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Article

Multiplicative Distance-Location Number of Graphs

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Abstract: For a set S of vertices in a connected graph G, the multiplicative distance of a vertex v with respect to S is defined by $d^*_S(v) = \prod_{x\in S, x\neq v} d(v,x)$. If $d^*_S(u) \neq d^*_S(v)$ for each pair u, v of distinct vertices of G, then S is called a multiplicative distance-locating set of G. The minimum cardinality of a multiplicative distance-locating set of G is called its multiplicative distance-location number $loc^*_d(G)$. If $d^*_S(u) \neq d^*_S(v)$ for each pair u, v of distinct vertices of $G-S$, then S is called an external multiplicative distance-locating set of G. The minimum cardinality of an external multiplicative distance-locating set of G is called its external multiplicative location number $loc^*_e(G)$. We prove the existence or non existence of multiplicative distance-locating sets in some well known classes of connected graphs. Also, we introduce a family of connected graphs such that $loc^*_d(G)$ exists. Moreover, there are infinite classes of connected graphs G for which $loc^*_d(G)$ exists as well as infinite classes of connected graphs G for which $loc^*_d(G)$ does not exist. Lower bound for the multiplicative distance-location number of a connected graph is established in terms of its order and diameter.

Keywords: Distance, locating set, Distance-locating set, Distance-location number, Multiplicative distance-locating set, Multiplicative distance-location number
Mathematics Subject Classification: 05C12

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1. Introduction

Let G be a connected graph and let $W = \{w_1, w_2, \cdots, w_k\}$ be an ordered set of vertices in G. If $v \in V(G)$, the k-vector $C_W(v)$ of v with respect to W is defined by $C_W(v) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$ where $d(v, w_i)$ is the distance between v and w_i $(1 \leq i \leq k)$. The set W is called a locating set if for every pair of distinct vertices $u, v \in V(G)$, $C_W(u) \neq C_W(v)$. It is also called a resolving set. The k-vector $C_W(v)$ is called the locating code of v with respect to W. The cardinality of a minimum locating set in G is called the location number of G and it is denoted by $loc(G)$. In 1975 and later in 1988, Slater [1, 2] introduced the concept of "Locating Set" of a graph G. Independently, in 1976, Harary and Melter [3] discovered these concepts as well but used the term Metric Dimension, rather than location number. Let S be a subset of $V(G)$. The distance of a vertex v with respect to S is defined by $d_S(v) = \sum_{\substack{x\in s\\ x\neq v}} d(v, x)$.

If $d_S(u) \neq d_S(v)$ for each pair u, v of distinct vertices of G, then S is called a distance-locating set of G. In 2003, Chartrand et al. [4] defined the concept "Distance-Locating Set" of a graph G and the minimum cardinality of a distance-locating set of G is the distance-location number $loc_d(G)$. They found the relationship between $loc(G)$ and $loc_d(G)$. Further, they found the existence of distance-locating sets in some well-known classes of connected graphs and proved that no distance-locating set in a connected graph G has cardinality 2 or 3.

Moreover, there are infinite classes of connected graphs G for which $loc_d(G)$ exists as well as infinite classes of connected graphs G for which $loc_d(G)$ doest not exist. Also, they found the lower bound for the distance-location number of a connected graph in terms of its order and diameter.

Motivated by the concept Distance-Locating Set of a graph G, we introduce the new concept "Multiplicative Distance-Locating Set" of a graph G and we define the cardinality of a minimum multiplicative distance-locating set of G as the multiplicative distance-location number $loc_d^*(G)$. Then we find the relationship between $loc(G)$ and $loc_d^*(G)$. Also, we find the existence of multiplicative distance-locating sets in some well-known classes of connected graphs and we prove that no multiplicative distance-locating set in a connected graph G has cardinality 2. Moreover, we find an infinite classes of connected graphs such as complete graph and star graph for which $loc_d^*(G)$ does not exist. Also, we find that the path P_3 is the only connected graph G of diameter 2 such that $loc_d^*(G)$ exists. We define the new class of connected graph $P_n \circledast C_m$, obtained by identifying any two consecutive vertices of C_m with the $(n-2)^{th}$ and $(n-1)^{th}$ vertices of P_n, for which $loc_d^*(G)$ exists. Also, we find the lower bound for the multiplicative distance-location number of a connected graph in terms of its order and diameter.

Chartrand et al. [4] defined the concept External Distance-Locating Set of G and the cardinality of a minimum external distance-locating set of G is the external location number $loc_e(G)$. Further, they found the relationship between $loc(G), loc_d(G)$ and $loc_d(G)$ and studied the external location number of some well-known classes of connected graphs.

Motivated by the concept External Distance-Locating Set of a graph G, we introduce the new concept "External Multiplicative Distance-Locating Set" of a graph G and the cardinality of a minimum external multiplicative distance-locating set of G is the external multiplicative location number $loc_e^*(G)$. Further, we find the external multiplicative location number of some well-known classes of connected graphs. Also, we introduce the new concept "Magic Locating Set" of a graph G and we determine the magic location number of some well-known classes of connected graphs.

2. The Multiplicative Distance-Location Number

For a set S of vertices in a connected graph G, the multiplicative distance of a vertex v in V with respect to S is defined by

$$d_S^*(v) = \prod_{x \in S, x \neq v} d(v, x).$$

If $S = \{v\}$, then we assign 0 to $d_S^*(v)$. If $d_S^*(u) \neq d_S^*(v)$ for each pair u, v of distinct vertices of G, then S is called a multiplicative distance-locating set of G. A minimum multiplicative distance-locating set of G is a multiplicative distance-locating set of minimum cardinality and this cardinality is the multiplicative distance-location number $loc_d^*(G)$. The following results describe a relationship between $loc_d^*(G)$ and $loc(G)$.

Proposition 1. *Every multiplicative distance-locating set of a graph is a locating set.*

Proof. On the contrary, assume that the multiplicative distance-locating set $S = \{w_1, w_2, \cdots, w_k\}$ is not a locating set. Then there exists two distinct vertices $u, v \in V(G)$ such that $C_S(u) = C_S(v)$. That is, $(d(u, w_1), d(u, w_2), \cdots, d(u, w_k)) = (d(v, w_1, d(v, w_2), \cdots, d(v, w_k))$. Then $\prod_{w_i \in S} d(u, w_i) = \prod_{w_i \in S} d(v, w_i)$. Therefore $d_S^*(u) = d_S^*(v)$, which is a contradiction to the fact that S is a multiplicative distance-locating set of G. □

Corollary 1. *If G is a connected graph such that $loc_d^*(G)$ exists, then $loc(G) \leq loc_d^*(G)$.*

The following Theorem 1 shows that the path P_n is the only connected graph of order n with $loc_d^*(G) = 1$.

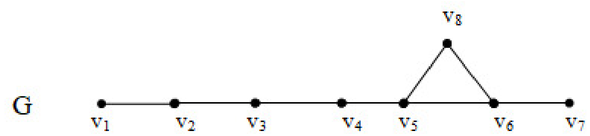
Theorem 1. *Let G be a connected graph of order $n \geq 2$. Then $loc^*_d(G) = 1$ if and only if $G = P_n$.*

Proof. Let $G = P_n : v_1, v_2, ..., v_n$ be a path on n vertices. Since $d(v_i, v_1) = i - 1$ for $1 \leq i \leq n$, $\{v_1\}$ is a minimum multiplicative distance-locating set of P_n. Therefore, $loc^*_d(G) = 1$. Conversely, suppose $loc^*_d(G) = 1$. Let $S = \{w\}$ be a minimum multiplicative distance-locating set for G. Then $d^*_S(v) = d(v, w)$ for each vertex v of G is a non-negative integer less than n. Since $d^*_S(v)$, $v \in V(G)$ are distinct, there exists a vertex u in G such that $d(u, w) = n - 1$. Consequently the diameter of G is $n - 1$. Hence $G = P_n$. □

Proposition 2. *No multiplicative distance-locating set in a connected graph G consists of two vertices of G.*

Proof. Suppose $S = \{u, v\}$ is a multiplicative distance-locating set for G. Then $d^*_S(u) = d(u, v) = d^*_S(v)$, which is a contradiction. □

In the case of distance-locating set of a graph G, there is no distance-locating set with exactly three vertices of G for any graph G. However, multiplicative distance-locating set of a graph may consists of three vertices of G. For example, for the following graph G,



$S = \{v_1, v_2, v_7\}$ is a minimum multiplicative distance locating set and hence $loc^*_d(G) = 3$. In order to give examples of other graphs G for which $loc^*_d(G)$ does not exist, we now make some observations. In a graph G, the nieghbourhood of a vertex v in G is the set $N(v)$ consisting of all vertices which are adjacent to v. The closed neighbourhood is $N[v] = N(v) \cup \{v\}$

Proposition 3. *If u and v are distinct vertices of a connected graph G such that either $N(u) = N(v)$ or $N[u] = N[v]$, then every multiplicative distance-locating set of G contains exactly one of u and v.*

Proof. Let $u, v \in G$ and $u \neq v$. Suppose S is a multiplicative distance-locating set for G.

Case 1. Suppose $N(u) = N(v)$.

If $u, v \in S$, then u and v are having the same distance apart from each vertex of G. Therefore,$d^*_S(u) = d^*_S(v)$, which is a contradiction to the fact that S is a multiplicative distance-locating set. Similar proof holds when $u, v \notin S$.

Case 2. Suppose $N[u] = N[v]$.

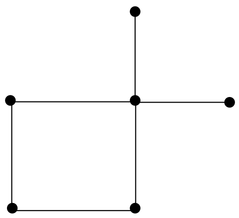
The proof is similar to the case 1. □

Corollary 2. *If G is a connected graph containing three distinct vertices u, v and w such that either $N(u) = N(v) = N(w)$ or $N[u] = N[v] = N[w]$, then $loc^*_d(G)$ does not exist.*

In a graph G, if $deg v = 1$, then v is called an end-vertex.

Corollary 3. *If G is a connected graph such that $loc^*_d(G)$ exists, then every vertex of G is adjacent to at most two end-vertices.*

The converse of Corollary 3 does not necessarily hold. For example, in the following graph G every vertex of G is adjacent to at most two end vertices, but $loc^*_d(G)$ does not exist.



We have verified this fact by considering all possible cases.

Corollary 4. *If $n \geq 3$, then $loc_d^*(K_n)$ and $loc_d^*(K_{1,n})$ do not exist.*

Proof. Immediately follows from Corollaries 2 and 3. □

Next we show that the path P_3 is the only connected graph G of diameter 2 such that $loc_d^*(G)$ exists.

Theorem 2. *If G is a connected graph of diameter 2 for which $loc_d^*(G)$ exists, then $G = P_3$.*

Proof. Let S be a minimum multiplicative distance-locating set for G and let $|S| = k$. Suppose $k = 1$. Since diameter of G is 2 and $|S| = 1$, by Theorem 1, $G = P_3$. By Proposition 2, $k \neq 2$. Suppose $k = 3$. Let $S = \{u, v, w\}$. Then $d_S^*(u) = d(u, v)\times d(u, w)$, $d_S^*(v) = d(u, v)\times d(v, w)$ and $d_S^*(w) = d(u, w)\times d(v, w)$. Since S is a multiplicative distance-locating set, $d_S^*(u) \neq d_S^*(v)$ and hence $d(u, w) \neq d(v, w)$. Since diameter of G is 2, without loss of generality, we may assume that $d(u, w) = 1$, $d(v, w) = 2$. Then $d_S^*(u) = d(u, v)$, $d_S^*(v) = 2d(u, v)$ and $d_S^*(w) = 2$ Since G has diameter 2, either $d(u, v) = 1$ or $d(u, v) = 2$. If $d(u, v) = 1$, then $d_S^*(v) = 2$. Therefore, $d_S^*(v) = d_S^*(w)$, which is a contradiction. If $d(u, v) = 2$, then $d_S^*(u) = 2$. Therefore, $d_S^*(u) = d_S^*(w)$, which is a contradiction. Thus, S is not a multiplicative distance-locating set for G. Hence assume that $k \geq 4$. Let $H = \langle S \rangle$ be the subgraph of G induced by S. Since G has diameter 2, for each $v \in V(H) = S$, it follows that $d_S^*(v)$ is uniquely determined by $k - deg_H(v)$ vertices of S since $d_H(u, v) = 1$ for the vertices u which are adjacent to v. Since H is nontrivial and diameter of G is 2, H contains at least two vertices x and y such that $deg_H(x) = deg_H(y)$. Suppose $deg_H(x) = deg_H(y) = a$. Since G has diameter 2, $d_S^*(x) = 2^{k-a-1}$ and $d_S^*(y) = 2^{k-a-1}$. Hence S is not a multiplicative distance-locating set for $k \geq 4$. □

A graph G is a k-partite graph if $V(G)$ can be partitioned into k subsets $V_1, V_2, \cdots, V_k$ such that uv is an edge of G if u and v are belong to different partite sets. If, in addition, every two vertices in different partite sets are joined by an edge, then G is a complete k-partite graph. If $|V_i| = n_i$ for $1 \leq i \leq k$, then we denote this complete k-partite graph by $K_{n_1,n_2,\cdots,n_k}$. The complete k-partite graphs are aslo referred to as complete multiplartite graphs.

The following result provides another well-known class of graphs for which $loc_d^*(G)$ does not exist.

Corollary 5. *If G is a complete multipartite graph of order at least 4 that is not complete, then $loc_d^*(G)$ does not exist.*

Now, we define a family of connected graph $P_n \circledast C_m$ such that $loc_d^*(G)$ exists.

Definition 1. *Let P_n be a path on n vertices and let C_m be a cycle on m vertices. The graph $P_n \circledast C_m$ is obtained by identifying any two consecutive vertices of C_m with the $(n-2)^{th}$ and $(n-1)^{th}$ vertices of P_n as shown in the following Figure 1,*

Theorem 3. *For $n > 4$, $m \geq 3$, $loc_d^*(P_n \circledast C_m) \leq 4$.*

Proof. **Case 1.** Suppose m is even, $m = 2k$. Let $\{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{k-1}, w_1, w_2, \ldots w_{k-1}\}$ be the vertices of $P_n \circledast C_m$ as shown in the above Figure 1.

Let $S = \{v_n, v_{n-2}, v_{n-3}, v_{n-4}\}$ be a subset of the vertex set of $P_n \circledast C_m$. Then

$$d_S^*(v_i) = d(v_i, v_n) \times d(v_i, v_{n-2}) \times d(v_i, v_{n-3}) \times d(v_i, v_{n-4}), \text{ for } 1 \leq i \leq n-5$$

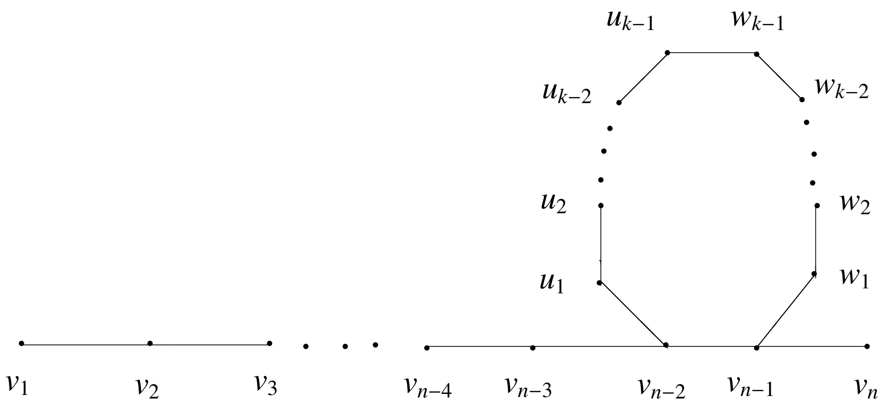


Figure 1

$$\begin{aligned}
&= (n-i)(n-i-2)(n-i-3)(n-i-4).\\
d_S^*(u_i) &= d(u_i, v_n) \times d(u_i, v_{n-2}) \times d(u_i, v_{n-3}) \times d(u_i, v_{n-4}), \text{ for } 1 \le i \le k-1\\
&= i(i+1)(i+2)^2.\\
d_S^*(w_i) &= d(w_i, v_n) \times d(w_i, v_{n-2}) \times d(w_i, v_{n-3}) \times d(w_i, v_{n-4}), \text{ for } 1 \le i \le k-1\\
&= (i+1)^2(i+2)(i+3).\\
d_S^*(v_{n-4}) &= d(v_{n-4}, v_n) \times d(v_{n-4}, v_{n-2}) \times d(v_{n-4}, v_{n-3}) = 4 \times 2 \times 1 = 8.\\
d_S^*(v_{n-3}) &= d(v_{n-3}, v_n) \times d(v_{n-3}, v_{n-2}) \times d(v_{n-3}, v_{n-4}) = 3 \times 1 \times 1 = 3.\\
d_S^*(v_{n-2}) &= d(v_{n-2}, v_n) \times d(v_{n-2}, v_{n-3}) \times d(v_{n-2}, v_{n-4}) = 2 \times 1 \times 2 = 4.\\
d_S^*(v_{n-1}) &= d(v_{n-1}, v_n) \times d(v_{n-1}, v_{n-2}) \times d(v_{n-1}, v_{n-3}) \times d(v_{n-1}, v_{n-4}) = 1 \times 1 \times 2 \times 3 = 6.\\
d_S^*(v_n) &= d(v_n, v_{n-2}) \times d(v_n, v_{n-3}) \times d(v_n, v_{n-4}) = 2 \times 3 \times 4 = 24.
\end{aligned}$$

Certainly, $d_S^*(v_i) \neq d_S^*(v_j)$ for each pair i, j of distinct integers $1 \le i, j \le n-5$ and $d_S^*(u_i) \neq d_S^*(u_j)$ and $d_S^*(w_i) \neq d_S^*(w_j)$ for each pair i, j of distinct integers $1 \le i, j \le k-1$. Since $\min\{d_S^*(v_i) : 1 \le i \le n-5, n > 5\} = 30$ and $\max\{d_S^*(v_i) : n-4 \le i \le n\} = 24$, $d_S^*(v_i) \neq d_S^*(v_j)$ for each pair i, j of distinct integers $1 \le i, j \le n$. The factors of $d_S^*(v_i)$, $d_S^*(u_i)$ and $d_S^*(w_i)$ are: $d_S^*(v_i) = x(x+1)(x+2)(x+4)$ where $x = n-i-4$, $d_S^*(u_i) = y(y+1)(y+2)^2$ where $y = i$ and $d_S^*(w_i) = z^2(z+1)(z+2)$ where $z = i+1$.

We claim that $d_S^*(v_i) \neq d_S^*(u_j)$ for every two integers $1 \le i \le n$, $1 \le j \le k-1$. First we discuss when i varies from 1 to $n-5$ and j varies from 1 to $k-1$. If $y < x-2$ or $y > x+4$,then $d_S^*(v_i) \neq d_S^*(u_j)$. We consider the following cases:

1. Suppose $d_S^*(v_i) = d_S^*(u_j)$. Then $x(x+1)(x+2)(x+4) = y(y+1)(y+2)^2$
   (a) Suppose $y = x-2$. Then $x(x+1)(x+2)(x+4) = (x-2)(x-1)x^2$. This implies $(x+1)(x+2)(x+4) = x(x-1)(x-2)$, which is a contradiction.
   (b) Suppose $y = x-1$. Then $x(x+1)(x+2)(x+4) = (x-1)x(x+1)^2$. This implies $x^2+6x+8 = x^2-1$. Therefore, $x = -3/2$, which is a contradiction to the fact that x is a positive integer.
   (c) Suppose $y = x$. Then $x(x+1)(x+2)(x+4) = x(x+1)(x+2)^2$. This implies that 4=2, which is a contradiction.
   (d) Suppose $y = x+1$. Then $x(x+1)(x+2)(x+4) = (x+1)(x+2)(x+3)^2$. Hence $x = -9/2$, which is a contradiction.
   (e) Suppose $y = x+2$. Then $x(x+1)(x+2)(x+4) = (x+2)(x+3)(x+4)^2$. Thus, $x = -2$, which is a contradiction.

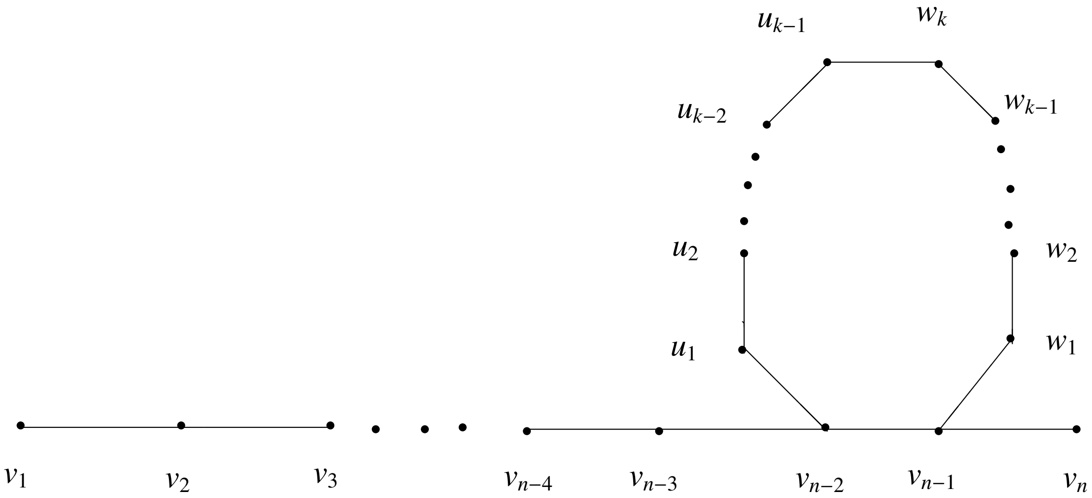


Figure 2

(f) Suppose $y = x + 3$. Then $x(x + 1)(x + 2)(x + 4) = (x + 3)(x + 4)(x + 5)^2$. This implies $10x^2 + 53x + 75 = 0$. Since $b^2 - 4ac = -191 < 0$, it has no real solution, which is a contradiction.

(g) Suppose $y = x + 4$. Then $x(x + 1)(x + 2)(x + 4) = (x + 4)(x + 5)(x + 6)^2$. This implies $14x^2 + 94x + 180 = 0$. Since $b^2 - 4ac = -1244 < 0$, it has no real solution, which is a contradiction.

Thus, $d_S^*(v_i) \neq d_S^*(u_j)$ for every two integers $1 \leq i \leq n - 5$, $1 \leq j \leq k - 1$. Since $\max\{d_S^*(v_i) : n - 4 \leq i \leq n\} = 24$ and $\{d_S^*(u_j) : 1 \leq j \leq k - 1\} = \{18, 96, \ldots\}$, $d_S^*(v_i) \neq d_S^*(u_j)$ for every two integers $1 \leq i \leq n$, $1 \leq j \leq k - 1$. Next we claim that $d_S^*(v_i) \neq d_S^*(w_j)$ for every two integers $1 \leq i \leq n$, $1 \leq j \leq k - 1$. First we discuss when i varies from 1 to $n - 5$ and j varies from 1 to $k - 1$. If $z < x - 2$ or $z > x + 4$, then $x(x + 1)(x + 2)(x + 4) \neq z^2(z + 1)(z + 2)$. Therefore, $d_S^*(v_i) \neq d_S^*(w_j)$.

2. Suppose $d_S^*(v_i) = d_S^*(w_j)$. Then $x(x+1)(x+2)(x+4) = z^2(z+1)(z+2)$. If $x-2 \leq z \leq x+4$, then the proof is similar to the above case. Thus, $d_S^*(v_i) \neq d_S^*(w_j)$ for every two integers $1 \leq i \leq n-5$, $1 \leq j \leq k - 1$. Since $\max\{d_S^*(v_i) : n - 4 \leq i \leq n\} = 24$ and $min\{d_S^*(w_j) : 1 \leq j \leq k - 1\} = 48$, $d_S^*(v_i) \neq d_S^*(w_j)$ for every two integers $1 \leq i \leq n$, $1 \leq j \leq k - 1$. Next we claim that $d_S^*(u_i) \neq d_S^*(w_j)$ for every two integers $1 \leq i, j \leq k - 1$. If $z < y - 3$ or $z > y + 2$, then $d_S^*(u_i) \neq d_S^*(w_j)$.

3. Suppose $d_S^*(u_i) = d_S^*(w_j)$. Then $y(y + 1)(y + 2)^2 = z^2(z + 1)(z + 2)$. If $y - 3 \leq z \leq y + 2$, then the proof is similar to the first case. Thus, $d_S^*(u_i) \neq d_S^*(w_j)$ for every two integers $1 \leq i, j \leq k - 1$. From the above three cases, $d_S^*(u) \neq d_S^*(v)$ for any two pair of distinct vertices u, v of $P_n \circledast C_m$. Therefore, S is a multiplicative distance-locating set for $P_n \circledast C_m$. Hence, $loc_d^*(P_n \circledast C_m) \leq 4$ when m is even.

Case 2. Suppose m is odd, $m = 2k + 1$.

Let $\{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{k-1}, w_1, w_2, \ldots, w_{k-1}, w_k\}$ be the vertices of $P_n \circledast C_m$ as shown in the Figure 2.

Let $S = \{v_n, v_{n-2}, v_{n-3}, v_{n-4}\}$ be a subset of the vertex set of $P_n \circledast C_m$. Then

$$\begin{aligned} d_S^*(v_i) &= d(v_i, v_n) \times d(v_i, v_{n-2}) \times d(v_i, v_{n-3}) \times (v_i, v_{n-4}), \text{ for } 1 \leq i \leq n - 5 \\ &= (n - i)(n - i - 2)(n - i - 3)(n - i - 4). \\ d_S^*(u_i) &= d(u_i, v_n) \times d(u_i, v_{n-2}) \times d(u_i, v_{n-3}) \times d(u_i, v_{n-4}), \text{ for } 1 \leq i \leq k - 1 \end{aligned}$$

$$\begin{aligned}
&= i(i+1)(i+2)^2.\\
d_S^*(w_i) &= d(w_i, v_n) \times d(w_i, v_{n-2}) \times d(w_i, v_{n-3}) \times d(w_i, v_{n-4}), \text{ for } 1 \le i \le k-1\\
&= (i+1)^2(i+2)(i+3).\\
d_S^*(w_k) &= d(w_i, v_n) \times d(w_i, v_{n-2}) \times d(w_i, v_{n-3}) \times d(w_i, v_{n-4}) = k(k+1)^2(k+2).\\
d_S^*(v_{n-4}) &= d(v_{n-4}, v_n) \times d(v_{n-4}, v_{n-2}) \times d(v_{n-4}, v_{n-3}) = 4 \times 2 \times 1 = 8.\\
d_S^*(v_{n-3}) &= d(v_{n-3}, v_n) \times d(v_{n-3}, v_{n-2}) \times d(v_{n-3}, v_{n-4}) = 3 \times 1 \times 1 = 3.\\
d_S^*(v_{n-2}) &= d(v_{n-2}, v_n) \times d(v_{n-2}, v_{n-3}) \times d(v_{n-2}, v_{n-4}) = 2 \times 1 \times 2 = 4.\\
d_S^*(v_{n-1}) &= d(v_{n-1}, v_n) \times d(v_{n-1}, v_{n-2}) \times d(v_{n-1}, v_{n-3}) \times d(v_{n-1}, v_{n-4}) = 1 \times 1 \times 2 \times 3 = 6.\\
d_S^*(v_n) &= d(v_n, v_{n-2}) \times d(v_n, v_{n-3}) \times d(v_n, v_{n-4}) = 2 \times 3 \times 4 = 24.
\end{aligned}$$

By case 1, $d_S^*(u) \neq d_S^*(v)$ for any two pair of distinct vertices u, v of $P_n \circledast C_m$. Therefore, S is a multiplicative distance-locating set for $P_n \circledast C_m$. Thus, $loc_d^*(P_n \circledast C_m) \le 4$ for m is odd. Hence $loc_d^*(P_n \circledast C_m) \le 4$. □

Corollary 6. *The multiplicative distance location number of $P_n \circledast C_m$ is either 3 or 4.*

Proof. Since $P_n \circledast C_m$ is not a path, the proof follows from Theorem 1 and Proposition 2 □

Now we establish a lower bound for the multiplicative distance-location number of a connected graph in terms of its order and diameter. Let G be a connected graph of order n. For a subset S of $V(G)$, define

$$D_{min}^*(S) = \min\{d_S^*(v) : v \in V(G)\},$$
$$D_{max}^*(S) = \max\{d_S^*(v) : v \in V(G)\}.$$

Observation 4. *Let G be a connected graph of order $n \ge 2$ and diameter $d \ge 2$. If S is a multiplicative distance-locating set for G with $|S| = k$, then $n - 1 \le D_{max}^*(S) - D_{min}^*(S) \le d^k$.*

Proof. Suppose S is a multiplicative distance-locating set for G. Then the numbers $d_S^*(v), v \in V(G)$ are distinct. The set $\{d_S^*(v) : v \in V(G)\}$ contains n distinct numbers. Therefore,

$$D_{max}^*(S) - D_{min}^*(S) \ge n - 1 \ldots. \quad (1)$$

Since S is a multiplicative distance-locating set, $diamG = d$ and $|S| = k$, $D_{min}^*(S) \ge 0$ and $D_{max}^*(S) \le d^k$. Thus,

$$D_{max}^*(S) - D_{min}^*(S) \le d^k \ldots. \quad (2)$$

Combining (1) and (2), we get, $n - 1 \le D_{max}^*(S) - D_{min}^*(S) \le d^k$. □

Observation 5. *If G is a connected graph of order $n \ge 2$ and diameter $d \ge 2$ for which $loc_d^*(G)$ exists, then $loc_d^*(G) \ge \left\lceil \frac{log(n-1)}{logd} \right\rceil$.*

Proof. Let S be a multiplicative distance-locating set for G and let $|S| = k$. By Observation 4, $n - 1 \le D_{max}^*(S) - D_{min}^*(S) \le d^k$. Therefore, $n - 1 \le d^k$. Thus $loc_d^*(G) \ge \left\lceil \frac{log(n-1}{logd} \right\rceil$. □

3. The External Multiplicative Location Number of a graph

We shown in Section 2, for many graphs G, $loc_d^*(G)$ does not exist. However, there is a closely related parameter that is defined for every graph. For a set S of vertices in a connected graph G, the multiplicative distance of a vertex v with respect to S is defined by

$$d_S^*(v) = \prod_{x \in S, x \neq v} d(v, x).$$

If $d_S^*(u) \neq d_S^*(v)$ for each pair u, v of distinct vertices of $G - S$, then S is called an external multiplicative distance-locating set of G. The cardinality of a minimum external multiplicative distance-locating set of G is called the external multiplicative location number of G. It is denoted by $loc_e^*(G)$.

Proposition 4. *Every external multiplicative distance-locating set is a locating set.*

Proof. Let S be an external multiplicative distance-locating set for G. Then the numbers $d_S^*(v), v \in V(G) - S$ are distinct. Let u, v be any two distinct vertices of G.

1. If $u, v \in S$, then $C_S(u) \neq C_S(v)$.
2. Suppose $u, v \in V(G) - S$. If $C_S(u) = C_S(v)$, then $d_S^*(u) = d_S^*(v)$, which is a contradiction. Therefore, $C_S(u) \neq C_S(v)$.
3. Suppose $u \in S$ and $v \in V(G) - S$. Then the locating code of the vertex u with respect to S, $C_S(u)$ contains the zero vector. Since $v \in V(G) - S$, $C_S(v)$ does not contain the zero vector. Therefore, $C_S(u) \neq C_S(v)$. Thus, from the above three cases, S is a locating set.

□

Since the vertex set of G is an external multiplicative distance-locating set, $loc_e^*(G)$ exists for every graph G. Since every multiplicative distance-locating set is an external multiplicative distance-locating set and since every external multiplicative distance-locating set is a locating set, we have the following.

Theorem 6. *For every connected graph G, $loc(G) \leq loc_e^*(G)$. Moreover, if G is a connected graph G of order $n \geq 2$ for which $loc_d^*(G)$ exists, then $1 \leq loc(G) \leq loc_e^*(G) \leq loc_d^*(G)$.*

In Theorem 1 the path P_n of order $n \geq 2$ is the only connected graph with $loc_d^*(G) = 1$. It is straight forward to show that the path P_n is also the only connected graph of order n with $loc_e^*(G) = 1$.

Theorem 7. *Let G be a connected graph G of order $n \geq 2$. Then $loc_e^*(G) = 1$ if and only if $G = P_n$.*

It is shown that the complete graph K_n of order $n \geq 2$ is the only connected graph of order n with $loc(G) = n - 1$, while $loc_d^*(K_n)$ does not exist by Corollary **??**. Certainly $loc_e^*(K_n) = n - 1$ for all n≥2 as well. On the other hand, K_n is not the only connected graph of order n with external multiplicative location number $n - 1$.

Theorem 8. *If G is an $r-$regular connected graph of order n and diameter 2, then $loc_e^*(G) = n - 1$.*

Proof. Suppose G is an $r-$regular connected graph of order n and diameter 2. Because of Theorem 7, it suffices to show that for each integer k with $2 \leq k \leq n - 2$, there is no external multiplicative distance-locating set of G consisting of k vertices. Assume to the contrary, that there is an external multiplicative distance-locating set of order k in G. Let $S = \{w_1, w_2, \ldots, w_k\}$ be an external multiplicative distance-locating set for G, where $2 \leq k \leq n - 2$. Let $V(G) - S = \{v_1, v_2, \ldots, v_{n-k}\}$. For each $v_i \in V(G) - S$, where $1 \leq i \leq n - k$, let s_i be the number of vertices of S that are adjacent to v_i, and let t_i be the number of vertices of S that are not adjacent to v_i. Thus, $s_i + t_i = k$, $0 \leq s_i \leq \min\{r, k\}$, and $0 \leq t_i \leq \min\{n - 1 - r, k\}$, for all i with $1 \leq i \leq n - k$. Since $diamG = 2$, $d_S^*(v_i) = 2t_i$ for all i with $1 \leq i \leq n - k$. Since S is an external multiplicative distance-locating set for G, all $n - k$ numbers $d_S^*(v_i)$ are distinct. Therefore, $t_i \neq t_j$. Since $s_i + t_i = k$, all $n - k$ numbers s_i are distinct and all $n - k$ numbers t_i are distinct. Let $H = \langle V(G) - S \rangle$ be the subgraph induced by $V(G) - S$. Then $deg_H(v_i) = r - s_i$ for all i with $1 \leq i \leq n - k$. Since the $n - k$ numbers s_i are distinct, $deg_H(v_i)$ are distinct, $1 \leq i \leq n - k$. Therefore, no two vertices of H have the same degree, which is impossible. Hence $loc_e^*(G) = n - 1$. □

As an immediate consequence of Theorem 8 we obtain the external location number of the Petersen graph.

Corollary 7. *The Petersen graph has external location number 9.*

Next we determine the external location number of the complete bipartite graphs.

Theorem 9. *For integers $r, s \geq 1$ with $r + s \geq 4$,*

$$loc_e^*(K_{r,s}) = \begin{cases} r+s-2 & \text{if } r \neq s, \\ r+s-1 & \text{if } r = s. \end{cases}$$

Proof. Assume that $r \leq s$. Let $V_1 = \{u_1, u_2, \ldots, u_r\}$ and $V_2 = \{v_1, v_2, \ldots, v_s\}$ be the partite sets of $K_{r,s}$. If $r = s$, then by Theorem 8 $loc_e^*(K_{r,s}) = r + s - 1$. So we may assume that $r < s$. Every external multiplicative distance-locating set of $K_{r,s}$ contains at least $r - 1$ vertices from V_1 and at least $s - 1$ vertices from V_2. Thus, $loc_e^*(K_{r,s}) \geq r + s - 2$. Let $W = (V_1 - \{u_r\}) \cup (V_2 - \{v_s\})$. Then $d_W^*(u_r) = 2^{(r-1)}$ and $d_W^*(v_s) = 2^{(s-1)}$. Suppose $d_W^*(u_r) = d_W^*(v_s)$. Then $r = s$, which is a contradiction. Therefore, $d_W^*(u_r) \neq d_W^*(v_s)$. Thus, W is an external multiplicative distance-locating set for $K_{r,s}$. Since $loc_e^*(K_{r,s}) \geq r + s - 2$, W is the minimum external multiplicative distance-locating set for $K_{r,s}$. Therefore, $loc_e^*(K_{r,s}) = r + s - 2$. □

4. The Magic Location Number of a graph

Let G be a connected graph with vertex set $V(G)$. Let S be a subset of $V(G)$. We define the distance of a vertex v with respect to S by

$$d_S^*(v) = \sum_{x \in S, x \neq v} d(v, x).$$

If $d_S(u) = d_S(v)$ for each pair u, v of distinct vertices of $G - S$, then S is called a magic locating set of G. The cardinality of a minimum magic locating set of G is called the magic location number of G. It is denoted by $loc_M(G)$. Every distance-locating set is not a magic locating set and every external distance-locating set is not a magic locating set.

The following Theorem 10 characterizes graph G with $loc_M(G) = 1$.

Theorem 10. *Let G be a connected graph of order $n \geq 2$. Then $loc_M(G) = 1$ if and only if G contains a vertex of degree $n - 1$.*

Proof. Suppose $\text{loc}_M(G) = 1$. Let $S = \{w\}$ be a minimum magic locating set for G. Then $d(u, w) = d(v, w)$ for each pair u, v of distinct vertices of $V(G) - S$. Assume that $d(u, w) = l > 1$ for some $u \in V(G) - S$, where $2 \leq l \leq n - 1$. Then there exists a path $P : u = u_0, u_1, u_2, \ldots, u_{l-1}, u_l = w$ in G. Therefore, $d(u, w) = l$ and $d(u_{l-1}, w) = 1$, which is a contradiction. Thus, $d(u, w) = 1$ for all $u \in V(G) - S$. Hence, w is adjacent to all the vertices of $G - S$. Therefore, $\deg(w) = n - 1$.

Conversely, assume that G contains a vertex of degree $n - 1$. Let w be a vertex of G such that $\deg(w) = n - 1$. Let $S = \{w\}$. Then $d_S(u) = 1$ for all $u \in V(G) - S$. Therefore, S is a minimum magic locating set for G. Hence, $\text{loc}_M(G) = 1$. □

Corollary 8. *The magic location number of the graphs $K_n, K_{1,n}, W_n, F_n$ is 1.*

In Theorem 11 and Theorem 12 we determine the magic location number of an even cycle and the complete bipartite graphs.

Theorem 11. *If C_n is an even cycle, then $loc_M(C_n) = 2$.*

Proof. Let $C_n : v_1, v_2, \ldots, v_k, \ldots, v_{2k}$ be an even cycle. Let $S = \{v_k, v_{2k}\}$ be a subset of $V(C_n)$. Then $d_S(v_i) = k$ for $1 \leq i < k$ and $d_S(v_i) = k$ for $k < i < 2k$. Hence S is a magic locating set for C_n. Therefore, $loc_M(C_n) \leq 2$. By Theorem 10 $loc_M(C_n) = 2$. □

Theorem 12. *For $m, n \geq 2$, $loc_M(K_{m,n}) = 2$.*

Proof. Let G be a complete bipartite graph with vertex set $V = V_1 \cup V_2$ where $V_1 = \{v_1, v_2, \ldots, v_m\}$ and $V_2 = \{u_1, u_2, \ldots, u_n\}$. Let $S = \{v_1, u_1\}$ be a subset of $V(G)$. Then $d_S(v_i) = 3$ for $1 \leq i \leq m$ and $d_S(v_i) = 3$ for $1 \leq i \leq n$. Hence S is a magic locating set for G. Therefore, $loc_M(K_{m,n}) \leq 2$. By Theorem 10 $loc_M(K_{m,n}) = 2$. □

Theorem 13 shows that the magic location number of a path P_n of order $n \geq 2$ is less than or equal to $n - 2$.

Theorem 13. *If P_n is a path on n vertices, then $loc_M(P_n) \leq n - 2$.*

Proof. Let $P_n : v_1, v_2, \ldots, v_k, \ldots, v_n$ be a path on n vertices. Let $S = \{v_2, v_3, \ldots, v_{n-1}\}$ be a subset of the vertex set of P_n. Then $d_S(v_1) = 1 + 2 + \ldots + (n-2) = \frac{(n-2)(n-1)}{2}$ and $d_S(v_n) = (n-2) + (n-3) + \ldots + 2 + 1 = \frac{(n-2)(n-1)}{2}$. Thus, S is a magic locating set for P_n. Therefore, $loc_M(P_n) \leq n - 2$. □

5. Conclusion

We have defined some new concepts Multiplicative Distance-locating set, multiplicative distance location number, External Multiplicative location number and magic location number of graphs. We have discussed the existence or non-existence of these numbers for some well known classes of connected graphs. For future study we would like to propose the following open problems.

1. If G is a connected graph of diameter 3 for which $loc_d^*(G)$ exists, then what about G?
2. For what values of n and m, $loc_d^*(P_n \divideontimes C_m) = 4$.
3. For what values of n and m, $loc_d^*(P_n \divideontimes C_m) = 3$.
4. Does equality holds in Theorem 13. Based on our experience we believe that equality holds.

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