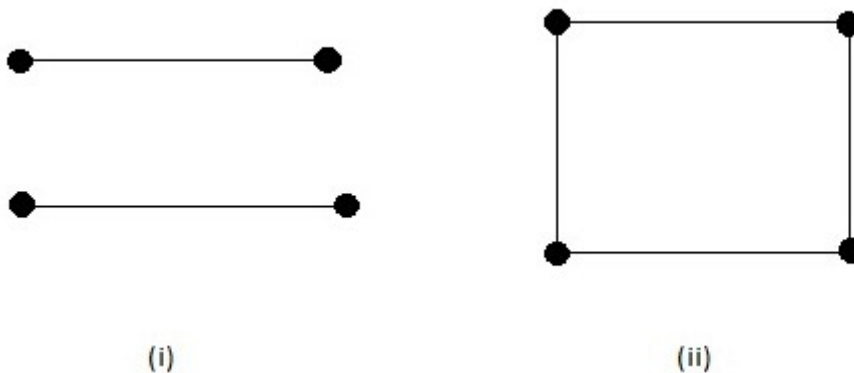

Figure 4. Orbital Regular Graph for $O((e, a^8)(e, a^i b))$

Next if we take different orbital $O((e, b)(b, a^8 b))$, we produce the connected orbital regular graph of three vertices.
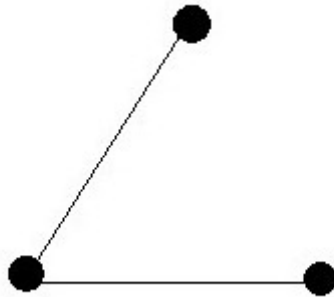
Figure 5. Orbital regular graph for $O((e, b)(b, a^8 b))$

Again, if we have orbital $O((e, b)(e, a^{i+2} b))$, $0 \leq i \leq 15$ and i is even. We get three orbitals where two orbitals have size two and one orbital have size four. This implies, $\Gamma_G^{*\Delta} = \bigcup_{i=1}^{2} K_2^i \bigcup C_4$.

(i)   (ii)

Figure 6. Orbital Regular Graph for $O((e, b)(e, a^{i+2} b))$

Here, we can produce the orbital regular graph for the rest of the orbitals and found that different orbitals of group G have different graph.

Theorem 4. *For a symmetric group $S_n, n > 2$ and n is even, each orbital of a group G have connected orbital regular graph except some orbitals which have disconnected graph with complete components.*

Proof. For $n = 2$, there is diagonal orbital of group G. This implies that the graph is empty.

For $n = 4$, If G acts regularly on Δ then $v_1, v_2 \in \Upsilon$ there exist $g \in G$ such that $(v_1, v_2)^g = (v_1^g, v_2^g)$. The vertex v_1 joined the vertex v_2 whenever $(v_1, v_2)^g = (v_1^g, v_2^g)$. According to [2] the elements of Δ are in the form

$$((e, (ab)), (e, (cd))), ((e, (ab)), ((ab), (cd))), ((e, (ab)), (e, (ab)(cd))),$$
$$(((e, (ab)), ((ab)(cd), (ac)(bd))), ((ab)(cd), (e, (ab)(cd))), (((ab)(cd)), ((ab), (ab)(cd))),$$
$$(((ab)(cd)), ((ab)(cd), (ac)(bd))), ((e, (ab)(cd)), ((ab), (ab)(cd))),$$
$$((e, (ab)), ((ab), (ab)(cd))), ((e, (ab)(cd)), ((ab)(cd), (ac)(bd))),$$
$$((ab), ((ab)(cd)), ((ab)(cd)), ((ac)(bd))).$$

Based on regular action, there are 26 orbitals of size one, five orbitals of size two and four orbitals of size four. Therefor, in the orbital of the form $((e, (ab)), (e, (ac)))$, the vertices of the form (e, ab) are adjacent to the vertices of the form (e, ac). Hence,we found that the orbital regular graph for the orbital is connected graph of two vertices, K_2. Similarly, for the remaining 25 orbitals of size one, we have $\Gamma_G^{*\Delta} = K_2$. However, for the orbitals $((e, (ab))(e, (ab)(cd)))$, $((e, (ab)(cd))(e, (ac)(bd)))$ and $((e, (ab)(cd))(e, (ad)(bc)))$ there are four vertices in each orbital and each vertex is adjacent to their next vertex to form a closed path, that is

$$(e, ab) \to (e, (ab)(cd)) \to ((ab)(cd), (cd)) \to ((ab)(cd)) \to (e, ab).$$

This implies, $\Gamma_G^{*\Delta} = C_4$.

Next, for the orbital of the form $((e, ab)((ab)(cd), (ad)(bc)))$, $((e, (ab)(cd))((ac)(bd)))$ and $((e, (ab)(cd))((ad)(bc)))$, we found that each orbitals contain connected graph of three vertices;

Orbitals	Vertices	Adjacent Vertices	$\Gamma_G^{*\Delta}$
$((e, ab)((ab)(cd), (ad)(bc)))$	(e, ab)	$((ab)(cd), (ad)(bc)), ((ab)(cd), (cd))$	connected graph of three vertices
$((e, (ab)(cd))((ac)(bd)))$	$(e, (ab)(cd))$	$((ac)(bd)), ((ac)(bd), (ad)(bc))$	connected graph of three vertices
$((e, (ab)(cd))((ad)(bc)))$	$((ad)(bc))$	$(e, (ab)(cd)), ((ab)(cd), (ad)(bc))$	connected graph of three vertices

Exception

Consider an orbitals of the type $((e, ab)(e, (ac)(bd)))$ and $((e, ab)((ab)(cd), (ac)(bd)))$ of size two, we found that there are four vertices in each orbitals. If G acts regularly on the set Δ then each orbital form two pairs (P_1, P_2) of connected graph but there is no edge between two pairs which shows that both the pairs are disconnected. This implies, $\Gamma_G^{*\Delta} = \bigcup_{i=1}^4 K_2$.

□

The Orbital regular graph for $((e, ab)(e, (ac)(bd)))$ and $((e, ab)((ab)(cd), (ac)(bd)))$ is presented in Figure 7.

P1 P2
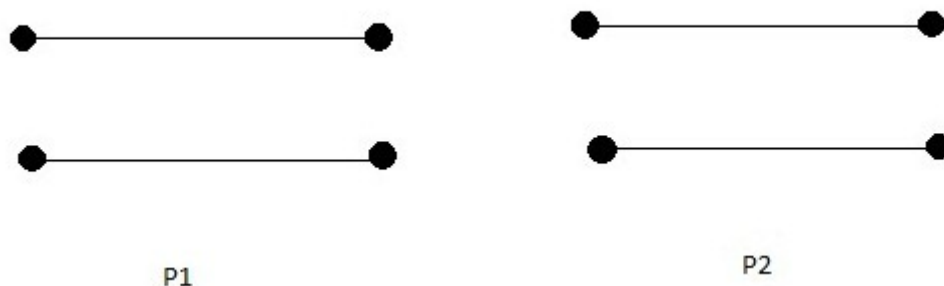
Figure 7. Orbital regular graph for $((e, ab)(e, (ac)(bd)))$ and $((e, ab)((ab)(cd), (ac)(bd)))$

Theorem 5. *For a Symmetric group $G = S_n$, n is odd, each orbital have $\Gamma_G^{*\Delta} = K_2$.*

Proof. Since n is odd, the number of elements in Δ are in the form $((e, (ab)), (e, (ac)))$, $((e, (ab)), (e, (bc))), ((e, (ac)), (e, (bc)))$. Based on regular action on Δ there are three central orbitals of size one, that is

$$\begin{aligned}
\Delta_1 &= ((e, (ab)), (e, (ac))), \\
\Delta_2 &= ((e, (ab)), (e, (bc))), \\
\Delta_3 &= ((e, (ac)), (e, (bc))).
\end{aligned}$$

This implies that each orbital have two adjacent vertices which follows that each orbital have $\Gamma_G^{*\Delta} = K_2$. $\qquad\square$

Theorem 6. *Let $G = \langle a, b : a^{2^\beta} = b^2 = e, ab = a^{-1}b \rangle$, β is even. Let $\Delta \subset \Upsilon \times \Upsilon$. If G acts regularly on Δ. Then*

Orbitals	$E(\Gamma_G^{*\Delta})$
$O((e, a^m b), (e, a^n b))$, $m \leq \beta + 3$, $1 \leq n \leq 2^\beta - 1$	$2^{\beta+2} - 2(n + 1)$, $1 \leq n \leq 2^{\beta-1} + 1$
$O((e, a^m b), (e, a^n b))$, $m \leq \beta + 4$, $1 \leq n \leq 2^\beta - 1$	$2^\beta - 2(n + 1)$, $0 \leq n \leq 2^{\beta-1} - 2$
$O((e, a^m b), (a^{2^{\beta-1}}, a^n b))$, $0 \leq m \leq 2^\beta - 1$ and $1 \leq n \leq 2^\beta - 1$	3
$O((e, a^{2^{\beta-1}})(e, a^n b))$, $0 \leq n \leq 2^\beta - 1$	$2^{\beta+1}$

Proof. Based on Theorem 2, for orbital $O((e, a^m b), (e, a^n b))$, $m \leq \beta + 3$ and $1 \leq n \leq 2^\beta - 1$ there are $2^\beta - (n + 1)$, $1 \leq n \leq 2^{\beta-1} + 1$ complete components of $\bigcup_{i=1}^{2} K_2$ and $2^{\beta-1}$ connected component of four vertices then the number of edges can be computed as follows:

$$\begin{aligned}
|E(\Gamma_G^{*\Delta})| &= 2(2^\beta - (n + 1)) + 2^{\beta-1}\binom{4}{2} - 2 \\
&= 2(2^\beta - (n + 1)) + 2^{\beta-1}(2^2) \\
&= 2(2^\beta - (n + 1)) + 2^{\beta+1} \\
&= 2^{\beta+2} - 2(n + 1).
\end{aligned}$$

Next, for $m \leq \beta + 4$ there are $2^{\beta-1} - (n + 1)$ complete component of $\bigcup_{i=1}^{2} K_2$, then

$$\begin{aligned}
|E(\Gamma_G^{*\Delta})| &= 2(2^{\beta-1} - (n + 1)) \\
&= 2^\beta - 2(n + 1).
\end{aligned}$$

Now, for orbital $O((e, a^m b), (a^{2^{\beta-1}}, a^n b))$, $0 \leq m \leq 2^\beta - 1$ and $1 \leq n \leq 2^\beta - 1$ there are one complete component of $\bigcup_{i=1}^{2} K_2$ and one complete component of K_2. Hence it follows that

$$|E(\Gamma_G^{*\Delta})| = 3$$

and for orbital $O((e, a^{2^{\beta-1}})(e, a^n b))$, $0 \leq n \leq 2^\beta - 1$ there are 2^β connected graph of three vertices. Thus

$$\begin{aligned}
|E(\Gamma_G^{*\Delta})| &= 2^\beta\binom{3}{2} - 1 \\
&= 2^{\beta+1}.
\end{aligned}$$

$\qquad\square$

Theorem 7. *Let $G = \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} \rangle$, n is even. Let $\Delta \subset \Upsilon \times \Upsilon$. If G acts regularly on Δ. Then*

Orbitals	$E(\Gamma_G^{*\Delta})$
$O((e, a^{2^{n-1}})(e, a^i b)), 0 \le i \le 2^n - 1$	$2^{m+2}, m \ge 2$
$O((e, a^i b)(a^i b, a^{i+2^{n-1}} b)), 0 \le i \le 2^n - 1$	$3(2^{n-2})$
$O((e, a^{2^{n-1}})(a^i b, a^{i+2^{n-1}} b)), 0 \le i \le 2^n - 1$	1
$O((a^{i'} b, a^{i+2^{n-1}} b)(a^{i'+2} b, a^{i+2^{n-1}+2} b)), 0 \le i \le 2^n - 1, 0 \le i' \le 2^{n-1}$	1
$O((e, a^i b)(e, a^{i+2} b)), 0 \le i \le 2^n - 1$	$2(2n - (k + 1) + 4), k \ge 1$
$(\text{exception})O((e, a^{2^{n-2}} b)(e, a^{2^{n-2}+2} b)$	2
$O((e, a^i b)(a^{2^{n-1}}, a^i b)), 0 \le i \le 2^n - 1$	$2(2n - (k + 1)) + 1, k \ge 1$
$(\text{exception})O((e, a^{2^{n-2}})(a^{2^{n-1}}, a^{2^{n-1}} b))$	1

Proof. Based on Theorem 3, we consider each case to compute number of edges of each orbital.

Case I We found that for the orbital $O((e, a^{2^{n-1}})(e, a^i b))$, there are 2^m, $m \ge 2$ connected graph of four vertices. This implies that

$$|E(\Gamma_G^{*\Delta})| = 2^m \binom{4}{2} - 2$$
$$= 2^{m+2}, m \ge 2.$$

Also, for $O((e, a^i b)(a^i b, a^{i+2^{n-1}} b))$ there are 2^{n-2} connected graph of three vertices. Thus,

$$|E(\Gamma_G^{*\Delta})| = 3(2^{n-2}).$$

Case II Consider $O((e, a^{2^{n-1}})(a^i b, a^{i+2^{n-1}} b))$ and $O((a^{i'} b, a^{i+2^{n-1}} b)(a^{i'+2} b, a^{i+2^{n-1}+2} b))$, then $\Gamma_G^{*\Delta} = K_2$. This implies

$$|E(\Gamma_G^{*\Delta})| = 1.$$

Case III For $O((e, a^i b)(e, a^{i+2} b))$, we found that each orbital have $(2n - (k + 1))$, $k \ge 1$ component of $\bigcup_{i=1}^2 K_2$ and one connected component of four vertices. Thus, we have

$$|E(\Gamma_G^{*\Delta})| = 2(2n - (k + 1)) + 4.$$

This case follows the exception, i.e., there exist one orbital of the type $((e, a^{2^{n-2}} b)(e, a^{2^{n-2}+2} b)$ which always have one component of $\bigcup_{i=1}^2 K_2$, then

$$|E(\Gamma_G^{*\Delta})| = 2.$$

Case IV For orbital $O((e, a^i b)(a^{2^{n-1}}, a^i b))$, we find

$$|E(\Gamma_G^{*\Delta})| = 2(2n - (k + 1)) + 1,$$

and (exception) $O((e, a^{2^{n-2}})(a^{2^{n-1}}, a^{2^{n-1}} b))$, we have

$$|E(\Gamma_G^{*\Delta})| = 1.$$

$\square$

4. Conclusion

In this research, we computed the orbitals of a group using a regular action to derive the orbital regular graph of some finite solvable groups. The cartesian product of the set of all pairs of commuting elements of size two in G with itself is used to obtain the orbitals of a group. We found the number of orbital edges for each orbital of a finite solvable group. We also observed that each group G orbital does have a different orbital regular graph.

Conflict of Interest

The authors declare no conflict of interest.

References

1. Goodman, F. M., 2002. *Algebra Abstract and Concrete Stressing Symmetry (2nd ed.)*. Prentice Hall.

2. Omer, S. M. S., Sarmin, N. H., and Erfanian, A., 2014. The orbit graph for some finite solvable groups. *AIP Conference Proceedings, 863*.

3. Fang, X. C., Li, C. H., and Praeger, C. E., 1998. On orbital regular graphs and Frobenius graphs. *Discrete Mathematics, 182*(1-3), pp.85-99.

4. Smith, S. M., 2007. Orbital digraphs of infinite primitive permutation groups. *Journal of Group Theory, 14*(6), pp.817-828.

5. Li, C. H., Lu, Z. P., and Marusic, D., 2004. On primitive permutation groups with small suborbits and their orbital graphs. *Journal of Algebra, 279*(2), pp.749-770.

6. Sheikh, A., 2017. Orbital diameter of the symmetric and alternating groups. *Journal of Algebraic Combinatorics, 45*(1), pp.1-32.

7. Besenk, M., 2018. The action of $SL(2, \mathbb{C})$ on hyperbolic 3-space and orbital graphs. *Graphs and Combinatorics, 34*(3), pp.545-554.

8. Nagnibeda, T., Perez, A. and Schreier, G., 2021. Schreier graphs of spinal groups. *International Journal of Algebra and Computation, 31*(6), pp.1191-1216.

9. Pogorelov, B. A., and Pudovkina, M. A., 2020. Classification of distance-transitive orbital graphs of overgroups of the Jevons group. *Discrete Mathematics and Applications, 30*(1), pp.7-22.

10. Rekvenyi, K., 2021. On the orbital diameter of groups of diagonal type. *arXiv preprint arXiv:2102.09867*.

11. Hoffmann, R., 1971. Interaction of orbitals through space and through bonds. *Accounts of Chemical Research, 4*(1), pp.1-9.

12. King, R. B., 2020. *Systematics of atomic orbital hybridization of coordination polyhedra: Role of f orbitals* (Doctoral dissertation, University of Georgia).

13. Rahaman, O., and Gagliardi, A., 2020. Deep learning total energies and orbital energies of large organic molecules using hybridization of molecular fingerprints. *Journal of Chemical Information and Modeling, 60*(12), pp.5971-5983.

14. Sole, P., 1994. The edge-forwarding index of orbital regular graphs. *Discrete Mathematics, 130*(1-3), pp.171-176.

15. Rotman, J. J., 2002. *Advanced Modern Algebra (1st ed.)*. Prentice Hall.