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Article

A note on Frobenius-Zorn's Theorem

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Abstract: We study the nonzero algebraic real algebras A with no nonzero joint divisor of zero. We prove that if $Z(A) \neq 0$ and A satisfies one of the Moufang identity, then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. We show also that if A is power-associative flexible and satisfies the identity $(a, a, [a, b]) = 0$, then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. Finally, we prove that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only algebraic real algebras with no nonzero divisor of zero satisfying the middle Moufang identity, or the right and left Moufang identities.

Keywords: Algebraic (quadratic) algebra, Joint divisor of zero, Moufang identities, Weakly alternative
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1. Introduction and Preliminaries

The study of real algebras without divisors of zero started since the discovery of $\mathbb{H}$ (Hamilton's quaternions) and $\mathbb{O}$ (Cayley's octonions).

Frobenius proves that $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are the only associative, quadratic real algebras without divisors of zero [1, 2]. Zorn shows that every alternative, quadratic, non associative real algebra without divisors of zero is isomorphic to the Cayley algebra $\mathbb{O}$ [1, 3]. So, every alternative, quadratic real algebra without divisors of zero is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ (Frobenius-Zorn theorem).

Many researchers have given minimum conditions for an algebra to be isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. The Frobenius-Zorn theorem will then be either improved or extended.

By definition an absolute valued real algebra is an algebra $\mathcal{A}$ endowed with an absolute value, i.e. a norm $\|\cdot\|$ on the vector space of $\mathcal{A}$ satisfying $\|xy\| = \|x\|\|y\|$ for all $x, y \in \mathcal{A}$. Clearly absolute valued algebras have no nonzero divisor of zero, and hence have no nonzero joint divisor of zero. Albert shows that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only absolute valued real algebraic algebras with unit [4, Theorem 2].

An algebra $\mathcal{A}$ is called division algebra, if $L_x : \mathcal{A} \to \mathcal{A} \;\; a \mapsto xa$ and $R_x : \mathcal{A} \to \mathcal{A} \;\; a \mapsto ax$ are bijective for all $x \in \mathcal{A}\backslash\{0\}$. Cuenca proves that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only division normed real algebras satisfying the middle Moufang identity [5, Theorem 2.3] and he also classified the division normed real algebras satisfying the left or right Moufang identity [5, Theorems 2.1 and 2.2]. In [6], Cuenca also studied the composition algebras satisfying Moufang identities.

Let $\mathcal{A}$ be a normed algebra. Recall that an element x of $\mathcal{A}$ is said to be a joint topological divisor of zero in $\mathcal{A}$ if there is a sequence x_n of norm-one elements of $\mathcal{A}$ satisfying $xx_n \to 0$ and $x_nx \to 0$.

If $\mathcal{A}$ is a weakly alternative normed real unital algebra with no nonzero joint topological divisor of zero, then $\mathcal{A}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ [7, Theorem III.5.13].

The algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only normed alternative real algebras with no nonzero joint topological divisor of zero [8, Theorem 2.5.50].

In [8, Theorem 2.5.29], Cabrera and Rodriguez prove also the following result.

Theorem 1. *Let $\mathcal{A}$ be a nonzero alternative real algebraic algebra with no nonzero joint divisor of zero. Then $\mathcal{A}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

In this paper, we prove that the Frobenius-Zorn theorem remains if we replace quadratic by algebraic and alternative by weakly alternative, middle Moufang identity, or left and right Moufang identities.

In this last part of this paragraph, we are going to discuss the assumptions on $\mathcal{A}$ in the main theorems.

We recall that if $\mathcal{A}$ is alternative, then $\mathcal{A}$ satisfies the three Moufang identity and $\mathcal{A}$ is weakly alternative. We precise also that every quadratic algebra is algebraic.

Let $\mathbb{A} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$. We recall that $\overset{\star}{\mathbb{A}}$ is obtained by endowing the normed space $\mathbb{A}$ with the product $x \cdot y = \overline{x}\,\overline{y}$, where $x \mapsto \overline{x}$ means the standard involution. We note that these algebras are algebraic, but are not quadratic.

We recall that $^*\mathbb{C}$ and $\mathbb{C}^*$ are obtained by endowing the normed space $\mathbb{C}$ with the products $x \cdot y = \overline{x}y$, and $x \cdot y = x\overline{y}$, respectively, where $x \mapsto \overline{x}$ means the standard involution. These algebras are algebraic and it is easy to verify that $^*\mathbb{C}$ satisfies right Moufang identity and $\mathbb{C}^*$ verifies the left Moufang identity. However, none of the algebras $^*\mathbb{C}$ and $\mathbb{C}^*$ is alternative. This implies that neither of the two identities of Moufang of the assertion 2 from Theorem 3 can be removed.

2. Notations and Preliminary

In this paper, A is a nonzero algebra over a field $\mathbb{K}$ of characteristic zero. We recall that $(.,.,.)$, $[.,.]$, $A(x_1, ..., x_n)$ denote respectively the associator, the commutator, and the subalgebra of A generated by $x_1, ..., x_n \in A$.

An element x of A is said to be a *divisor of zero* in A if there exists $y \in A \setminus \{0\}$ such that $xy = 0$ or $yx = 0$, and that x is said to be a *joint divisor of zero* in A if there is $y \in A \setminus \{0\}$ such that $xy = 0$ and $yx = 0$.

An element a of an algebra is said to be isotropic whenever $a \neq 0 = a^2$. We note that isotropic elements are nonzero joint divisors of zero.

Let A be an algebra over $\mathbb{K}$. The centre of A is defined as the subset of A consisting of those elements $a \in A$ such that

$$[a, A] = (a, A, A) = (A, a, A) = (A, A, a) = 0$$

and is denoted by $Z(A)$. Elements of $Z(A)$ are called central elements of A. We note that $Z(A)$ becomes an associative and commutative subalgebra of A.

An algebra A is called alternative (resp. flexible) if $x^2y = x(xy)$ and $yx^2 = (yx)x$ (resp. $(x, y, x) = 0$) for all $x, y \in A$.

An algebra A is called power-associative (resp. algebraic) if $A(x)$ is associative (resp. $A(x)$ is finite dimensional) for all $x \in A$.

We also remember that an algebra A satisfies:

(i) the middle Moufang identity, if $(ab)(ca) = [a(bc)]a$, for all $a, b, c \in A$,

(ii) the right Moufang identity, if $b[(ac)a] = [(ba)c]a$, for all $a, b, c \in A$,

(iii) the left Moufang identity, if $[a(ba)]c = a[b(ac)]$, for all $a, b, c \in A$.

An algebra A is called non-commutative Jordan, if A is flexible and satisfies $(x^2, y, x) = 0$ for all $x, y \in A$. A non commutative Jordan $\mathbb{K}$-algebra A is said to be weakly alternative if it satisfies the identity $(x, x, [x, y]) = 0$ for all $x, y \in A$ [7].

An algebra A is called quadratic algebra, if A contains a unit element $\mathbf{1}$ and $\mathbf{1}, x, x^2$ are linearly dependent for all $x \in A$.

Let $\mathcal{A}$ be a quadratic algebra over $\mathbb{K}$. By definition, for each $a \in A$, there are $t(a)$ and $n(a)$ of $\mathbb{K}$ such that

$$a^2 - t(a)a + n(a)\mathbf{1} = 0. \tag{1}$$

If a belongs to $A \setminus (\mathbb{K}\mathbf{1})$, the scalars $t(a)$ and $n(a)$ in (1) are uniquely determined. Otherwise, we have $a = \alpha\mathbf{1}$ for a unique $\alpha \in \mathbb{K}$, and we set $t(a) := 2\alpha$ and $n(a) := \alpha^2$, so that (1) is fulfilled. The mappings $a \to t(a)$ and $a \to n(a)$, from A to $\mathbb{K}$, defined in this way are called the trace function and the algebraic norm function on A, respectively.

It follows from [8, Proposition 2.5.12] that $A = \mathbb{K}\mathbf{1} \oplus V$, where $V := ker(t)$. So, for every $a \in \mathcal{A}$, we have unique writing $a = \lambda\mathbf{1} + x$ for $\lambda \in \mathbb{K}$ and $x \in V$. We recall that λ is called the scalar part of a, and x is called the vector part of a. If for $x, y \in V$, we denote by $-(x, y)$ and $x \times y$ the scalar and vector parts of xy, respectively, that is to say, if we write

$$xy = -(x, y)\mathbf{1} + x \times y, \quad \text{with} \quad (x, y) \in \mathbb{K} \quad \text{and} \quad x \times y \in V,$$

then we are provided with a bilinear form $(.,.) : V \times V \to \mathbb{K}$ and a bilinear mapping $\times : V \times V \to V$. We note that, for $x \in V$, we have $x^2 \in \mathbb{K}\mathbf{1}$ (by (2.5.9) in [8]) and also $x^2 = -(x, x)\mathbf{1} + x \times x$, so that, since $x \times x$ lies in V, we get $x \times x = 0$. Therefore $(V, \times)$ is an anticommutative algebra.

Every right alternative algebra is power-associative [9, Lemma 1]. The Mikheev's theorem proves that if A is right alternative algebra, then $(x, x, y)^4 = 0$ [10]. By switching to the opposite algebra, we deduce the following corollary:

Corollary 1. *Let A be an algebra over $\mathbb{K}$ without isotropic elements. Then the following assertions are equivalent:*

1. *A is a right alternative algebra,*
2. *A is a left alternative algebra,*
3. *A is an alternative algebra.*

3. Main Results

Proposition 1. *Let A be a nonzero algebraic real algebra with no nonzero joint divisor of zero, and with nonzero centre. If A satisfies one of the Moufang identities, then A is a quadratic alternative algebra.*

Proof. Since the centre of A is nonzero, it follows from [8, Proposition 2.5.33] that A is unital.

We now study each one of the Moufang identities separately.

Case (m): A is a middle Moufang algebra.

Taking $b = \mathbf{1}$ in the middle Moufang identity $(ab)(ca) = [a(bc)]a$ we get $a(ca) = (ac)a$, that is, A is a flexible algebra, and consequently, taking $c = a$ we obtain that $[a^2, a] = 0$. Moreover, taking $c = b = a$ in the middle Moufang identity we see that $a^2a^2 = (aa^2)a$. Since

$$(a^2a)a - (aa^2)a = [a^2, a]a = 0,$$

it follows that $a^2a^2 = (a^2a)a$, that is, $(a^2, a, a) = 0$. Thus A is a power-associative algebra [11], and hence, by [8, Proposition 2.5.10(ii)], A is quadratic.

Let us write $A = \mathcal{A}(V, \times, (.,.))$. Keeping in mind that A is flexible, by [8, Proposition 2.5.18(ii)] we know that

$$(.,.) \quad \text{is symmetric and} \quad (x \times y, z) = (x, y \times z) \quad \text{for all} \quad x, y, z \in V \tag{2}$$

For any $x, y \in V$, we have that

$$xy = -(x, y)\mathbf{1} + x \times y \quad \text{and} \quad yx = -(y, x)\mathbf{1} + y \times x,$$

and hence

$$\begin{aligned} (xy)(yx) &= ((x, y)(y, x) - (x \times y, y \times x))\mathbf{1} \\ &\quad - (x, y)y \times x - (y, x)x \times y + (x \times y) \times (y \times x) \\ &= ((x, y)^2 + (x \times y, x \times y))\mathbf{1}, \end{aligned} \tag{3}$$

where in the last equality we have used the symmetry of $(.,.)$, and the anticommutativity of $\times$. On the other hand,

$$xy^2 = x(- (y, y)\mathbf{1} + y \times y) = x(- (y, y)\mathbf{1}) = -(y, y)x,$$

where in the second equality we have used the anticommutativity of $\times$. Therefore,

$$(xy^2)x = -(y, y)x^2 = (y, y)(x, x)\mathbf{1} - (y, y)x \times x = (y, y)(x, x)\mathbf{1}, \tag{4}$$

where in the last equality we have used the anticommutativity of $\times$.

Since $(xy)(yx) = (xy^2)x$ (by the middle Moufang identity), it follows from (3) and (4) that

$$(x \times y, x \times y) = (x, x)(y, y) - (x, y)^2.$$

Therefore, by [8, Proposition 2.5.18(iii)], the algebraic norm fonction n on A admits composition. Moreover n is nondegenerate because of [8, Lemma 2.5.15 and Proposition 2.5.18(i)]. Finally, by [8, Corollary 2.5.19(ii)], we conclude that A is alternative.

Case (r): A is a right Moufang algebra.

Taking $c = \mathbf{1}$ in the right Moufang identity $b[(ac)a] = [(ba)c]a$ we get $ba^2 = (ba)a$, that is, A is a right alternative algebra. Therefore, by Corollary 1, A is alternative. Moreover, by [8, Proposition 2.5.10(ii)], we conclude that A is quadratic.

Case (l): A is a left Moufang algebra.

This case follows from Case (r) by passing to the opposite algebra. □

Combining the above proposition with Theorem 1 we obtain the following.

Theorem 2. *Let A be a nonzero algebraic real algebra with no nonzero joint divisor of zero and with nonzero centre. If A satisfies one of the Moufang identities, then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

The following result provides us with sufficient conditions for an algebra satisfying one of the Moufang identities to have a nonzero centre.

Proposition 2. *Let A be an algebra over $\mathbb{K}$ with no nonzero divisor of zero. We have:*

(i) *If A satisfies the middle or right Moufang identity, and if A has a right unit, then A is unital.*

(ii) *If A satisfies the middle or left Moufang identity, and if A has a left unit, then A is unital.*

Proof. (i) Suppose that A has a right unit e.

If A satisfies the middle Moufang identity, then, taking $a = b = e$ in that identity, we get $ec = e(ec)$, hence $e(c - ec) = 0$, so $c = ec$, and as a result e is the unit of A.

If A satisfies the right Moufang identity, then, taking $a = b = e$ in that identity, we get $e(ec) = ec$, hence $e(ec - c) = 0$, so $ec = c$, and as a result e is the unit of A.

The proof of (ii) is similar. □

Proposition 2 yields the next consequence of Theorem 2.

Corollary 2. *Let A be a nonzero algebraic real algebra with no nonzero divisor of zero. Suppose that A satisfies one of the following conditions:*

(i) *A is a middle or right Moufang algebra, and A has a right unit.*

(ii) *A is a middle or left Moufang algebra, and A has a left unit.*

Then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.

Theorem 3. *Let A be a nonzero algebraic real algebra with no nonzero divisor of zero. Then the following assertions are equivalent:*

(i) *A is a middle Moufang algebra,*

(ii) *A is a right and left Moufang algebra,*

(iii) *A is alternative,*

(iv) *A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

Proof. The implications $(iii) \Rightarrow (i)$, $(iii) \Rightarrow (ii)$, and $(iv) \Rightarrow (iii)$ are clear, whereas the implication$(iii) \Rightarrow (iv)$ follows from Theorem 1.

$(i) \Rightarrow (iii)$. Let $x \in A \setminus \{0\}$. Then the subalgebra $A(x)$ is a finite-dimensional division real algebra. Since every finite-dimensional real algebra can be provided with an algebra norm [8, Proposition 1.1.7], it follows from [5, Theorem 2.3] that $A(x)$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$, hence $A(x)$ is power-associative, and so $A(x)$ is associative. Now, it follows from the arbitrariness of x in $A \setminus \{0\}$ that A is power-associative. Therefore, by [8, Proposition 2.5.10(ii)], A is quadratic, and hence A has a unit. Finally, A is alternative because of Proposition 1.

$(ii) \Rightarrow (iii)$. Argue as in the above paragraph, with [5, Theorems 2.1 and 2.2] instead of [5, Theorem 2.3]. □

Proposition 3. *Let A be a nonzero flexible quadratic algebra over $\mathbb{K}$ with no nonzero joint divisor of zero. If A satisfies the identity $(a, a, [a, b]) = 0$, then A is alternative.*

Proof. Let us write $A = \mathcal{A}(V, \times, (.,.))$. Since A is flexible it follows from [8, Proposition 2.5.18(ii)] that

$$(.,.) \quad \text{is symmetric and} \quad (x \times y, z) = (x, y \times z) \quad \text{for all} \quad x, y, z \in V. \tag{5}$$

For any $x, y \in V$, we have that

$$xy = -(x, y)\mathbf{1} + x \times y \quad \text{and} \quad yx = -(y, x)\mathbf{1} + y \times x,$$

and hence the symmetry of $(.,.)$ and the anticommutativity of $\times$, yield $[x,y] = 2x \times y$. Moreover, keeping in mind the second assertion in (5) and the anticommutativity of $\times$, we see that

$$(x, x \times y) = (x \times x, y) = 0 \quad \text{and} \quad (x, x \times (x \times y)) = (x \times x, x \times y) = 0.$$

Therefore

$$x^2[x,y] = (-(x,x)\mathbf{1})(2x \times y) = -2(x,x)x \times y,$$

$$x[x,y] = x(2x \times y) = -2(x, x \times y)\mathbf{1} + 2x \times (x \times y) = 2x \times (x \times y)$$

and

$$\begin{aligned} x(x[x,y]) &= x(2x \times (x \times y)) \\ &= -2(x, x \times (x \times y))\mathbf{1} + 2x \times (x \times (x \times y)) \\ &= 2x \times (x \times (x \times y)). \end{aligned}$$

Hence

$$\begin{aligned} 0 &= (x, x, [x,y]) \\ &= x^2[x,y] - x(x[x,y]) \\ &= -2(x,x)x \times y - 2x \times (x \times (x \times y)), \end{aligned}$$

and consequently

$$x \times (x \times (x \times y)) = -(x,x)x \times y. \tag{6}$$

For the sake of comfortability, let us set $m_{x,y} := x \times (x \times y) - (x,y)x + (x,x)y$. Note that

$$(x, m_{x,y}) = (x, x \times (x \times y)) - (x,y)(x,x) + (x,x)(x,y) = 0$$

and that, keeping in mind (6), also

$$x \times m_{x,y} = x \times (x \times (x \times y)) - (x,y)x \times x + (x,x)x \times y = 0.$$

Therefore

$$xm_{x,y} = -(x, m_{x,y})\mathbf{1} + x \times m_{x,y} = 0$$

and

$$m_{x,y}x = -(m_{x,y}, x)\mathbf{1} + m_{x,y} \times x = -(x, m_{x,y})\mathbf{1} - x \times m_{x,y} = 0.$$

Thus $m_{x,y}$ is a joint divisor of zero of A, and consequently $m_{x,y} = 0$. Finally, looking at [8, Proposition 2.5.18(iv)], we conclude that A is alternative. □

Theorem 4. *Let A be a nonzero power-associative flexible algebraic real algebra with no nonzero joint divisor of zero. If A satisfies the identity $(a, a, [a,b]) = 0$, then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

Proof. By [8, Proposition 2.5.10(ii)], A is quadratic. Suppose that A satisfies the identity $(a, a, [a,b]) = 0$. Then, by Proposition 3, A is alternative, and hence A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ because of Theorem 1. □

Recall the definition of a weakly alternative algebra, as well as the fact that non-commutative Jordan algebras are power-associative [8, Proposition 2.4.19], we have the following corollary.

Corollary 3. *Every nonzero weakly alternative algebraic real algebra with no nonzero joint divisor of zero is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.*

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Conflict of Interest

The authors declare no conflict of interest.

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