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Article

Metric Dimension of Corona Product and Join Graph of Zero Divisor Graphs of Direct Product of Finite Fields

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Abstract: The metric dimension of a graph is the smallest number of vertices such that all vertices are uniquely determined by their distances to the chosen vertices. The corona product of graphs G and H is the graph $G \odot H$ obtained by taking one copy of G, called the center graph, $|V(G)|$ copies of H, called the outer graph, and making the j^{th} vertex of G adjacent to every vertex of the j^{th} copy of H, where $1 \leqslant j \leqslant |V(G)|$. The Join graph $G + H$ of two graphs G and H is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. In this paper, we determine the Metric dimension of Corona product and Join graph of zero divisor graphs of direct product of finite fields.

Keywords: Zero divisor graph, Metric dimension, Corona product, Join graph, finite fields
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1. Introduction and Preliminaries

The concept of graph of a commutative ring R, was introduced by Beck [1] in 1988. Beck considered all the elements of the ring R as the vertices of the graph and two distinct vertices x and y are adjacent in the graph if and only if $x \cdot y = 0$, the additive identity of the ring R. This definition of graph given by Beck was modified by Anderson and Livingston [2] in 1999. Anderson and Livingston considered only the non-zero zero divisors of commutative ring R to be the vertex set of the graph known as zero divisor graph of R denoted by $\Gamma(R)$ in which two distinct vertices x and y are adjacent in $\Gamma(R)$ if and only if $x \cdot y = 0$. In this paper we consider the zero divisor graph of the reduced ring, $R = F_1 \times F_2 \times \cdots \times F_n$, of direct product of finite fields. Let $Z^*(R)$ be the set of non-zero zero-divisors of the ring $R = F_1 \times F_2 \times \cdots \times F_n$ and $\Gamma(R)$ denote the graph with vertex set as $Z^*(R)$ and edge set as $\{rs : r \cdot s = 0, r, s \in Z^*(R)\}$.

The metric dimension of a graph is the smallest number of vertices such that all vertices are uniquely determined by their distances to the chosen vertices. For an ordered subset $W = \{w_1, w_2, \cdots, w_k\}$ of vertices in a connected graph G and a vertex v of G, the metric representation of v with respect to W is the $k - vector$

$$r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k)).$$

The set W is a resolving set for G if $r(u|W) = r(v|W)$ implies that $u = v$ for all pairs u, v of vertices of G. In other words, a set $W \subset V(G)$ is called a resolving set, if for each two distinct vertices $u, v \in V(G)$

there exists $w \in W$ such that $d(u,w) \neq d(v,w)$, where $d(x,y)$ is the distance between the vertices x and y. The minimum cardinality of a resolving set for G is called the metric dimension of G, and denoted by $\beta(G)$.

The problem of Metric dimension was introduced by Slater [3] and also independently introduced by Harary and Melter [4]. In [5–7] graphs of order n with metric dimension $1, n-3, n-2$ and $n-1$ have been characterized. In [8], metric dimensions for many particular classes of graphs have been determined.

The corona product of graphs G and H is the graph $G \odot H$ obtained by taking one copy of G, called the center graph, $|V(G)|$ copies of H, called the outer graph, and making the j^{th} vertex of G adjacent to every vertex of the j^{th} copy of H, where $1 \leqslant j \leqslant |V(G)|$.

The Join graph $G + H$ of two graphs G and H is the graph with vertex set

$$V(G+H) = V(G) \cup V(H)$$

and edge set

$$E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

In this paper, we determine the Metric dimension of Corona product and Join graph of zero divisor graphs of direct product of finite fields.

2. Results on Metric Dimension of Zero Divisor Graph of Direct Product of Finite Fields

In [9, Proposition 6.2], Raja, Pirzada & Redmond, proved that for any $k \geqslant 2$, $loc(\Gamma(\prod_{i=1}^{k} \mathbb{Z}_2)) \leqslant k$. They also proved that [9, Theorem 6.3], $loc(\Gamma(\prod_{i=1}^{5} \mathbb{Z}_2)) = 5$. In [10], it is proved that if $n = 2, 3, 4$, and order of each field is two, then $\beta(\Gamma(R)) = n - 1$.

Theorem 1. [10, Theorem 2.4] *If $R = F_1 \times \cdots \times F_n, n \geqslant 2$ with $|F_i| \geqslant 3,\ (1 \leq i \leq n)$, then the metric dimension $\beta(\Gamma(R)) = |V(\Gamma(R))| - (2^n - 2)$.*

Consider the reduced rings $R_1 = F_1 \times \cdots \times F_n,\ (n \geqslant 2)$ and $R_2 = J_1 \times \cdots \times J_m,\ (m \geqslant 2)$ where $F_i,\ (1 \leqslant i \leqslant n),\ J_k,\ (1 \leqslant k \leqslant m)$ are finite fields with $|F_i| \geqslant 2,\ |J_k| \geqslant 2$.

In [11], the following result is proved.

Theorem 2. [11] *Let $R_1 = F_1 \times \cdots \times F_n,\ (n \geqslant 2)$ and $R_2 = J_1 \times \cdots \times J_m,\ (m \geqslant 2)$ where $F_i,\ (1 \leqslant i \leqslant n),\ J_k,\ (1 \leqslant k \leqslant m)$ are finite fields with $|F_i| \geqslant 2,\ |J_k| \geqslant 2$. Then,*

- *(i) the diameter of Corona product of $\Gamma(R_1)$ and $\Gamma(R_2)$ is 3 if $n = 2,\ m = 2,\ |F_1| = |F_2| = |J_1| = |J_2| = 2$,*
- *(ii) the diameter of Corona product of $\Gamma(R_1)$ and $\Gamma(R_2)$ is 4 if $n = 2,\ m = 2, |F_1| \geqslant 3$ or $|F_2| \geqslant 3$ and $|J_1| \geqslant 3$ or $|J_2| \geqslant 3$,*
- *(iii) the diameter of Corona product of $\Gamma(R_1)$ and $\Gamma(R_2)$ is 5 if $n \geqslant 3,\ m \geqslant 3$,*
- *(iv) the girth of Corona product of $\Gamma(R_1)$ and $\Gamma(R_2)$ is 3 if $n = 2, m = 2, |F_1|, |F_2| \geqslant 3,\ |J_1|, |J_2| \geqslant 3$,*
- *(v) the girth of Corona product of $\Gamma(R_1)$ and $\Gamma(R_2)$ is 3 if $n \geqslant 3$ or $m \geqslant 3$.*

3. Metric Dimension of Corona Product of Zero Divisor Graphs of Direct Product of Finite Fields

In this section, we determine the Metric dimension of Corona product of zero divisor graphs of direct product of finite fields.

Theorem 3. *If $R_1 = F_1 \times \cdots \times F_n,\ (n \geqslant 2)$ and $R_2 = J_1 \times \cdots \times J_m,\ (m \geqslant 2)$ where $F_i,\ (1 \leqslant i \leqslant n),\ J_k,\ (1 \leqslant k \leqslant m)$ are finite fields with $|F_i| = 2,\ |J_k| = 2$, then the Metric dimension of Corona product of $\Gamma(R_1)$ and $\Gamma(R_2)$ is $\beta\Big(\Gamma(R_1) \circ \Gamma(R_2)\Big) \leqslant n + (2^n - 2) \cdot m$.*

Proof. According to [9, Proposition 6.2], the metric dimension $\beta(\Gamma(R_1)) \leqslant n$ and $\beta(\Gamma(R_2)) \leqslant m$. Let

$$W_0 = \Big\{v_i \in V(\Gamma(R_1)) : v_i \text{ contains } '0' \; in \; i^{th} \text{ position and } '1' \text{ in remaining positions, } (1 \leqslant i \leqslant n)\Big\}.$$

We first prove that W_0 is resolving set for the center graph $\Gamma(R_1)$.

Let $x, y \in V(\Gamma(R_1))$. Let $r(x|W_0) = r(y|W_0)$. Suppose x contains r_1 number of $0's$ and y contains r_2 number of $0's$, $(1 \leqslant r_1, r_2 \leqslant n-1)$.

Case 1. $r_1 \neq r_2$.

Let $r_1 < r_2$. Then there exists a position i such that y contains $'0'$ in the i^{th} position and x contains $'1'$ in the i^{th} position. Then both y and $v_i \in W_0$ are adjacent to a vertex z with a $'1'$ in the i^{th} position and $'0'$ everywhere else. Also y is not adjacent to $v_i \implies d(y, v_i) = 2$. Suppose x contains $'0'$ in the j^{th} position $(j \neq i)$. Then x is adjacent to a vertex w with a $'1'$ in the j^{th} position and $'0'$ everywhere else. Also w is adjacent to z and x is not adjacent to z and v_i. Therefore, $d(x, v_i) = 3 \implies r(x|W_0) \neq r(y|W_0)$. This is a contradiction.

Case 2. $r_1 = r_2$ but x and y differ in atleast one position of $'0'$. Suppose y contains $'0'$ in the i^{th} position and x contains $'1'$ in the i^{th} position. Then from case (i), $r(x|W_0) \neq r(y|W_0)$. This is a contradiction. Thus, $r(x|W_0) = r(y|W_0) \implies x = y \implies W_0$ is a resolving set for $\Gamma(R_1)$. Let

$$W_k = \Big\{v_i \in V(\Gamma(R_1)) : v_i \text{ contains } '0' \; in \; i^{th} \text{ position and } '1' \text{ in remaining positions, } (1 \leqslant i \leqslant m)\Big\},$$

with $1 \leqslant k \leqslant |V(\Gamma(R_1))|$. As proved above, similarly we can prove that W_k is resolving set for the outer graph of k^{th} copy of $\Gamma(R_2)$. Let $W = W_0 \cup \Big[\cup_{j=1}^{|V(\Gamma(R_1))|} W_j\Big]$. Then, W is resolving set for the Corona product of $\Gamma(R_1) \circ \Gamma(R_2)$ and for $x \in V\Big(\Gamma(R_1) \circ \Gamma(R_2)\Big)$, the metric representation of x with respect to W is the vector of length $n + |V(\Gamma(R_1))| \cdot m$. Since, $V(\Gamma(R_1)) = 2^n - 2$, therefore, the Metric dimension of Corona product of $\Gamma(R_1)$ and $\Gamma(R_2)$ is $\beta\Big(\Gamma(R_1) \circ \Gamma(R_2)\Big) \leqslant n + (2^n - 2) \cdot m$. □

Remark 1. *(i) If $n = 2, 3, 4$ and $m = 2, 3, 4$ and order of each field is two, then according to [10],* $\beta\Big(\Gamma(R_1) \circ \Gamma(R_2)\Big) = (n-1) + (2^n - 2) \cdot (m-1)$.

(ii) If $n = 5$ and $m = 5$, and order of each field is two, then according to [9, Theorem 6.3], $\beta\Big(\Gamma(R_1) \circ \Gamma(R_2)\Big) = 5 + (2^5 - 2) \cdot 5$.

Theorem 4. *If $R_1 = F_1 \times \cdots \times F_n$, $(n \geqslant 2)$ and $R_2 = J_1 \times \cdots \times J_m$, $(m \geqslant 2)$ where F_i, $(1 \leqslant i \leqslant n)$, J_k, $(1 \leqslant k \leqslant m)$ are finite fields with $|F_i| \geqslant 3$, $|J_k| \geqslant 3$, then the Metric dimension of Corona product of $\Gamma(R_1)$ and $\Gamma(R_2)$ is*

$$\beta\Big(\Gamma(R_1) \circ \Gamma(R_2)\Big) = \Big(|V(\Gamma(R_1))| - (2^n - 2)\Big) + |V(\Gamma(R_1))| \cdot \Big(|V(\Gamma(R_2))| - (2^m - 2)\Big).$$

Proof. Consider the reduced rings $R_1 = F_1 \times \cdots \times F_n$, $(n \geqslant 2)$ and $R_2 = J_1 \times \cdots \times J_r$, $(r \geqslant 2)$ where F_i, $(1 \leqslant i \leqslant n)$, J_r, $(1 \leqslant r \leqslant m)$ are finite fields with $|F_i| \geqslant 3$, $|J_r| \geqslant 3$. Let

$$S_0 = \Big\{(a_1, \cdots, a_n) \in V(\Gamma(R_1)) : a_i \in \{0, 1\} \text{ with not all } a_i = 0 \text{ and not all } a_i = 1, \; 1 \leqslant i \leqslant n\Big\}.$$

Then $|S_0| = 2^n - 2$. Let $W_0 = V(\Gamma(R_1)) \setminus S_0$. According to [10, Theorem 2.4], W_0 is minimum resolving set for $\Gamma(R_1)$ and $|W_0| = |V(\Gamma(R))| - (2^n - 2)$. Now consider the Corona product of $\Gamma(R_1)$ and $\Gamma(R_2)$. There are $|V(\Gamma(R_1))|$ copies of $\Gamma(R_2)$ in the Corona product graph $\Gamma(R_1) \circ \Gamma(R_2)$ and the k^{th} vertex of $\Gamma(R_1)$ is adjacent to each and every vertex of k^{th} copy of $\Gamma(R_2)$. Let

$$S_j = \Big\{(a_1, \cdots, a_m) \in V(\Gamma(R_2)) : a_i \in \{0, 1\} \text{ with not all } a_i = 0 \text{ and not all } a_i = 1, \; 1 \leqslant i \leqslant m\Big\}$$

and $1 \leqslant j \leqslant |V(\Gamma(R_1))|$. Then $|S_j| = 2^m - 2$. Let $W_j = V(\Gamma(R_2)) \setminus S_j$. According to [10, Theorem 2.4], W_j is minimum resolving set for $\Gamma(R_2)$ and $|W_j| = |V(\Gamma(R_2))| - (2^m - 2)$. Let

$$W = W_0 \cup \Big[\cup_{j=1}^{|V(\Gamma(R_1))|} W_j\Big],$$

and

$$S = S_0 \ \cup \ \left[\cup_{j=1}^{|V(\Gamma(R_1))|} S_j\right].$$

Then $W = V\Big(\Gamma(R_1) \circ \Gamma(R_2)\Big) \setminus S$ and $|W| = \Big(|V(\Gamma(R_1))| - (2^n - 2)\Big) + |V(\Gamma(R_1))| \cdot \Big(|V(\Gamma(R_2))| - (2^m - 2)\Big)$ and $|S| = (2^n - 2) + |V(\Gamma(R_1))| \cdot (2^m - 2)$. We claim that W is minimum resolving set for the corona product graph $\Gamma(R_1) \circ \Gamma(R_2)$.

For $x \in V\Big(\Gamma(R_1) \circ \Gamma(R_2)\Big)$, the metric representation of x with respect to W is the vector of length $\Big(|V(\Gamma(R_1))| - (2^n - 2)\Big) + |V(\Gamma(R_1))| \cdot \Big(|V(\Gamma(R_2))| - (2^m - 2)\Big)$.

Clearly, $r(x|W)$ is distinct for each $x \in W$, as the vector $r(x|W)$ contains exactly one $'0'$ entry with the position of $'0'$ in distinct co-ordinate for each metric representation of $x \in W$. It is enough to show that for each $y \in S$, the metric representation $r(y|W)$ is distinct.

For $y_1, \ y_2 \in S = S_0 \ \cup \ \left[\cup_{j=1}^{|V(\Gamma(R_1))|} S_j\right]$, if $y_1 \in S_{j_1}, \ y_2 \in S_{j_2}, \ j_1 \neq j_2$, then clearly $d(y_1, w) \neq d(y_2, w), \ \forall \ w \in W_j$, since W_{j_1} is in the j_1^{th} copy of $\Gamma(R_2)$ and W_{j_2} is in the j_2^{th} copy of $\Gamma(R_2)$ in corona product $\Gamma(R_1) \circ \Gamma(R_2)$. Now suppose there exists distinct $y_1, \ y_2 \in S_j \ (0 \leqslant j \leqslant |V(\Gamma(R_1))|)$ such that $r(y_1|W) = r(y_2|W)$. In other words, $d(y_1, w) = d(y_2, w), \ \forall \ w \in W = W_0 \ \cup \ \left[\cup_{j=1}^{|V(\Gamma(R_1))|} W_j\right]$.

Since the entries in y_1 and y_2 are only $0's$ and $1's$ and $y_1, \ y_2 \in S_j$, this implies y_1 and y_2 will differ in at least one position containing $'0'$. Suppose y_1 contains $'0'$ in the r^{th} co-ordinate position and y_2 contains $'1'$ in the r^{th} co-ordinate position. Then y_1 is adjacent to a vertex $w_1 \in W_j$, with a non-zero entry other than $'1'$ in the r^{th} position and $'0'$ in the remaining positions (as each field contains at least three elements). This implies $d(y_1, w_1) = 1 = d(y_2, w_1)$ as $r(y_1|W) = r(y_2|W)$. But since y_2 and w_1 both contain non-zero entry in the k^{th} position, this implies $d(y_2, w_1) = 2 \ or \ 3$. This is a contradiction to $d(y_1, w_2) = 1 = d(y_2, w_1)$. Therefore, the metric representation $r(y|W)$ is distinct for each $y \in S$. Thus, W is a resolving set for the corona product graph $\Gamma(R_1) \circ \Gamma(R_2)$.

Now consider $W' = W \setminus \{x\}, \ x \in W$. Suppose x contains $r, 0's$ in positions $i_1, i_2, \cdots, i_r$ and non-zero entries in remaining positions with at least one non-zero entry other than 1, and let $y \in S = S_0 \ \cup \left[\cup_{j=1}^{|V(\Gamma(R_1))|} S_j\right]$ with $r, 0's$ in positions $i_1, i_2, \cdots, i_r$ and $'1'$ in remaining positions. This implies $x \neq y$ and x, y contains $r, 0's$ in same co-ordinate positions $i_1, i_2, \cdots, i_r$. This implies the vertices adjacent(non-adjacent) to x are also adjacent(non-adjacent) to y, implies $d(x, u) = d(y, u), \forall u \in V(\Gamma(R_1) \circ \Gamma(R_2)) \setminus \{x, y\}$ implies $d(x, w) = d(y, w), \forall w \in W'$ implies $r(x|W') = r(y|W')$. Thus, $W' = W \setminus \{x\}$ is not a resolving set for each $x \in W$. This implies W is a minimal resolving set. Thus, the metric dimension, $\beta\Big(\Gamma(R_1) \circ \Gamma(R_2)\Big) \leqslant |W| = \Big(|V(\Gamma(R_1))| - (2^n - 2)\Big) + |V(\Gamma(R_1))| \cdot \Big(|V(\Gamma(R_2))| - (2^m - 2)\Big)$.

Now there are exactly $C(n, r)$ vertices in $S_j \ (0 \leqslant j \leqslant |V(\Gamma(R_1))|)$, such that any two vertices with r zero entries will differ in at least one position containing $'0'$. Since $|S_j| = 2^n - 2$ if $j = 0$ and $|S_j| = 2^m - 2$ if $(1 \leqslant j \leqslant |V(\Gamma(R_1))|)$, any two vertices in S_j will either differ in the number of $'0'$ entries or if the two vertices contain same number of $0's$, then the two vertices will differ in at least one position containing $'0'$. Thus, if $S'_j \subset V(\Gamma(R_1))$ with $|S'_j| = (2^n - 2) + 1$ when $j = 0$ or $S'_j \subset V(\Gamma(R_2))$ with $|S'_j| = (2^m - 2) + 1$ where $1 \leqslant j \leqslant |V(\Gamma(R_1))|$, then there exists at least two vertices, say, $x, y \in S'_j$ such that both x, y contains same number of $0's$ and position of $0's$ is also same in both x and y. This implies positions of non-zero entries is also same, but since x and y are distinct, they will differ in at least one position containing non-zero entry. Since x, y both contains $0's$ in same co-ordinate positions, by a similar argument above, we get, $d(x, u) = d(y, u), \ \forall \ u \in V\Big(\Gamma(R_1) \circ \Gamma(R_2)\Big) \setminus \{x, y\}$.

Let S' be the set obtained from $S = S_0 \ \cup \ \left[\cup_{j=1}^{|V(\Gamma(R_1))|} S_j\right]$, by replacing the set S_j with S'_j. Then $|S'| = |S| + 1$. Therefore, if $T = V\Big(\Gamma(R_1) \circ \Gamma(R_2)\Big) \setminus S'$ with

$$|T| = \Big(|V(\Gamma(R_1))| - (2^n - 2)\Big) + |V(\Gamma(R_1))| \cdot \Big(|V(\Gamma(R_2))| - (2^m - 2)\Big) - 1,$$

then there exists at least two vertices, say,

$$x, y \in V\Big(\Gamma(R_1) \circ \Gamma(R_2)\Big) \setminus T,$$

such that, $d(x, v) = d(y, v),\ \forall\ v \in T$, implies $r(z|T) = r(w|T)$. This implies T cannot be a resolving set with $|T| = \Big(|V(\Gamma(R_1))| - (2^n - 2)\Big) + |V(\Gamma(R_1))| \cdot \Big(|V(\Gamma(R_2))| - (2^m - 2)\Big) - 1$ implies $\beta\Big(\Gamma(R_1) \circ \Gamma(R_2)\Big) > |T|$ implies $\beta\Big(\Gamma(R_1) \circ \Gamma(R_2)\Big) > \Big(|V(\Gamma(R_1))| - (2^n - 2)\Big) + |V(\Gamma(R_1))| \cdot \Big(|V(\Gamma(R_2))| - (2^m - 2)\Big) - 1$ implies $\beta\Big(\Gamma(R_1) \circ \Gamma(R_2)\Big) \geqslant \Big(|V(\Gamma(R_1))| - (2^n - 2)\Big) + |V(\Gamma(R_1))| \cdot \Big(|V(\Gamma(R_2))| - (2^m - 2)\Big)$.

Hence, the metric dimension,

$$\beta\Big(\Gamma(R_1) \circ \Gamma(R_2)\Big) = \Big(|V(\Gamma(R_1))| - (2^n - 2)\Big) + |V(\Gamma(R_1))| \cdot \Big(|V(\Gamma(R_2))| - (2^m - 2)\Big).$$

□

4. Metric Dimension of Join Graph of Zero Divisor Graphs of Direct Product of Finite Fields

Theorem 5. *If $R_1 = F_1 \times \cdots \times F_n,\ (n \geqslant 2)$ and $R_2 = J_1 \times \cdots \times J_m,\ (m \geqslant 2)$ where $F_i,\ (1 \leqslant i \leqslant n),\ J_k,\ (1 \leqslant k \leqslant m)$ are finite fields with $|F_i| = 2,\ |J_k| = 2$, then the Metric dimension of Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ is $\beta\Big(\Gamma(R_1) + \Gamma(R_2)\Big) \leqslant n + m$.*

Proof. According to [9, Proposition 6.2], the metric dimension $\beta(\Gamma(R_1)) \leqslant n$ and $\beta(\Gamma(R_2)) \leqslant m$. Let

$$W_1 = \Big\{v_i \in V(\Gamma(R_1)) : v_i \text{ contains;}'\ 0' \text{ in } i^{\text{th}} \text{ position and } '1' \text{ in remaining positions, } (1 \leqslant i \leqslant n)\Big\}.$$

In the proof of Theorem 3, we proved that, W_1 is resolving set for $\Gamma(R_1)$. Let

$$W_2 = \Big\{v_i \in V(\Gamma(R_1)) : v_i \text{ contains } '0' \text{ in } i^{\text{th}} \text{ position and } '1' \text{ in remaining positions, } (1 \leqslant i \leqslant m)\Big\}.$$

Similarly, W_2 is resolving set for $\Gamma(R_2)$.

Let $W = W_1\ \cup\ W_2$. Then, clearly W is resolving set for the Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ and for $x \in V\Big(\Gamma(R_1) + \Gamma(R_2)\Big)$, the metric representation of x with respect to W is the vector of length $n + m$. Hence, the Metric dimension of Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ is
$\beta\Big(\Gamma(R_1) + \Gamma(R_2)\Big) \leqslant n + m$. □

Remark 2. *(i) If $n = 2, 3, 4$ and $m = 2, 3, 4$ and order of each field is two, then according to* [10], $\beta\Big(\Gamma(R_1) + \Gamma(R_2)\Big) = (n - 1) + (m - 1)$.

(ii) If $n = 5$ and $m = 5$, and order of each field is two, then according to [9, Theorem 6.3], $\beta\Big(\Gamma(R_1) + \Gamma(R_2)\Big) = 5 + 5 = 10$.

Theorem 6. *If $R_1 = F_1 \times \cdots \times F_n,\ (n \geqslant 2)$ and $R_2 = J_1 \times \cdots \times J_m,\ (m \geqslant 2)$ where $F_i,\ (1 \leqslant i \leqslant n),\ J_k,\ (1 \leqslant k \leqslant m)$ are finite fields with $|F_i| \geqslant 3,\ |J_k| \geqslant 3$, then the Metric dimension of Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ is $\beta\Big(\Gamma(R_1) + \Gamma(R_2)\Big) = \Big(|V(\Gamma(R_1))| - (2^n - 2)\Big) + \Big(|V(\Gamma(R_2))| - (2^m - 2)\Big)$.*

Proof. Let

$$S_1 = \Big\{(a_1, \cdots, a_n) \in V(\Gamma(R_1)) : a_i \in \{0, 1\} \text{ with not all } a_i = 0 \text{ and not all } a_i = 1\Big\}.$$

Then $|S_1| = 2^n - 2$. Let $W_1 = V(\Gamma(R_1)) \setminus S_1$. According to [10, Theorem 2.4], W_1 is minimum resolving set for $\Gamma(R_1)$ and $|W_1| = |V(\Gamma(R))| - (2^n - 2)$. Let

$$S_2 = \Big\{(a_1, \cdots, a_m) \in V(\Gamma(R_2)) : a_i \in \{0, 1\} \text{ with not all } a_i = 0 \text{ and not all } a_i = 1\Big\}.$$

Then $|S_2| = 2^m - 2$. Let $W_2 = V(\Gamma(R_2)) \setminus S_2$. Then by [10, Theorem 2.4], W_2 is minimum resolving set for $\Gamma(R_2)$ and $|W_2| = |V(\Gamma(R))| - (2^m - 2)$.

Consider the Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ and let $W = W_1 \cup W_2$. For each $x \in V(\Gamma(R_1)),\ r(x|W_1)$ is distinct, and $d(x, z) = 1$ for all $z \in W_2$ in the Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$. Similarly, for each $y \in V(\Gamma(R_2)),\ r(x|W_2)$ is distinct, and $d(y, z) = 1$ for all $z \in W_1$ in the Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$.

Thus, $W = W_1 \cup W_2$ is resolving set for the Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ and for $w \in V\Big(\Gamma(R_1)+\Gamma(R_2)\Big)$, the metric representation of w with respect to W is the vector of length $\Big(|V(\Gamma(R_1))| - (2^n - 2)\Big) + \Big(|V(\Gamma(R_2))| - (2^m - 2)\Big)$.

Now consider $W' = W \setminus \{x\},\ w \in W$. Either $x \in W_1$ or $x \in W_2$. Without loss of generality suppose $x \in W_1$. Suppose x contains $r, 0's$ in positions $i_1, i_2, \cdots, i_r$ and non-zero entries in remaining positions with at least one non-zero entry other than 1, and let $y \in S_1$ with $r, 0's$ in positions $i_1, i_2, \cdots, i_r$ and $'1'$ in remaining positions. This implies $x \neq y$ and x, y contains $r, 0's$ in same co-ordinate positions $i_1, i_2, \cdots, i_r$. This implies the vertices adjacent(non-adjacent) to x are also adjacent(non-adjacent) to y, implies $d(x, u) = d(y, u), \forall u \in V\Big(\Gamma(R_1) + \Gamma(R_2)\Big) \setminus \{x, y\}$ implies $d(x, w) = d(y, w), \forall w \in W'$ implies $r(x|W') = r(y|W')$. Similarly, if $x \in W_2$ then $r(x|W') = r(y|W')$. Thus, $W' = W \setminus \{x\}$ is not a resolving set for each $x \in W$. This implies W is a minimal resolving set. Thus, the metric dimension is $\beta\Big(\Gamma(R_1) + \Gamma(R_2)\Big) \leqslant |W| = \Big(|V(\Gamma(R_1))| - (2^n - 2)\Big) + \Big(|V(\Gamma(R_2))| - (2^m - 2)\Big)$.

Now there are exactly $C(n, r)$ vertices in S_j $(j = 1, 2)$, such that any two vertices with r zero entries will differ in at least one position containing $'0'$. Since $|S_j| = 2^n - 2$ if $j = 1$ and $|S_j| = 2^m - 2$ if $j = 2$, any two vertices in S_j will either differ in the number of $'0'$ entries or if the two vertices contain same number of $0's$, then the two vertices will differ in at least one position containing $'0'$. Thus, if $S'_j \subset V(\Gamma(R_1))$ with $|S'_j| = (2^n - 2) + 1$ where $j = 1$ or $S'_j \subset V(\Gamma(R_2))$ with $|S'_j| = (2^m - 2) + 1$ where $j = 2$, then there exists at least two vertices, say, $x, y \in S'_j$ such that both x, y contains same number of $0's$ and position of $0's$ is also same in both x and y. This implies positions of non-zero entries is also same, but since x and y are distinct, they will differ in at least one position containing non-zero entry. Since x, y both contains $0's$ in same co-ordinate positions, by a similar argument above, we get, $d(x, u) = d(y, u),\ \forall\ u \in V\Big(\Gamma(R_1) + \Gamma(R_2)\Big) \setminus \{x, y\}$.

Let S' be the set obtained from $S = S_1 \cup S_2$, by replacing the set S_j with S'_j. Then $|S'| = |S| + 1$. Therefore, if $T = V\Big(\Gamma(R_1)+\Gamma(R_2)\Big)\setminus S'$ with $|T| = \Big(|V(\Gamma(R_1))|-(2^n-2)\Big)+\Big(|V(\Gamma(R_2))|-(2^m-2)\Big)-1$, then there exists at least two vertices, say, $x, y \in V\Big(\Gamma(R_1) + \Gamma(R_2)\Big) \setminus T$, such that, $d(x, v) = d(y, v),\ \forall\ v \in T$, implies $r(z|T) = r(w|T)$. This implies T cannot be a resolving set with $|T| = \Big(|V(\Gamma(R_1))| - (2^n - 2)\Big) + \Big(|V(\Gamma(R_2))| - (2^m - 2)\Big) - 1$ implies $\beta\Big(\Gamma(R_1) + \Gamma(R_2)\Big) > |T|$ implies $\beta\Big(\Gamma(R_1) + \Gamma(R_2)\Big) > \Big(|V(\Gamma(R_1))| - (2^n - 2)\Big) + \Big(|V(\Gamma(R_2))| - (2^m - 2)\Big) - 1$ implies $\beta\Big(\Gamma(R_1)+\Gamma(R_2)\Big) \geqslant \Big(|V(\Gamma(R_1))|-(2^n-2)\Big)+\Big(|V(\Gamma(R_2))|-(2^m-2)\Big)$.

Hence, the metric dimension is $\beta\Big(\Gamma(R_1)+\Gamma(R_2)\Big) = \Big(|V(\Gamma(R_1))|-(2^n-2)\Big)+\Big(|V(\Gamma(R_2))|-(2^m-2)\Big)$. □

5. Conclusion

We have determined the Metric dimension of Corona product and Join graph of zero divisor graphs of direct product of finite fields.

Author Contributions

Both authors Dr. Nithya Sai Narayana and Subhash Mallinath Gaded wrote the main manuscript text and reviewed the manuscript. Both authors have contributed equally to the work. Both authors read and approved the final manuscript.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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