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Article

The Index Weighted Hermitian Adjacency Matrices for Mixed Graphs

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Abstract: Based on the Hermitian adjacency matrices of second kind introduced by Mohar [1] and weighted adjacency matrices introduced in [2], we define a kind of index weighted Hermitian adjacency matrices of mixed graphs. In this paper we characterize the structure of mixed graphs which are cospectral to their underlying graphs, then we determine a upper bound on the spectral radius of mixed graphs with maximum degree Δ, and characterize the corresponding extremal graphs.

Keywords: Mixed graph, Spectral radius, Eigenvalue, Adjacency matrix, Cospectrality
Mathematics Subject Classification: 05C50, 15A18

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1. Introduction and Preliminaries

In this paper, we consider only simple and finite graphs. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We say that two vertices u and v are adjacent if they are joined by an edge. A mixed graph M_G is obtained from a simple graph G by orienting each edge of some subset $E_0 \in E(G)$ and we call G the underlying graph of M_G. We write an undirected edge as $\{u, v\}$ and an arc form u to v as (u, v). For a vertex set V' of $V(M_G)$, let $M_G[V']$ be a mixed subgraph of M_G induced on V'. We say a mixed graph is connected if its underlying graph is connected. The degree of a vertex in a mixed graph M_G is defined to be the degree of this vertex in the underlying graph. We divide the vertices adjacent to v in M_G into three sets: $N^0_{M_G}(v) = \{u \in V(M_G) : \{u, v\} \in E(M_G)\}$; $N^+_{M_G}(v) = \{u \in V(M_G) : (v, u) \in E(M_G)\}$ and $N^-_{M_G}(v) = \{u \in V(M_G) : (u, v) \in E(M_G)\}$.

Recently, Yu, Geng and Zhou [3] defined a kind of new Hermitian adjacency matrix of a mixed graph, written by $H_k(M_G) = (h_{uv})$, which is as follows:

$$h_{uv} = \begin{cases} e^{\frac{2\pi \mathbf{i}}{k}}, & \text{if } (u, v) \text{ is an arc from } u \text{ to } v; \\ e^{-\frac{2\pi \mathbf{i}}{k}}, & \text{if } (v, u) \text{ is an arc from } v \text{ to } u; \\ 1, & \text{if } \{u, v\} \text{ is an undirected edge}; \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathbf{i}$ is the imaginary unit and k is a positive integer. $H_k(M_G)$ unifies some well known matrices. In fact, when $k = 1$, $H_1(M_G) = A(G)$ is the adjacency matrix of G; when when $k = 4$, $H_4(M_G) = H(M_G)$ is the Hermitian adjacency matrix of M_G, proposed independently by Guo and Mohar [4], Liu and Li [5]; When $k = 6$, $H_6(M_G)$ is is the Hermitian adjacency matrix of second kind for M_G, proposed by Mohar [1] recently.

The author in [2] gave the following definition of a new weighted adjacency matrix of a graph weighted by its degrees.

Definition 1. *Let $G = (V(G), E(G))$ be a graph. Denote by d_u the degree of a vertex u in G. Let $f(x, y)$ be a function symmetric in x and y. The weighted adjacency matrix $A_f(G) = (a_{uv})$ of G is defined as follows:*

$$a_{uv} = \begin{cases} f(d_u, d_v), & \text{if } uv \in E(G); \\ 0, & \text{otherwise.} \end{cases}$$

$A_f(G)$ introduced in [2] unifies many matrices weighted by topological index. When $f(x, y) = 1$, $A_f(G)$ is the adjacency matrix of G; when $f(x, y) = \sqrt{\frac{x+y-2}{xy}}$, $A_f(G)$ is the *ABC* matrix and was introduced by Estrada [6] and was studied in [7]; When $f(x, y) = \frac{2}{x+y}$, $A_f(G)$ is the *harmonic* matrix and was introduced in [8]; When $f(x, y) = \frac{x^2+y^2}{2xy}$, $A_f(G)$ is the *AG* matrix and was studied in [9], [10] and [11].

In this paper, inspired by [3], we generalize weighted adjacency matrix to mixed graphs and define a index weighted Hermitian matrix $H_f^k(M_G) = (h_{uv})$ for a mixed graph M_G, where

$$h_{uv} = \begin{cases} f(d_u, d_v) \cdot \omega, & \text{if } (u, v) \text{ is an arc from } u \text{ to } v; \\ f(d_u, d_v) \cdot \bar{\omega}, & \text{if } (v, u) \text{ is an arc from } v \text{ to } u; \\ f(d_u, d_v), & \text{if } \{u, v\} \text{ is an undirected edge;} \\ 0, & \text{otherwise,} \end{cases}$$

$\omega = cos(\frac{2\pi}{k}) + \mathbf{i}sin(\frac{2\pi}{k})$ for a positive integer k and $f(x, y) > 0$ is a symmetric and real function. It can be seen that when $f(x, y) = 1$, $H_f^k(M_G)$ is the matrix $H^k(M_G)$, which is defined in [3]; when $k = 1$, $H_f^k(M_G)$ is the matrix $A_f(M_G)$, introduced in [2]. It's easy to check that if all edges of M_G are undirected edges, then $H_f^k(M_G) = A_f(M_G)$. Note that When k is a positive integer and $f(x, y) > 0$ is a symmetric and real function, $H_f^k(M_G)$ is Hermitian and the eigenvalues of $H_f^k(M_G)$ are real. With fixed f, k and a mixed graph M_G, let $\lambda_1(M_G) \geq \lambda_2(M_G) \geq \cdots \lambda_n(M_G)$ be n eigenvalues of $H_f^k(M_G)$ with $|V(M_G)| = n$. The collection $\{\lambda_1(M_G), \lambda_2(M_G), \cdots, \lambda_n(M_G)\}$ is called the H_f^k-spectrum of M_G, we say M_G is H_f^k-cospectral to its underlying graph if $H_f^k(M_G)$ and $H_f^k(G)$ have the same H_f^k-spectrum. We call the spectral radius of $H_f^k(M_G)$ the H_f^k-spectrum radius of M_G, denoted by $\rho_f^k(M_G)$.

In the following paper, we assume that k is a positive integer and $f(x, y) > 0$ is a symmetric and real function. We define several particular partitions of $V(M_G)$ which will be used in our proof.

Definition 2. *Let M_G be a mixed graph,*

(i) *For a partition $V_1 \cup V_2 \cup \cdots \cup V_k$(possible empty) of $V(M_G)$, we call it a $\mathscr{A}$-partition of $V(M_G)$ if every undirected edge $\{u, v\}$ with $u \in V_i$ and $v \in V_j$ such that $i - j \equiv 0 (mod\ k)$ and every arc (u, v) with $u \in V_i$ and $v \in V_j$ such that $i - j \equiv 1 (mod\ k)$.*

(ii) *When k is even. For a partition $V_1 \cup V_2 \cup \cdots \cup V_k$(possible empty) of $V(M_G)$, we call it a $\mathscr{B}$-partition of $V(M_G)$ if every undirected edge $\{u, v\}$ with $u \in V_i$ and $v \in V_j$ such that $i - j \equiv \frac{k}{2} (mod\ k)$ and every arc (u, v) with $u \in V_i$ and $v \in V_j$ such that $i - j \equiv \frac{k}{2} + 1 (mod\ k)$.*

(iii) *When k is odd. For a partition $V_1 \cup \cdots \cup V_k \cup U_1 \cup \cdots \cup U_k$(possible empty) of $V(M_G)$, we call it a $\mathscr{C}$-partition of $V(M_G)$ if every undirected edge $\{u, v\}$ is between U_i and V_i for some $i \in \{1, 2, \ldots, k\}$ and every arc (u, v) is between U_i,V_j or V_i,U_j, where $i - j \equiv 1 (mod\ k)$.*

When $k = 6$, three partitions can be seen in Figures 1 and 2. By the Definition 2, $V(M_G)$ has a $\mathscr{B}$*-partition* only if k is an even positive integer and it's easy to check that a $\mathscr{C}$*-partition* of $V(M_G)$ is a $\mathscr{A}$*-partition*. The main result of paper is as follows:

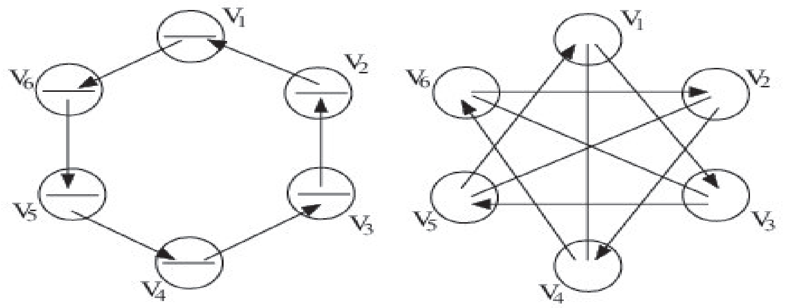

Figure 1. $\mathscr{A}$*-partition*(left) and $\mathscr{B}$*-partition*(right) when k is Equal to 6

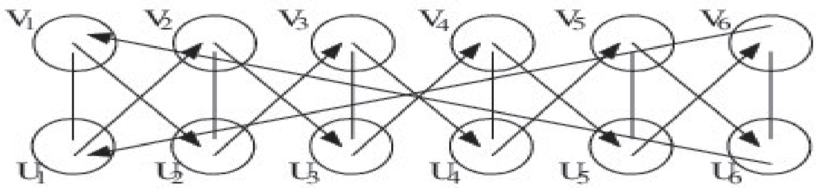

Figure 2. $\mathscr{C}$*-partition* when k is Equal to 6

Theorem 1. *Assume that k is a positive integer and $f(x,y) > 0$ is a symmetric real function, not decreasing in variable x. Then for any connected mixed graph M_G with maximum degree Δ, we have $\rho_f^k(M_G) \leq f(\Delta,\Delta)\Delta$, where equality holds if and only if M_G such that G is Δ-regular and $V(M_G)$ admits a $\mathscr{A}$-partition or a $\mathscr{B}$-partition.*

The structure of the paper is as follows. In Section 2 we show a disturbance on $E(M_G)$ which remains the H_f^k-spectrum of M_G. In Section 3 we characterize a family of mixed graphs which are H_f^k-cospectral to their underlying graph and in Section 4 we prove Theorem 1

2. Switching Equivalence for $\mathbf{M_G}$

In this section, we focus on some sufficient conditions of H_f^k-cospectrality of mixed graphs.

Supposed that $V(M_G)$ can be partitioned into k(possible empty) sets $V_1 \cup V_2 \cup \cdots \cup V_k$. We say the partition is admissible if it such that:

(i) $i - j \equiv 0$ or 1 or 2(mod k) for every arc (u, v) in M_G, where $u \in V_i$ and $v \in V_j$;

(ii) $i - j \equiv 0$ or 1(mod k) for every undirected edge $\{u, v\}$ in M_G, where $u \in V_i$ and $v \in V_j$.

We say a mixed graph M_G is admissible if it has an admissible partition.

For a mixed graph M_G, a *three-way switching* for M_G with respect to its admissible partition $V_1 \cup V_2 \cup \cdots \cup V_k$ is a operation of changing M_G into M'_G by making the changes in what follows (see Figure 3):

(i) replacing each undirected edge $\{u, v\}$ with an arc (u, v), where $u \in V_i$, $v \in V_j$ and $j - i \equiv 1$(mod k);

(ii) replacing each arc (u, v) with an undirected edge $\{u, v\}$, where $u \in V_i$, $v \in V_j$ and $i - j \equiv 1$(mod k);

(iii) replacing each arc (u, v) with an arc (v, u), where $u \in V_i$, $v \in V_j$ and $i - j \equiv 2$(mod k).

Theorem 2. *If a mixed graph $M_G^{'}$ is obtained from an admissible mixed graph M_G by a three-way switching, then $M_G^{'}$ is H_f^k-cospectral to M_G.*

Proof. Let $V_1 \cup V_2 \cup \cdots \cup V_k$ be an admissible partition of $V(M_G)$, Let $n_i = |V_i|$ for $i = \{1, 2, \ldots, k\}$ and $D = diag\{\omega I_{n_1}, \omega^2 I_{n_2}, \ldots, , \omega^k I_{n_k}\}$, let $H_f^k(M_G) = (h_{ij})_{n\times n}$ and $H_f^k(M'_G) = (h'_{ij})_{n\times n}$. In order to prove the Theorem, it suffices to prove that $H_f^k(M'_G) = D^{-1}H_f^k(M_G)D$.

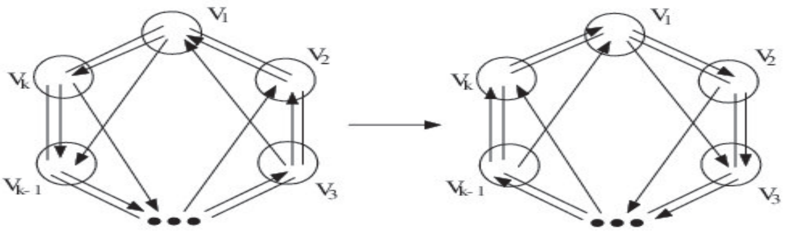

Figure 3. Three Way Switching for an Admissible Partition

It's easy to check that for every vertices pair u, v with $u \in V_i$ and $v \in V_j$, where $i \geq j$, $(D^{-1}H_f^k(M_G)D)_{u,v} = \omega^{k+j-i}h_{uv}$. If u is not adjacent to v in M_G, then $\omega^{k+j-i}h_{uv} = 0 = h'_{uv}$ since M_G and M'_G have the same underlying graph.

When u,v are adjacent, we consider following cases.

Case 1. $i - j \equiv 0 \pmod k$, in this case we have $h'_{uv} = h_{uv}$ since the switching doesn't change the edges in $[M_G(V_i)]$.

Case 2. $i - j \equiv 1 \pmod k$, note that in this case, the edge between u,v in M_G is a undirected edge $\{u, v\}$ or a arc (u, v) from u to v. If $\{u, v\}$ is a undirected edge, then by by the switching, it follows that (v, u) is a arc from v to u in M'_G and $h'_{u,v} = f(d_u, d_v)\omega^{-1} = \omega^{k+j-i}h_{uv}$. If (u, v) is a arc from u to v, then by the switching it follows that $\{u, v\}$ is a undirected edge in M'_G and $h'_{u,v} = \omega^{-1}f(d_u, d_v)\omega = \omega^{k+j-i}h_{uv}$.

Case 3. $i - j \equiv 2 \pmod k$, note that in this case, the edge between u,v in M_G must be a arc (u, v) from u to v. Then by the switching it follows that (v, u) is a arc from v to u in M'_G and $h'_{u,v} = f(d_u, d_v)\omega^{-1} = \omega^{-2}f(d_u, d_v)\omega^1 = \omega^{k+j-i}h_{uv}$.

Together with Case 1, 2 and 3, we complete the proof of Theorem. □

Note that V_i can be empty for a admissible partition $V_1 \cup V_2 \cup \cdots \cup V_k$ of M_G, so the *three-way switching* can induces some particular equivalent switchings for M_G.

Corollary 1. *If $V(M_G)$ has a partition $V_1 \cup V_2$ such that there exists no arc from V_1 to V_2, then the graph obtained from M_G by replacing all undirected edges $\{u, v\}$ with $u \in V_1$ and $v \in V_2$ by (u, v) and replacing all arcs (u, v) with $u \in V_2$ and $v \in V_1$ by $\{u, v\}$ is H_f^k-cospectral with M_G.*

Corollary 2. *If $V(M_G)$ has a partition $V_1 \cup V_2$ such that $[V_1, V_2]$ only contains arcs from V_2 to V_1, then the graph obtained from M_G by replacing all arcs (u, v) with $u \in V_2$ and $v \in V_1$ by (v, u) is H_f^k-cospectral with M_G.*

Corollary 3. *If $k = 3$ and M_G has a partition $V_1 \cup V_2$,then the graph obtained from M_G by following changes is H_f^k-cospectral with M_G:*

(i) replacing all arcs (u, v) with $u \in V_2$ and $v \in V_1$ by $\{u, v\}$;
(ii) replacing all undirected edges $\{u, v\}$ with $u \in V_2$ and $V \in v_1$ by (v, u);
(iii) replacing all arcs (u, v) with $u \in V_1$ and $V \in v_2$ by (v, u).

3. $\mathbf{H_f^k}$-cospectrality with Underlying Graph

In this section, we focus on the H_f^k-cospectrality of M_G and its underlying graph G.

Theorem 3. *Let M_G be a connected mixed graph. Then the following statements are equivalent:*

(i) G and M_G are H_f^k-cospectral.
(ii) $\lambda_1(G) = \lambda_1(M_G)$.
(iii) M_G admits a $\mathscr{A}$-partition.

Proof. It's obvious that (i) implies (ii). Note that M_G could be changed into G by a *three-way switching* if it admits a $\mathscr{A}$*-partition*. So (iii) implies (i) by Theorem 2. In order to complete the proof, it suffices to prove that (ii) implies (iii). Assume that (ii) holds. Let $H_f^k(M_G) = (h_{uv})_{n\times n}$ and $H_f^k(G) = A_f(G) = (a_{uv})_{n\times n}$, let $Y = (y_1, y_2, \dots, y_n)^T \in C^n$ be a normalized eigenvector of $H_f^k(M_G)$ corresponding to $\lambda_1(M_G)$. Note that $|h_{u,v}| = a_{u,v}$ for every $u, v \in |V(M_G)|$. Then we have

$$\begin{aligned}\lambda_1(M_G) =& \bar{Y}^T \cdot H_f^k(M_G) \cdot Y \\ =& \sum_{u\in V(M_G)} \sum_{v\in V(M_G)} \bar{y_u} h_{u,v} y_v \\ \leq& \sum_{u\in V(M_G)} \sum_{v\in V(M_G)} |\bar{y_u} y_v| \cdot |h_{u,v}| \\ =& \sum_{u\in V(M_G)} \sum_{v\in V(M_G)} |y_u||y_v| \cdot a_{u,v} \leq \lambda_1(G). \end{aligned} \tag{1}$$

The equality in (1) must holds since $\lambda_1(M_G) = \lambda_1(G)$, which require that

$$h_{u,v}\bar{y}_u y_v = |h_{u,v}\bar{y}_u y_v| \tag{2}$$

for all edges uv in G. Without loss of generality, we can assume that a vertex $v_0 \in V(M_G)$ such that $y_{v_0} \in R^+$, then $\frac{|\bar{y}_{v_0}|}{\bar{y}_{v_0}} = 1$ and $h_{v_0,v} y_v = |h_{v_0,v} y_v|$ for all $y \in N_G(v_0)$ by (4), it follows that $\frac{y_v}{|y_v|} = \omega^{-1}$ if $v \in N^+_{M_G}(v_0)$; $\frac{y_v}{|y_v|} = 1$ if $v \in N^0_{M_G}(v_0)$ and $\frac{y_v}{|y_v|} = \omega$ if $v \in N^-_{M_G}(v_0)$.

Note that G is connected, by repeating the above steps, it follows that $\frac{y_v}{|y_v|} = \omega^t$ with $t \in Z$ for all $v \in V(M_G)$. Let $V_i = \{v \in V(M_G) : \frac{y_v}{|y_v|} = \omega^t,\ \textit{where}\ t \equiv i(\textit{mod}\ k)\}$, where $i \in \{1, 2, \dots, k\}$. it's easy to check that $V_1, V_2, \dots, V_k$ form a $\mathscr{A}$*-partition* of M_G, which complete the proof. □

With Theorem 3, we have following corollary.

Corollary 4. *A mixed tree M_T is H_f^k-cospectral to its underlying graph.*

Proof. We construct a $\mathscr{A}$*-partition* of $V(M_T)$ to prove this corollary. Choose a vertex $v_0 \in V(M_T)$ and denote by $P(u)$ the unique (v_0, u)-path in T for all $u \in V(M_T)$, then we define a function f on $V(G)$, where the value of $f(u)$ equals to the number of arcs toward v_0 in $P(u)$ minus the number of arcs toward u in $P(u)$.

Let $V_i = \{u \in V(M_T) : f(u) \equiv i(\textit{mod}\ k)\}$, where $i \in \{1, 2, \dots, k\}$. Obviously $V(M_T) = V_1 \cup V_2 \cup \cdots \cup V_k$ and it's easy to check that for every undirected edge $\{u, v\}$ with $u \in V_i$ and $v \in V_j$, $f(u) - f(v) = 0$ so $i = j$; for every arc (u, v) with $u \in V_i$ and $v \in V_j$, $f(u) - f(v) = 1$ so $i - j \equiv 1(\textit{mod}\ k)$. It implies that $V_1 \cup V_2 \cup \cdots \cup V_k$ forms a $\mathscr{A}$*-partition* of $V(M_T)$ and complete the proof. □

Then the main Theorem in [12] can be directly generalized to mixed trees.

Corollary 5. *Assume that $f(x, y)$ is increasing and convex in variable x, Then the mixed trees in n vertices owns largest H_f^k-spectral radius if and only if its underlying graph is S_n or a double star $S_{d,n-d}$ for some $d \in \{2, \dots, n-2\}$.*

Corollary 6. *Assume that $f(x, y)$ has the form $P(x, y)$ or $\sqrt{P(x, y)}$, where $P(x, y)$ is a symmetric polynomial with nonnegative coefficients and zero constant term. Then the mixed tree on $n(n \geq 9)$ vertices owns the smallest H_f^k-spectral radius if and only if its underlying graph is P_n.*

4. Proof of Theorem 1

Lemma 1. *Let M_G be a mixed graph. Then $|\lambda_i(M_G)| \leq \lambda_1(G)$ for all $i \in \{1, 2 \dots, n\}$.*

Proof. Let $Y = (y_1, y_2, \ldots, y_n)^T \in C^n$ be a normalized eigenvector of $H_f^k(M_G)$ corresponding to $\lambda_i(M_G)$. Then we have

$$\begin{aligned}|\lambda_i(M_G)| =&|\bar{Y}^T \cdot H_f^k(M_G) \cdot Y| \\ =&|\sum_{u\in V(M_G)}\sum_{v\in V(M_G)} \bar{y}_u h_{u,v} y_v| \\ \leq& \sum_{u\in V(M_G)}\sum_{v\in V(M_G)} |\bar{y}_u y_v| \cdot |h_{u,v}| \\ =& \sum_{u\in V(M_G)}\sum_{v\in V(M_G)} |y_u||y_v| \cdot a_{u,v} \leq \lambda_1(G).\end{aligned}$$

Which completes the proof. □

Note that $|\lambda_n(M_G)| > |\lambda_1(M_G)|$ is possible for a mixed graph M_G. By Lemma 1, it follows that $\rho_f^k(G) = \rho_f^k(M_G)$ if and only if $\lambda_1(M_G) = \lambda_1(G)$ or $-\lambda_n(M_G) = \lambda_1(G)$. In the following two lemmas we focus on the condition of the second equality.

Lemma 2. *Let k be an odd positive integer and M_G be a connected mixed graph, then $-\lambda_n(M_G) = \lambda_1(G)$ if and only if $V(M_G)$ admits a $\mathscr{C}$-partition.*

Proof. If $V(M_G)$ admits a $\mathscr{C}$*-partition*, say $V_1 \cup \cdots \cup V_k \cup U_1 \cup \cdots \cup U_k$. Then G is a bipartite graph and $(V_1 \cup U_1),\ (V_1 \cup U_1), \ldots,\ (V_k \cup U_k)$ form a $\mathscr{A}$*-partition* of $V(M_G)$, by Perron-Frobinius Theorem and Theorem 3 it follows that $\lambda_n(M_G) = \lambda_n(G) = -\lambda_1(G)$.

If $-\lambda_n(M_G) = \lambda_1(G)$ holds, Let $H_f^k(M_G) = (h_{uv})_{n\times n}$ and $H_f^k(G) = A_f(G) = (a_{uv})_{n\times n}$, let $Y = (y_1, y_2, \ldots, y_n)^T \in C^n$ be a normalized eigenvector of $H_f^k(M_G)$ corresponding to $\lambda_n(M_G)$. Then we have

$$\begin{aligned}-\lambda_n(M_G) =& - \bar{Y}^T \cdot H_f^k(M_G) \cdot Y \\ =& - \sum_{u\in V(M_G)}\sum_{v\in V(M_G)} \bar{y}_u h_{uv} y_v \\ \leq& \sum_{u\in V(M_G)}\sum_{v\in V(M_G)} |\bar{y}_u y_v| \cdot |h_{uv}| \\ =& \sum_{u\in V(M_G)}\sum_{v\in V(M_G)} |y_u||y_v| \cdot a_{uv} \leq \lambda_1(G). \qquad (3)\end{aligned}$$

The equality in (3) must holds because $-\lambda_n(M_G) = \lambda_1(G)$, which requires that

$$h_{u,v}\bar{y}_u y_v = -|h_{u,v}\bar{y}_u y_v|, \qquad (4)$$

for all edges uv in G. Without loss of generality, we can assume that a vertex $v_0 \in V(M_G)$ such that $y_{v_0} \in R^+$, then $\frac{|\bar{y}_{v_0}|}{\bar{y}_{v_0}} = 1$ and $h_{v_0,v} y_v = |h_{v_0,v} y_v|$ for all $y \in N_G(v_0)$ by (4), it follows that $\frac{y_v}{|y_v|} = -\omega^{-1}$ if $v \in N^+_{M_G}(v_0)$; $\frac{y_v}{|y_v|} = -1$ if $v \in N^0_{M_G}(v_0)$ and $\frac{y_v}{|y_v|} = -\omega$ if $v \in N^-_{M_G}(v_0)$.

Note that G is connected, by repeating the above steps, it follows that $\frac{y_v}{|y_v|} \in \{\pm\ \omega^t, t \in Z\}$ for all $v \in V(M_G)$. Let $V_i = \{v \in V(M_G) : \frac{y_v}{|y_v|} = \omega^t,\ \textit{where}\ t \equiv i (mod\ k)\}$ and $U_i = \{v \in V(M_G) : \frac{y_v}{|y_v|} = -\omega^t,\ \textit{where}\ t \equiv i (mod\ k)\}$ where $i \in \{1, 2, \ldots, k\}$. it's easy to check that $V_1 \cup \cdots \cup V_k \cup U_1 \cup \cdots \cup U_k$ form a $\mathscr{C}$*-partition* of $V(M_G)$, which complete the proof. □

Note that a $\mathscr{C}$*-partition* $V_1, \ldots, V_k, U_1, \ldots, U_k$ of $V(G)$ forms into a $\mathscr{A}$*-partition* $(V_1 \cup U_1),\ (V_1 \cup U_1), \ldots,\ (V_k \cup U_k)$. So we have following corollary.

Corollary 7. *Let k be an odd positive integer and M_G be a connected mixed graph, then $\rho(G) = \rho(M_G)$ if and only if $V(M_G)$ admits a $\mathscr{A}$-partition.*

Lemma 3. *Let k be an even positive integer and G be a connected, regular graph, then a mixed graph M_G with underlying graph G such that $-\lambda_n(M_G) = \lambda_1(G)$ if and only if M_G admits a $\mathscr{B}$-partition.*

Proof. Assume that G is Δ-regular, let $H_f^k(M_G) = (h_{uv})_{n\times n}$ and $H_f^k(G) = A_f(G) = (a_{uv})_{n\times n}$, let $X = (x_1, x_2, \ldots, x_n)^T \in R^n$ be a normalized eigenvector of $A_f(G)$ corresponding to $\lambda_1(G)$ and $Y = (y_1, y_2, \ldots, y_n)^T \in C^n$ be a normalized eigenvector of $H_f^k(M_G)$ corresponding to $\lambda_n(M_G)$.

Assume that $-\lambda_n(M_G) = \lambda_1(G)$, then by the similar proof it follows that

$$h_{u,v}\bar{y}_u y_v = -|h_{u,v}\bar{y}_u y_v|, \tag{5}$$

for all edges uv in G. Without loss of generality, we can assume that a vertex $v_0 \in V(M_G)$ such that $y_{v_0} \in R^+$, then $\frac{|\bar{y}_{v_0}|}{\bar{y}_{v_0}} = 1$ and $h_{v_0,v}y_v = |h_{v_0,v}y_v|$ for all $y \in N_G(v_0)$ by (5), it follows that $\frac{y_v}{|y_v|} = -\omega^{-1} = \omega^{\frac{k}{2}-1}$ if $v \in N^+_{M_G}(v_0)$; $\frac{y_v}{|y_v|} = -1 = \omega^{\frac{k}{2}}$ if $v \in N^0_{M_G}(v_0)$ and $\frac{y_v}{|y_v|} = -\omega = \omega^{\frac{k}{2}+1}$ if $v \in N^-_{M_G}(v_0)$.

Note that G is connected, by repeating the above steps, it follows that $\frac{y_v}{|y_v|} \in \{\omega^t, t \in Z\}$ for all $v \in V(M_G)$. Let $V_i = \{v \in V(M_G) : \frac{y_v}{|y_v|} = \omega^t,\ \textit{where}\ t \equiv i (\textit{mod}\ k)\}$, where $i \in \{1, 2, \ldots, k\}$. it's easy to check that $V_1 \cup \cdots \cup V_k$ forms a *$\mathscr{B}$-partition* of $V(M_G)$.

Assume that M_G admits a *$\mathscr{B}$-partition* $V_1 \cup V_2 \cup \cdots \cup V_k$, we construct a vector $Z \in C^n$ with $z_u = x_u \cdot w^i$, where $u \in V_i$. Then for any $u \in V(M_G)$ with $u \in V_i$,

$$\begin{aligned}
(H_f^k(G)\cdot Z)_u =& f(\Delta,\Delta)\Big(\sum_{v\in N^0_{M_G}(u)} z_v + \sum_{v\in N^+_{M_G}(u)} \omega z_v + \sum_{v\in N^-_{M_G}(u)} \omega^{-1} z_v\Big)\\
=& f(\Delta,\Delta)\Big(\sum_{v\in N^0_{M_G}(u)} x_v\omega^{i+\frac{k}{2}} + \sum_{v\in N^+_{M_G}(u)} \omega x_v \omega^{i+\frac{k}{2}-1}\\
&+ \sum_{v\in N^-_{M_G}(u)} \omega^{-1} x_v \omega^{i+\frac{k}{2}+1}\Big)\\
=& -f(\Delta,\Delta)\omega^i \sum_{v\in N_G(u)} x_v\\
=& -\lambda_1(G)\omega^i x_u = -\lambda_1 z_u.
\end{aligned}$$

It follows that $H_f^k(G)\cdot Z = -\lambda_1(G)Z$, so $-\lambda_n(M_G) = \lambda_1(G)$. □

Combined with Theorem 3 we have following corollary.

Corollary 8. *Let k be a even integer positive and G be a connected, regular graph, then a mixed graph M_G such that $\rho_f^k(M_G) = \rho_f^k(G)$ if and only if $V(G)$ admits a $\mathscr{A}$-partition or a $\mathscr{B}$-partition.*

Lemma 4. *Let k be a positive integer and $f(x, y) > 0$ be a symmetric real function, not decreasing in variable x, then for any proper subgraph H of G, we have $\rho_f^k(H) < \rho_f^k(G)$.*

Proof. We have following two claims.

Claim 1. *For any proper, spanning and connected subgraph H of G, $\rho_f^k(H) < \rho_f^k(G)$.*

Let $y_1 > 0, y_2 > 0$ be the Perron vector of $\rho(G)$ and $\rho(H)$, respectively. Note that $A_f(G)-A_f(G') \geq 0$ and $A_f(G) - A_f(G') \neq 0$ since $f(x, y)$ is symmetric, not decreasing in x and H is a proper subgraph of G. Then

$$(\rho_f^k(G) - \rho_f^k(H)){y_1}'y_2 = {y_1}'(A_f(G) - A_f(H))y_2 > 0.$$

It follows that $\rho_f^k(G) > \rho_f^k(H)$.

Claim 2. *For any proper, induced subgraph H of G, $\rho_f^k(H) < \rho_f^k(G)$.*

Let $y > 0$, $y_1 > 0$ be the Perron vector of $\rho_f^k(G)$ and $\rho_f^k(H)$, respectively. We can assume that

$$A_f(G) = \begin{pmatrix} A_f(H) + K & L \\ L' & M \end{pmatrix}$$

where $K \geq 0$ is symmetric. Then

$$A_f(G)\begin{pmatrix} y_1 \\ 0 \end{pmatrix} = \begin{pmatrix} (A_f(H) + K)y_1 \\ L'y_1 \end{pmatrix} = \begin{pmatrix} \rho_f^k(H)y_1 + Ky_1 \\ L'y_1 \end{pmatrix} \geq \rho_f^k(H)\begin{pmatrix} y_1 \\ 0 \end{pmatrix}.$$

If

$$A_f(G)\begin{pmatrix} y_1 \\ 0 \end{pmatrix} = \rho_f^k(H)\begin{pmatrix} y_1 \\ 0 \end{pmatrix},$$

then $\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \geq 0$ is a eigenvector of $A_f(G)$ corresponding to $\rho_f^k(G)$, which is contradict to Perron-Frobinius Theorem. So we have $A_f(G)\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \geq \rho_f^k(H)\begin{pmatrix} y_1 \\ 0 \end{pmatrix}$ and $A_f(G)\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \neq \rho_f^k(H)\begin{pmatrix} y_1 \\ 0 \end{pmatrix}$. It follows that $(\rho_f^k(G) - \rho_f^k(H))y'\begin{pmatrix} y_1 \\ 0 \end{pmatrix} = y'(A_f(G)\begin{pmatrix} y_1 \\ 0 \end{pmatrix} - \rho_f^k(H)\begin{pmatrix} y_1 \\ 0 \end{pmatrix}) \geq 0$, so $\rho_f^k(G) > \rho_f^k(H)$.

For any proper subgraph H of G, we can assume that H is connected. Then H must be a spanning subgraph of some induced subgraph H' of G. If $H' = G$, then by Claim 1 it follows that $\rho_f^k(H) < \rho_f^k(H') = \rho_f^k(G)$. If $H' \neq G$, then by Claim 2 it follows that $\rho_f^k(H) \leq \rho_f^k(H') < \rho_f^k(G)$. It completes the proof. □

Proof of Theorem 1. Note that if G is a Δ-regular graph, then $A_f(G) = f(\Delta, \Delta)A(G)$ and $\rho_f^k(G) = f(\Delta, \Delta)\Delta(G)$. Since every graph with maximum degree Δ must be a subgraph of a Δ-regular graph, by Lemma 4 it follows that $\rho_f^k(G) \leq f(\Delta, \Delta)\Delta(G)$ for all graphs G with maximum degree Δ, where equality holds if and only if G is Δ-regular.

For a mixed graph M_G with maximum degree Δ, by Lemma 1 it follows that $\rho_f^k(M_G) \leq f(\Delta, \Delta)\Delta(G)$, by Corollary 7, 8 it follows that equality holds if and only if underlying graph G is $\Delta(G)$-regular and M_G admits a $\mathscr{A}$*-partition* or M_G admits a $\mathscr{B}$*-partition* when k is even. □

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Conflict of Interest

The authors declare no conflict of interest.

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