



## $D$ -irregularity strength of a graph

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### ABSTRACT

We initiate to study a  $D$ -irregular labeling, which generalizes both non-inclusive and inclusive  $d$ -distance irregular labeling of graphs. Let  $G = (V(G), E(G))$  be a graph,  $D$  a set of distances, and  $k$  a positive integer. A mapping  $\varphi$  from  $V(G)$  to the set of positive integers  $\{1, 2, \dots, k\}$  is called a  $D$ -irregular  $k$ -labeling of  $G$  if every two distinct vertices have distinct weights, where the weight of a vertex  $x$  is defined as the sum of labels of vertices whose distance from  $x$  belongs to  $D$ . The least integer  $k$  for which  $G$  admits a  $D$ -irregular labeling is the  $D$ -irregularity strength of  $G$  and denoted by  $s_D(G)$ . In this paper, we establish several fundamental properties on  $D$ -irregularity strength for arbitrary graphs. We also determine this parameter exactly for families of graphs with small diameter or small maximum degree.

*Keywords:* Graph labeling,  $D$ -irregular  $k$ -labeling,  $D$ -irregularity strength, Distance

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## 1. Introduction

Throughout the paper, only undirected simple finite graphs are considered. For undefined terminology and notation, we follow [8]. Let  $G$  be a graph. We will denote by  $V(G)$  and  $E(G)$  the *vertex* and *edge set* of  $G$ , respectively. The *distance* between two vertices  $x, y \in V(G)$  will be denoted by  $d(x, y)$ .

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Graph labeling has been an intriguing topic, extensively studied over the last six decades. To date, over 350 graph labelings techniques have been developed in over 3600 papers [11], one of which is a distance antimagic labeling, firstly established by Kamatchi and Arumugam [14]. A bijection  $\varphi : V(G) \rightarrow \{1, 2, \dots, |V(G)|\}$  is said to be a *distance antimagic labeling* of a graph  $G$  if the *vertex weights*, given by

$$wt(x) = \sum_{z \in N(x)} \varphi(z),$$

are all distinct, where  $N(x)$ , the *open neighborhood* of  $x$ , is the set of all vertices adjacent to  $x$ . A *distance antimagic graph* is a graph that has a distance antimagic labeling.

Simanjuntak et al. [16] introduced a more generalized concept, called a *D-antimagic labeling*, where  $D$  is a non-empty subset of the set  $\{0, 1, \dots, \text{diam}(G)\}$ . Here, the difference is that the *weight* of a vertex  $x$  is now defined as

$$wt(x) = \sum_{z \in N_D(x)} \varphi(z),$$

where  $N_D(x) = \{z : d(x, z) \in D\}$  is the *D-neighborhood* of  $x$ . A graph that admits a *D-antimagic* labeling is called *D-antimagic*. Notably, a  $\{1\}$ -antimagic graph corresponds to a distance antimagic graph. Some graph families have been proved to be *D-antimagic*, among others, see [9, 12, 13, 14, 16, 17, 24].

Instead of looking at bijection type labelings, Chartrand et al. [7] considered an *edge k-labeling*  $\varphi : E(G) \rightarrow \{1, 2, \dots, k\}$  of a graph  $G$ , where distinct edges may receive an identical label, such that the *vertex weights*,

$$wt(x) = \sum_{xy \in E(G)} \varphi(xy),$$

are all distinct. They named such labelings *irregular assignments* and the *irregularity strength*,  $s(G)$ , is the minimum positive integer  $k$  such that  $G$  has an irregular assignment using labels not exceeding  $k$ . This topic has gained significant interest, with numerous papers published; see [2, 3, 15] for some recent results.

Combining distance-based labelings and irregular assignments, Slamin [18] introduced a *non-inclusive distance irregular k-labeling* of a graph  $G$  as a mapping  $\varphi : V(G) \rightarrow \{1, 2, \dots, k\}$  such that for every two distinct vertices  $x, y \in V(G)$  there is  $wt(x) \neq wt(y)$ , where

$$wt(x) = \sum_{z \in N(x)} \varphi(z),$$

is the *weight* of a vertex  $x$ . The *non-inclusive distance irregularity strength*,  $\text{dis}(G)$ , of  $G$  is the smallest integer  $k$  such that  $G$  has a non-inclusive distance irregular labeling. There are not many graphs whose non-inclusive distance irregularity strengths are known. Among those that are known include paths, complete graphs, cycles, wheels, fan and book graphs [5, 18], disjoint union of paths, suns, helms and friendships [20].

Later on, Bača et al. [4] modified such a labeling and introduced an *inclusive distance irregular labeling* of graphs. In this case, the label of the vertex is also included in the computation of its vertex weight, while the non-inclusive is not. Furthermore, the two labelings were generalized by Bong et al. [6] to *non-inclusive* and *inclusive d-distance irregular labelings*, where  $d$  is an integer arbitrarily taken from 1 up to the diameter of the graph. A (non-)inclusive 1-distance irregular labeling is also called a (non-)inclusive distance irregular labeling. For more information on these subjects one can see [19, 21, 22, 23].

In this paper, we propose a new invariant, called a  $D$ -irregular labeling, which generalizes the previous labelings mentioned in [4, 6, 18]. This generalization is analogous to  $D$ -antimagic labelings by Simanjuntak et al. [16]. The formal definition is provided below.

**Definition 1.1.** Let  $G = (V(G), E(G))$  be a graph with diameter  $\text{diam}(G)$ . For a non-empty set of distances  $D \subseteq \{0, 1, \dots, \text{diam}(G)\}$  and a positive integer  $k$ , a function  $\varphi : V(G) \rightarrow \{1, 2, \dots, k\}$  is called a  $D$ -irregular  $k$ -labeling of  $G$  if  $wt(x) \neq wt(y)$  for every two distinct vertices  $x, y \in V(G)$ , where the weight of a vertex  $x$  is defined as

$$wt(x) = \sum_{z \in N_D(x)} \varphi(z).$$

We set  $wt(x) = 0$  if  $N_D(x) = \emptyset$ . The  $D$ -irregularity strength of  $G$ , denoted by  $s_D(G)$ , is the minimum  $k$  such that  $G$  admits a  $D$ -irregular labeling. If such an integer  $k$  does not exist then we define  $s_D(G) = \infty$ .

In a case when  $G$  is disconnected, we have that  $\text{diam}(G) = \infty$ , and  $D \subseteq \{0, 1, \dots\}$ . Suppose that  $G_1, G_2, \dots, G_m$ ,  $m \geq 2$ , be the components of  $G$ ,  $\max_{1 \leq i \leq m} \text{diam}(G_i) = \partial$ , and  $D' \subseteq D \cap \{\partial + 1, \partial + 2, \dots\}$ . Then  $s_D(G) = s_{D-D'}(G)$ . Moreover, from Definition 1.1, it follows that

- (a) A  $\{1\}$ -irregular labeling is a non-inclusive distance irregular labeling.
- (b) A  $\{0, 1\}$ -irregular labeling is an inclusive distance irregular labeling.
- (c) For  $d \in \{1, 2, \dots, \text{diam}(G)\}$ , a  $\{1, 2, \dots, d\}$ -irregular labeling is a non-inclusive  $d$ -distance irregular labeling.
- (d) For  $d \in \{1, 2, \dots, \text{diam}(G)\}$ , a  $\{0, 1, \dots, d\}$ -irregular labeling is an inclusive  $d$ -distance irregular labeling.

The rest of the paper is organized as follows. In Section 2, we present fundamental properties of  $D$ -irregularity strength for arbitrary graphs. In Section 3, we compute the exact values on this parameter for certain graphs with small diameter. Section 4 addresses graphs with small maximum degree. Finally, Section 5 presents concluding remarks.

## 2. Basic properties

We start with the following theorem which provides the necessary and sufficient conditions for a graph to possess finite  $D$ -irregularity strength.

**Theorem 2.1.** *Let  $G$  be a graph. Then  $s_D(G) = \infty$  if and only if there exist two distinct vertices  $x, y \in V(G)$  such that  $N_D(x) = N_D(y)$ . Moreover, if  $s_D(G) < \infty$  then  $s_D(G) \leq 2^{|V(G)|-1}$ .*

**Proof.** Let  $x, y$  be two vertices of  $G$  with  $N_D(x) = N_D(y)$ . Then, in any vertex labeling  $\varphi$  of  $G$ ,

$$wt(x) = \sum_{z \in N_D(x)} \varphi(z) = \sum_{z \in N_D(y)} \varphi(z) = wt(y),$$

meaning that  $\varphi$  can not be  $D$ -irregular.

Now suppose that  $N_D(x) \neq N_D(y)$  for every two distinct vertices  $x, y \in V(G)$ . Let us denote the vertices of  $G$  arbitrarily by the symbols  $x_1, x_2, \dots, x_{|V(G)|}$ . Define a vertex  $(2^{|V(G)|-1})$ -labeling  $\varphi$  of  $G$

as follows:

$$\varphi(x_i) = 2^{i-1} \quad \text{for } i = 1, 2, \dots, |V(G)|.$$

We define a labeling  $\beta$  such that

$$\beta_{ij} = \begin{cases} 1 & \text{if } x_i \in N_D(x_j), \\ 0 & \text{if } x_i \notin N_D(x_j), \end{cases}$$

where  $i = 1, 2, \dots, |V(G)|$  and  $j = 1, 2, \dots, |V(G)|$ . Then,

$$wt(x_j) = \sum_{x_i \in N_D(x_j)} \varphi(x_i) = \sum_{i: x_i \in N_D(x_j)} 2^{i-1} = \sum_{i=1}^{|V(G)|} \beta_{ij} 2^{i-1}. \quad (1)$$

To prove that all the vertex weights are distinct, it suffices to show that the sums  $\sum_{i=1}^{|V(G)|} \beta_{ij} 2^{i-1}$  are all distinct for  $j = 1, 2, \dots, |V(G)|$ . However, this is evident if we consider that the ordered  $|V(G)|$ -tuple  $(\beta_{1j}, \beta_{2j}, \dots, \beta_{|V(G)|j})$  corresponds to the binary code representation of the sum (1). Because  $N_D(x) \neq N_D(y)$  for every two distinct vertices  $x, y \in V(G)$  we can get that the  $|V(G)|$ -tuples are distinct for distinct vertices. Hence  $s_D(G) \leq 2^{|V(G)|-1}$ .  $\square$

In a non-trivial graph  $G$ , it is evident that if  $D = \{0, 1, \dots, \text{diam}(G)\}$  then  $N_D(x) = V(G) = N_D(y)$  for any two vertices  $x, y \in V(G)$ , and so by Theorem 2.1  $G$  has no  $D$ -irregular labeling. Hence, the next corollary holds.

**Corollary 2.2.** *Let  $G$  be a non-trivial graph. Then  $s_{\{0,1,\dots,\text{diam}(G)\}}(G) = \infty$ .*

For a graph  $G$  and a set of distances  $D$ , the *complement* of  $D$ , denoted by  $D^c$ , is a set in which  $D \cup D^c = \{0, 1, \dots, \text{diam}(G)\}$  and  $D \cap D^c = \emptyset$ . In the next theorem, we give a relationship between  $D$ -irregularity strength and  $D^c$ -irregularity strength of graphs.

**Theorem 2.3.** *Let  $G$  be a non-empty graph. Then*

- (a)  $s_D(G) = \infty$  if and only if  $s_{D^c}(G) = \infty$ .
- (b) If  $G$  is connected and  $s_D(G) < \infty$  then  $s_{D^c}(G) = s_D(G)$ .
- (c) Let  $G$  be a disconnected graph with components  $G_1, G_2, \dots, G_m$  for  $m \geq 2$ , and  $s_D(G) < \infty$ . If there exists a  $D$ -irregular  $s_D(G)$ -labeling  $\varphi$  of  $G$  with the property that  $wt(x) - wt(y) \neq \sum_{z \in V(G_i)} \varphi(z) - \sum_{z \in V(G_j)} \varphi(z)$  for every two distinct vertices  $x \in V(G_i)$  and  $y \in V(G_j)$ ,  $i \neq j$ , then  $s_{D^c}(G) \leq s_D(G)$ .

**Proof.** (a) If  $s_D(G) = \infty$  then by Theorem 2.1 there exist two distinct vertices  $x, y \in V(G)$  such that  $N_D(x) \neq N_D(y)$ . If  $G$  is connected then

$$\begin{aligned} N_{D^c}(x) &= N_{\{0,1,\dots,\text{diam}(G)\}-D}(x) \\ &= V(G) - N_D(x) = V(G) - N_D(y) = N_{\{0,1,\dots,\text{diam}(G)\}-D}(y) = N_{D^c}(y), \end{aligned}$$

and so  $s_{D^c}(G) = \infty$  by Theorem 2.1.

Next, let  $G$  be disconnected with components  $G_1, G_2, \dots, G_m$  for some  $m \geq 2$ . If  $x, y \in V(G_i)$  for some  $i$ , then

$$\begin{aligned} N_{D^c}(x) &= N_{\{0,1,\dots\}-D}(x) = N_{(\{0,1,\dots,\text{diam}(G_i)\}-D) \cup \{\text{diam}(G_i)+1, \text{diam}(G_i)+2, \dots\}}(x) \\ &= N_{\{0,1,\dots,\text{diam}(G_i)\}-D}(x) = V(G_i) - N_D(x) = V(G_i) - N_D(y) \\ &= N_{\{0,1,\dots,\text{diam}(G_i)\}-D}(y) = N_{(\{0,1,\dots,\text{diam}(G_i)\}-D) \cup \{\text{diam}(G_i)+1, \text{diam}(G_i)+2, \dots\}}(y) \\ &= N_{\{0,1,\dots\}-D}(y) = N_{D^c}(y), \end{aligned}$$

thus  $s_{D^c}(G) = \infty$  by Theorem 2.1. If  $x \in V(G_i)$  and  $y \in V(G_j)$  with  $\text{diam}(G_i) \geq \text{diam}(G_j)$  for  $i \neq j$ , then  $N_D(x) = N_D(y) = \emptyset$ , and so  $D \subseteq \{\text{diam}(G_i) + 1, \text{diam}(G_i) + 2, \dots\}$ . As  $G$  is a non-empty graph, we may assume that  $G_i$  contains at least two vertices. Indeed, for every two distinct vertices  $x, z \in V(G_i)$ ,  $N_{D^c}(x) = N_{D^c}(z) = N_{\{0,1,\dots\}-D}(z) = N_{(\{0,1,\dots,\text{diam}(G_i)\}-D) \cup \{\text{diam}(G_i)+1, \text{diam}(G_i)+2, \dots\}}(z) = N_{\{0,1,\dots,\text{diam}(G_i)\}-D}(z) = V(G_i)$ , hence  $s_{D^c}(G) = \infty$  by Theorem 2.1.

Conversely, if  $s_{D^c}(G) = \infty$ , similar reasoning shows that  $s_D(G) = \infty$ .

(b) Let  $G$  be connected and  $s_D(G) < \infty$ . Let  $\varphi$  be a  $D$ -irregular  $s_D(G)$ -labeling of  $G$ . We define a new vertex labeling  $\varphi'$  of  $G$  in such a way that

$$\varphi'(x) = \varphi(x) \quad \text{for } x \in V(G).$$

We show that  $\varphi'$  is a  $D^c$ -irregular  $s_D(G)$ -labeling of  $G$ .

Evidently  $\varphi'(x) \leq s_D(G)$  for all  $x \in V(G)$ . For the vertex weights under the labeling  $\varphi'$  we have

$$wt(x) = \sum_{z \in N_{D^c}(x)} \varphi'(z) = \sum_{z \in N_{\{0,1,\dots,\text{diam}(G)\}-D}(x)} \varphi'(z) = \sum_{z \in V(G)} \varphi(z) - \sum_{z \in N_D(x)} \varphi(z).$$

Note that the sums  $\sum_{z \in N_D(x)} \varphi(z)$  are distinct for each  $x \in V(G)$  since  $\varphi$  is  $D$ -irregular. This implies that  $wt(x) \neq wt(y)$  for every two distinct vertices  $x, y \in V(G)$ , and so  $s_{D^c}(G) \leq s_D(G)$ .

Since  $s_D(G) < \infty$  we have  $s_{D^c}(G) < \infty$  according to (1). Then, by similar arguments, we can prove that  $s_D(G) \leq s_{D^c}(G)$ . Therefore  $s_{D^c}(G) = s_D(G)$ .

(c) Let  $G$  be a disconnected graph with components  $G_1, G_2, \dots, G_m$  for  $m \geq 2$ , and  $s_D(G) < \infty$ . Assume  $\varphi$  be a  $D$ -irregular  $s_D(G)$ -labeling of  $G$  satisfying the stated condition. Define a new vertex labeling  $\varphi'$  of  $G$  such that

$$\varphi'(x) = \varphi(x) \quad \text{for } x \in V(G).$$

We shall show that  $\varphi'$  is a  $D^c$ -irregular  $s_D(G)$ -labeling of  $G$ .

Clearly  $\varphi'(x) \leq s_D(G)$  for all  $x \in V(G)$ . Let us consider the weight of any vertex  $x$  in the component  $G_i$ ,  $i = 1, 2, \dots, m$ , under the labeling  $\varphi'$ . We have

$$\begin{aligned} wt(x) &= \sum_{z \in N_{D^c}(x)} \varphi'(z) = \sum_{z \in N_{\{0,1,\dots\}-D}(x)} \varphi'(z) \\ &= \sum_{z \in N_{(\{0,1,\dots,\text{diam}(G_i)\}-D) \cup \{\text{diam}(G_i)+1, \text{diam}(G_i)+2, \dots\}}(x)} \varphi'(z) \\ &= \sum_{z \in N_{\{0,1,\dots,\text{diam}(G_i)\}-D}(x)} \varphi'(z) \\ &= \sum_{z \in V(G_i)} \varphi(z) - \sum_{z \in N_D(x)} \varphi(z). \end{aligned}$$

Consider any two distinct vertices  $x, y \in V(G)$ . Evidently if  $x$  and  $y$  are in the same component then  $wt(x) \neq wt(y)$  since  $\sum_{z \in N_D(x)} \varphi(z) \neq \sum_{z \in N_D(y)} \varphi(z)$ . Now, suppose that  $x$  and  $y$  are in the different components, say  $x \in V(G_i)$  and  $y \in V(G_j)$  for  $i \neq j$ . By the property in (3), it follows that  $wt(x) \neq wt(y)$ . So all the vertex weights are distinct, and hence  $s_{D^c}(G) \leq s_D(G)$ .  $\square$

It is evident that every  $D$ -antimagic labeling is also a  $D$ -irregular labeling. Consequently, we derive the following property, which provides a better upper bound for  $s_D(G)$  in a case that  $G$  is  $D$ -antimagic.

**Proposition 2.4.** *Let  $G$  be a  $D$ -antimagic graph on  $n$  vertices. Then  $s_D(G) \leq n$ .*

The next proposition, which is straightforward, shows that the upper bound in Proposition 2.4 is sharp.

**Proposition 2.5.** *Let  $G$  be a graph on  $n$  vertices. If one of the following conditions holds:*

- (a)  $G$  is connected and  $D \in \{\{0\}, \{1, 2, \dots, \text{diam}(G)\}\}$ ,
  - (b)  $G \cong mK_2$ ,  $m \geq 2$ , and  $D \in \{\{0\}, \{1\}\}$ , or
  - (c)  $G$  is disconnected,  $G \not\cong mK_2$ ,  $m \geq 2$ , and  $D = \{0\}$ ,
- then  $G$  is  $D$ -antimagic and  $s_D(G) = n$ .

In the next theorem, we provide lower bounds for  $D$ -irregularity strength of arbitrary graphs. For a vertex  $x$  in a graph  $G$ , the  $D$ -degree,  $d_D(x)$  of  $x$  is the cardinality of the set  $N_D(x)$ .

**Theorem 2.6.** *Let  $G$  be a graph with  $s_D(G) < \infty$ . Let  $\delta_D(G)$  and  $\Delta_D(G)$  be the minimum and the maximum  $D$ -degree of vertices of  $G$ , respectively. Let  $n_{D,i}$  be the number of vertices in  $G$  with  $D$ -degree  $i$  for  $i = \delta_D(G), \delta_D(G) + 1, \dots, \Delta_D(G)$ . Then,*

$$s_D(G) \geq \max_{\delta_D(G) \leq i \leq \Delta_D(G)} \left\{ \left\lceil \frac{\delta_D(G) + \sum_{j=\delta_D(G)}^i n_{D,j} - 1}{i} \right\rceil \right\}.$$

**Proof.** For some  $t$ , let

$$\left\lceil \frac{\delta_D(G) + \sum_{j=\delta_D(G)}^t n_{D,j} - 1}{t} \right\rceil = \max_{\delta_D(G) \leq i \leq \Delta_D(G)} \left\{ \left\lceil \frac{\delta_D(G) + \sum_{j=\delta_D(G)}^i n_{D,j} - 1}{i} \right\rceil \right\}.$$

In any  $D$ -irregular labeling of a graph  $G$ , the smallest weight of vertices with  $D$ -degrees  $\delta_D(G), \delta_D(G) + 1, \dots, t$  is at least  $\delta_D(G)$ , and the largest among them must be at least  $\delta_D(G) + \sum_{j=\delta_D(G)}^t n_{D,j} - 1$ . Such largest weight is obtained from the sum of at most  $t$  labels. Therefore

$$s_D(G) \geq \left\lceil \frac{\delta_D(G) + \sum_{j=\delta_D(G)}^t n_{D,j} - 1}{t} \right\rceil = \max_{\delta_D(G) \leq i \leq \Delta_D(G)} \left\{ \left\lceil \frac{\delta_D(G) + \sum_{j=\delta_D(G)}^i n_{D,j} - 1}{i} \right\rceil \right\}.$$

$\square$

Notice that if  $D = \{1\}$ ,  $D = \{0, 1\}$ , and  $D = \{0, 1, \dots, d\}$ ,  $1 \leq d \leq \text{diam}(G)$ , then we acquire the lower bounds for  $s_D(G)$  which were proved in [21, Theorem 2.1], [19, Theorem 2], and [23, Lemma 2.1], respectively.

The next two theorems demonstrate key properties that will be instrumental in determining the  $D$ -irregularity strength of a graph.

**Theorem 2.7.** *Let  $G$  be a graph with  $s_D(G) < \infty$ . Let  $x, y$  be two distinct vertices of  $G$  such that  $N_D(x) - \{y\} = N_D(y) - \{x\}$ . Then  $x$  and  $y$  must receive distinct labels in any  $D$ -irregular labeling of  $G$ . Furthermore, if  $G$  is connected then  $x$  and  $y$  must receive distinct labels in any  $D^c$ -irregular labeling of  $G$ .*

**Proof.** Let  $x, y$  be two distinct vertices of  $G$  with  $N_D(x) - \{y\} = N_D(y) - \{x\}$ . As  $s_D(G) < \infty$ ,  $x \in N_D(y)$  and  $y \in N_D(x)$ . Next, in any  $D$ -irregular labeling  $\varphi$  of  $G$ ,

$$wt(x) = \sum_{z \in N_D(x)} \varphi(z) = \varphi(y) + \sum_{z \in N_D(x) - \{y\}} \varphi(z),$$

and

$$wt(y) = \sum_{z \in N_D(y)} \varphi(z) = \varphi(x) + \sum_{z \in N_D(y) - \{x\}} \varphi(z).$$

Since  $wt(x) \neq wt(y)$  and  $\sum_{z \in N_D(x) - \{y\}} \varphi(z) = \sum_{z \in N_D(y) - \{x\}} \varphi(z)$  then  $\varphi(x) \neq \varphi(y)$ .

Now let  $G$  be connected. Then, in any  $D^c$ -irregular labeling  $\varphi$  of  $G$ ,

$$\begin{aligned} wt(x) &= \sum_{z \in N_{D^c}(x)} \varphi(z) = \sum_{z \in N_{\{0,1,\dots,\text{diam}(G)\}-D}(x)} \varphi(z) = \sum_{z \in V(G)} \varphi(z) - \sum_{z \in N_D(x)} \varphi(z) \\ &= \sum_{z \in V(G)} \varphi(z) - \varphi(y) - \sum_{z \in N_D(x) - \{y\}} \varphi(z), \end{aligned}$$

and

$$\begin{aligned} wt(y) &= \sum_{z \in N_{D^c}(y)} \varphi(z) = \sum_{z \in N_{\{0,1,\dots,\text{diam}(G)\}-D}(y)} \varphi(z) = \sum_{z \in V(G)} \varphi(z) - \sum_{z \in N_D(y)} \varphi(z) \\ &= \sum_{z \in V(G)} \varphi(z) - \varphi(x) - \sum_{z \in N_D(y) - \{x\}} \varphi(z). \end{aligned}$$

Since  $wt(x) \neq wt(y)$  and  $\sum_{z \in N_D(x) - \{y\}} \varphi(z) = \sum_{z \in N_D(y) - \{x\}} \varphi(z)$  then  $\varphi(x) \neq \varphi(y)$ .  $\square$

**Theorem 2.8.** *Let  $G$  be a graph,  $D = D' \cup D''$ ,  $D' \cap D'' = \emptyset$ , and  $s_D(G) < \infty$ . Let  $x, y$  be two distinct vertices of  $G$  such that  $x \notin N_D(y)$  and  $y \notin N_D(x)$ . If  $N_{D'}(x) = N_{D'}(y)$ ,  $N_{D''}(x) \neq N_{D''}(y)$  and  $|N_{D''}(x)| = |N_{D''}(y)| = 1$  then vertices in  $N_{D''}(x)$  and  $N_{D''}(y)$  must receive distinct labels in any  $D$ -irregular labeling of  $G$ .*

**Proof.** Let  $x, y$  be two distinct vertices of  $G$  with  $x \notin N_D(y)$  and  $y \notin N_D(x)$  such that  $N_{D'}(x) = N_{D'}(y)$ ,  $N_{D''}(x) \neq N_{D''}(y)$  and  $|N_{D''}(x)| = |N_{D''}(y)| = 1$ . Let  $N_{D''}(x) = \{x'\}$  and  $N_{D''}(y) = \{y'\}$ , where  $x' \neq y'$ . Then, in any  $D$ -irregular labeling  $\varphi$  of  $G$ ,

$$\begin{aligned} wt(x) &= \sum_{z \in N_D(x)} \varphi(z) = \sum_{z \in N_{D' \cup D''}(x)} \varphi(z) = \sum_{z \in N_{D'}(x)} \varphi(z) + \sum_{z \in N_{D''}(x)} \varphi(z) \\ &= \sum_{z \in N_{D'}(x)} \varphi(z) + \varphi(x'), \end{aligned}$$

and

$$\begin{aligned} wt(y) &= \sum_{z \in N_D(y)} \varphi(z) = \sum_{z \in N_{D' \cup D''}(y)} \varphi(z) = \sum_{z \in N_{D'}(y)} \varphi(z) + \sum_{z \in N_{D''}(y)} \varphi(z) \\ &= \sum_{z \in N_{D'}(y)} \varphi(z) + \varphi(y'). \end{aligned}$$

Since  $wt(x) \neq wt(y)$  and  $\sum_{z \in N_{D'}(x)} \varphi(z) = \sum_{z \in N_{D'}(y)} \varphi(z)$  then  $\varphi(x') \neq \varphi(y')$ . □

Notice that when  $D = \{1\}$ , Theorem 2.7 yields the result mentioned in [18, Observation 2], and also when  $D = \{0, 1\}$ ,  $D' = \{1\}$ , and  $D'' = \{0\}$ , Theorem 2.8 provides the property proved in [4, Lemma 3.1].

In [16], Simanjuntak et al. defined a *distance- $D$  graph* of a graph  $G$ , denoted by  $G_D$ , as a simple graph with the same vertices as  $G$ , where two vertices are adjacent if their distance in  $G$  is an element of  $D$ . Note that, since  $G_D$  is a simple graph, any loops that would arise when  $0 \in D$  are excluded from the graph.

**Theorem 2.9.** *Let  $G$  be a graph.*

- (a) *If  $D$  does not contain 0 then  $s_{\{1\}}(G_D) = s_D(G)$ .*
- (b) *If  $D$  contains 0 then  $s_{\{0,1\}}(G_D) = s_D(G)$ .*

**Proof.** Let  $\varphi$  be a  $D$ -irregular  $s_D(G)$ -labeling of a graph  $G$ . We define a vertex labeling  $\varphi'$  of  $G_D$  in the following way.

$$\varphi'(x') = \varphi(x),$$

for any vertex  $x' \in V(G_D)$  corresponding to a vertex  $x \in V(G)$ .

If  $0 \notin D$ , then we have

$$\sum_{y \in N(x')} \varphi'(y) = \sum_{y \in N_D(x)} \varphi(y),$$

which implies that  $s_{\{1\}}(G_D) \leq s_D(G)$ . Using similar arguments, we can also show that  $s_D(G) \leq s_{\{1\}}(G_D)$ , so  $s_{\{1\}}(G_D) = s_D(G)$ . This proves (1).

Furthermore, if  $0 \in D$  then

$$\varphi'(x) + \sum_{y \in N(x')} \varphi'(y) = \sum_{y \in N_D(x)} \varphi(y),$$

which leads to  $s_{\{0,1\}}(G_D) \leq s_D(G)$ . Again, using similar reasoning, we can demonstrate that  $s_D(G) \leq s_{\{0,1\}}(G_D)$ , thus  $s_{\{0,1\}}(G_D) = s_D(G)$ . This completes the proof of (2). □

### 3. Graphs with small diameter

This section showcases the  $D$ -irregularity strength for families of graphs with diameter at most three, including complete graphs, the join graph  $G + K_1$ , complete bipartite graphs, and double stars, for all possible sets  $D$ .

A *complete graph*  $K_n$  is the graph with  $n$  vertices in which any two of them are joined by an edge. Evidently, the graph  $K_1$  is the only graph of diameter zero, and the graph  $K_n$ ,  $n \geq 2$ , is the only graph with diameter one.

A *complete bipartite graph*  $K_{m,n}$  is the graph whose vertices can be partitioned into two subsets  $X$  and  $Y$  with  $|X|=m$  and  $|Y|=n$ , such that  $xy$  is an edge if and only if  $x \in X$  and  $y \in Y$ . The graph  $K_{m,n}$ ,  $1 \leq m \leq n$ ,  $(m,n) \neq (1,1)$ , has diameter two. The complete bipartite graph  $K_{1,n}$  is called a *star*, the only tree with diameter two.

The *join product* between two given graphs  $G$  and  $H$ , denoted by  $G + H$ , is the graph obtained from  $G$  and  $H$  by joining an edge from every vertex of  $G$  to every vertex of  $H$ . The graph  $G + H$

has diameter one or two, with diameter one if and only if  $G$  and  $H$  are complete graphs. Therefore, the graph  $G + K_1$  has diameter two if and only if  $G$  is not a complete graph.

A *double star*  $S_{m,n}$  is the graph obtained from two stars  $K_{1,m}$  and  $K_{1,n}$  by connecting the two vertices of degrees  $m$  and  $n$  in  $K_{1,m}$  and  $K_{1,n}$ , respectively. The graph  $S_{m,n}$ ,  $1 \leq m \leq n$ , is the only tree with diameter three.

We first examine complete graphs. It is evident that  $s_D(K_1) = 1$ . We shall show that  $K_1$  is the only graph with  $D$ -irregularity strength one. Beforehand, let us prove in the next proposition a generalization of the well-known property that a simple graph on  $n \geq 2$  vertices whose all the vertex degrees are distinct does not exist.

**Proposition 3.1.** *There is no simple graph on  $n \geq 2$  vertices whose vertex  $D$ -degrees are all distinct.*

**Proof.** Assume that such a graph  $G$  exists. Note that the  $D$ -degree of a vertex is at least 0 and at most  $n$ . Thus,  $n$  numbers in the set  $\{0, 1, \dots, n\}$  have to be the  $D$ -degree of vertices of  $G$ .

We will show that  $n$  cannot be a vertex  $D$ -degree. Assume otherwise, that  $d_D(x) = n$  for some vertex  $x \in V(G)$ . This means that  $N_D(x) = V(G)$ , which forces  $G$  to be connected and  $D = \{0, 1, \dots, \text{diam}(G)\}$ . But then, this implies that  $N_D(y) = V(G)$  for any vertex  $y \neq x \in V(G)$ , and so  $N_D(x) = N_D(y)$ , a contradiction. Therefore the vertex  $D$ -degrees of  $G$  must be  $0, 1, \dots, n - 1$ .

Consider two distinct vertices  $x, y \in V(G)$  with  $d_D(x) = 0$  and  $d_D(y) = n - 1$ . Since  $d_D(x) = 0$ , we get  $0 \notin D$ . This implies that  $N_D(y) = V(G) - \{y\}$ , and so  $x \in N_D(y)$ . However, this is impossible since  $d_D(x) = 0$ . Hence, the statement holds. □

From Proposition 3.1 it follows that for any graph  $G$  of order  $n \geq 2$ , labeling all vertices of  $G$  with 1 will always result in at least two vertices with the same weight. This brings us to the characterization of graphs with  $D$ -irregularity strength one.

**Theorem 3.2.** *Let  $G$  be a graph. Then  $s_D(G) = 1$  if and only if  $G \cong K_1$ .*

**Question 3.3.** *Characterize all graphs with  $D$ -irregularity strength two.*

For  $n \geq 2$ , the  $D$ -irregularity strength of the complete graph  $K_n$  which can be directly derived from Proposition 2.5 and Corollary 2.2, is presented in the following theorem. Note that the cases when  $D = \{1\}$  and  $D = \{0, 1\}$  were also proved in [18] and [4], respectively.

**Theorem 3.4** ([4, 18]). *Let  $K_n$  be a complete graph for  $n \geq 2$ . Then*

$$s_D(K_n) = \begin{cases} n & \text{if } D \in \{\{0\}, \{1\}\}, \\ \infty & \text{if } D = \{0, 1\}. \end{cases}$$

Next, we deal with the join graph  $G + K_1$ . To begin, let us restate the results on  $s_D(G + K_1)$  for  $D = \{1\}$  and  $D = \{0, 1\}$ .

**Theorem 3.5** ([22]). *Let  $G$  be a graph. If  $s_{\{1\}}(G) = \infty$  then  $s_{\{1\}}(G + K_1) = \infty$ . Moreover, if*

$s_{\{1\}}(G) < \infty$  then

$$s_{\{1\}}(G + K_1) = \begin{cases} s_{\{1\}}(G) & \text{if there exists a } \{1\}\text{-irregular } s_{\{1\}}(G)\text{-labeling } \varphi \text{ of } G \\ & \text{such that } \sum_{x \in V(G)} \varphi(x) > \max\{wt(x) : x \in V(G)\} + 1, \\ s_{\{1\}}(G) + 1 & \text{otherwise.} \end{cases}$$

**Theorem 3.6** ([4]). *Let  $G$  be a graph with maximum degree  $\Delta(G)$ . Then*

$$s_{\{0,1\}}(G + K_1) = \begin{cases} s_{\{0,1\}}(G) & \text{if } \Delta(G) < |V(G)| - 1, \\ \infty & \text{if } \Delta(G) = |V(G)| - 1. \end{cases}$$

Corollary 2.2, Proposition 2.5, and Theorems 2.3, 3.5 and 3.6 allow us to completely obtained exact values on  $D$ -irregularity strength for the graph  $G + K_1$ .

**Theorem 3.7.** *Let  $G$  be a not complete graph on  $n \geq 2$  vertices with maximum degree  $\Delta(G)$ . Then*

(a) *If  $D \in \{\{0\}, \{1, 2\}\}$  then  $s_D(G + K_1) = n + 1$ .*

(b) *If one of the following conditions is satisfied:*

- $D \in \{\{2\}, \{0, 1\}\}$  and  $\Delta(G) < |V(G)| - 1$ , or
- $D \in \{\{1\}, \{0, 2\}\}$  and there exists a  $\{1\}$ -irregular  $s_{\{1\}}(G)$ -labeling  $\varphi$  of  $G$  such that  $\sum_{x \in V(G)} \varphi(x) > \max\{wt(x) : x \in V(G)\} + 1$ ,

*then  $s_D(G + K_1) = s_D(G)$ .*

(c) *If  $D \in \{\{1\}, \{0, 2\}\}$  and  $\sum_{x \in V(G)} \varphi(x) = \max\{wt(x) : x \in V(G)\} + 1$  for every  $\{1\}$ -irregular  $s_{\{1\}}(G)$ -labeling  $\varphi$  of  $G$ , then  $s_D(G + K_1) = s_D(G) + 1$ .*

(d) *If one of the following conditions is satisfied:*

- $D \in \{\{1\}, \{0, 2\}\}$  and  $s_D(G) = \infty$ ,
- $D \in \{\{2\}, \{0, 1\}\}$  and  $\Delta(G) = |V(G)| - 1$ , or
- $D = \{0, 1, 2\}$ ,

*then  $s_D(G + K_1) = \infty$ .*

Now, consider the complete bipartite graph  $K_{m,n}$  for  $1 \leq m \leq n$ ,  $(m, n) \neq (1, 1)$ . By Proposition 2.5, it is clear that  $s_{\{0\}}(K_{m,n}) = s_{\{1,2\}}(K_{m,n}) = m + n$ . From [4, Theorem 3.1] and Theorem 2.3, we know that  $s_{\{0,1\}}(K_{m,n}) = s_{\{2\}}(K_{m,n}) = n$  when  $m < n$ , and  $s_{\{0,1\}}(K_{m,n}) = s_{\{2\}}(K_{m,n}) = n + 2$  when  $m = n \geq 2$ . Moreover,  $s_D(K_{m,n}) = \infty$  whenever  $D \in \{\{1\}, \{0, 2\}, \{0, 1, 2\}\}$ , which follows immediately from [18, Observation 1], Theorem 2.3 and Corollary 2.2. Thus, we arrive at the following theorem.

**Theorem 3.8** ([4, 18]). *Let  $K_{m,n}$  be a complete bipartite graph for  $1 \leq m \leq n$ ,  $(m, n) \neq (1, 1)$ . Then*

$$s_D(K_{m,n}) = \begin{cases} m + n & \text{if } D \in \{\{0\}, \{1, 2\}\}, \\ n & \text{if } D \in \{\{2\}, \{0, 1\}\} \text{ and } m < n, \\ n + 2 & \text{if } D \in \{\{2\}, \{0, 1\}\} \text{ and } m = n \geq 2, \\ \infty & \text{if } D \in \{\{1\}, \{0, 2\}, \{0, 1, 2\}\}. \end{cases}$$

**Corollary 3.9.** *Let  $K_{1,n}$  be a star for  $n \geq 2$ . Then*

$$s_D(K_{1,n}) = \begin{cases} n+1 & \text{if } D \in \{\{0\}, \{1, 2\}\}, \\ n & \text{if } D \in \{\{2\}, \{0, 1\}\}, \\ \infty & \text{if } D \in \{\{1\}, \{0, 2\}, \{0, 1, 2\}\}. \end{cases}$$

We now focus on the double star. Let us denote the vertex and edge set of the double star  $S_{m,n}$  respectively by  $V(S_{m,n}) = \{x, x_i : i = 1, 2, \dots, m\} \cup \{y, y_j : j = 1, 2, \dots, n\}$  and  $E(S_{m,n}) = \{xy\} \cup \{xx_i : i = 1, 2, \dots, m\} \cup \{yy_j : j = 1, 2, \dots, n\}$ .

**Theorem 3.10** ([6]). *Let  $S_{m,n}$  be a double star for  $1 \leq m \leq n$ . Then*

$$s_{\{0,1\}}(S_{m,n}) = \begin{cases} n & \text{if } m < n, \\ n+1 & \text{otherwise.} \end{cases}$$

**Lemma 3.11.** *If  $D \in \{\{1\}, \{0, 2, 3\}\}$  then  $s_D(S_{1,1}) = 2$ .*

**Proof.** Obviously,  $s_{\{1\}}(S_{1,1}) = 2$ . Combining this with Theorem 2.3,  $s_{\{0,2,3\}}(S_{1,1}) = 2$ .  $\square$

**Lemma 3.12.** *Let  $S_{m,n}$  be a double star for  $1 \leq m < n$ . If  $D \in \{\{0, 1\}, \{0, 3\}, \{1, 2\}, \{2, 3\}\}$  then  $s_D(S_{m,n}) = n$ .*

**Proof.** From Theorems 2.3 and 3.10, we get  $s_{\{0,1\}}(S_{m,n}) = s_{\{2,3\}}(S_{m,n}) = n$ . Let  $D = \{0, 3\}$ . By Theorem 2.8,  $y_i$  and  $y_j$ ,  $i \neq j$ , must receive distinct labels, thus  $s_{\{0,3\}}(S_{m,n}) \geq n$ . Define a labeling  $\varphi : V(S_{m,n}) \rightarrow \{1, 2, \dots, n\}$  such that

$$\begin{aligned} \varphi(x) &= 1, \\ \varphi(y) &= 2, \\ \varphi(x_i) &= i \quad \text{for } i = 1, 2, \dots, m, \\ \varphi(y_j) &= j \quad \text{for } j = 1, 2, \dots, n. \end{aligned}$$

It is easy to see that all vertex labels are at most  $n$ . For the vertex weights we have

$$\begin{aligned} wt(x) &= 1, \\ wt(y) &= 2, \\ wt(x_i) &= \frac{n(n+1)}{2} + i \quad \text{for } i = 1, 2, \dots, m, \\ wt(y_j) &= \frac{m(m+1)}{2} + j \quad \text{for } j = 1, 2, \dots, n. \end{aligned}$$

As  $m < n$ , we get that  $\frac{m(m+1)}{2} + n < \frac{n(n+1)}{2} + 1$ , which implies that the vertex weights are all distinct. Therefore  $s_{\{0,3\}}(S_{m,n}) = n$ . By combining this with Theorem 2.3 we also have  $s_{\{1,2\}}(S_{m,n}) = n$ .  $\square$

**Lemma 3.13.** *Let  $S_{m,n}$  be a double star for  $1 \leq m \leq n$ . If one of the following conditions holds:*

- (a)  $D \in \{\{2\}, \{0, 1, 3\}\}$  and  $m < n$ , or  
 (b)  $D \in \{\{0, 1\}, \{2, 3\}\}$  and  $m = n$ ,  
 then  $s_D(S_{m,n}) = n + 1$ .

**Proof.** Let  $D = \{2\}$  and  $m < n$ . Suppose that  $\varphi$  is a  $\{2\}$ -irregular labeling of  $S_{m,n}$ . By Theorem 2.7,  $\varphi(y_i) \neq \varphi(y_j)$  for  $i \neq j$ , so  $s_{\{2\}}(S_{m,n}) \geq n$ .

Assume that  $s_{\{2\}}(S_{m,n}) = n$ . Then  $\{\varphi(y_j) : j = 1, 2, \dots, n\} = \{1, 2, \dots, n\}$ , which leads to  $wt(x) = \frac{n(n+1)}{2}$  and  $\{wt(y_j) : j = 1, 2, \dots, n\} = \{\frac{n(n+1)}{2} - n + \varphi(x), \frac{n(n+1)}{2} - n + 1 + \varphi(x), \dots, \frac{n(n+1)}{2} - 1 + \varphi(x)\}$ . As  $1 \leq \varphi(x) \leq n$ , the vertex weights are not distinct. Therefore,  $s_{\{2\}}(S_{m,n}) \geq n + 1$ .

Let  $\varphi$  be a vertex labeling of  $S_{m,n}$  defined such that

$$\begin{aligned}\varphi(x) &= n + 1, \\ \varphi(y) &= n, \\ \varphi(x_i) &= i \quad \text{for } i = 1, 2, \dots, m, \\ \varphi(y_j) &= j \quad \text{for } j = 1, 2, \dots, n.\end{aligned}$$

Then we obtain the vertex weights as follows.

$$\begin{aligned}wt(x) &= \frac{n(n+1)}{2}, \\ wt(y) &= \frac{m(m+1)}{2}, \\ wt(x_i) &= \frac{m(m+1)}{2} + n - i \quad \text{for } i = 1, 2, \dots, m, \\ wt(y_j) &= \frac{n(n+1)}{2} + n + 1 - j \quad \text{for } j = 1, 2, \dots, n.\end{aligned}$$

It is not difficult to verify that the vertex weights are distinct and the labels used in the labeling  $\varphi$  are not greater than  $n + 1$ , which means that  $\varphi$  is a  $\{2\}$ -irregular  $(n + 1)$ -labeling of  $S_{m,n}$  for  $m < n$ . Hence  $s_{\{2\}}(S_{m,n}) = n + 1$ . Furthermore, combining this with Theorem 2.3 we also have  $s_{\{0,1,3\}}(S_{m,n}) = n + 1$  for  $m < n$ .

Next, we have  $s_{\{0,1\}}(S_{n,n}) = s_{\{2,3\}}(S_{n,n}) = n + 1$  by Theorems 2.3 and 3.10.  $\square$

**Lemma 3.14.** Let  $S_{n,n}$  be a double star for  $n \geq 2$ . If  $D \in \{\{0, 3\}, \{1, 2\}\}$  then  $s_D(S_{n,n}) = n + 2$ .

**Proof.** Let  $D = \{0, 3\}$ . Define a vertex labeling  $\varphi : V(S_{n,n}) \rightarrow \{1, 2, \dots, n + 2\}$  of  $S_{n,n}$  in the following way:

$$\begin{aligned}\varphi(x) &= 1, \\ \varphi(y) &= 2, \\ \varphi(x_i) &= i \quad \text{for } i = 1, 2, \dots, n, \\ \varphi(y_i) &= i + 2 \quad \text{for } i = 1, 2, \dots, n.\end{aligned}$$

Obviously, the largest label that appears on the vertices is  $n + 2$ . Moreover, we have the following vertex weights.

$$\begin{aligned} wt(x) &= 1, \\ wt(y) &= 2, \\ wt(x_i) &= \frac{n(n+5)}{2} + i \quad \text{for } i = 1, 2, \dots, n, \\ wt(y_i) &= \frac{n(n+1)}{2} + i + 2 \quad \text{for } i = 1, 2, \dots, n. \end{aligned}$$

It is easy to see that the vertex weights are distinct. Therefore  $s_{\{0,3\}}(S_{n,n}) \leq n + 2$ .

Next, we shall show that  $s_{\{0,3\}}(S_{n,n}) \geq n + 2$ . For a contradiction, assume that there is a  $\{0, 3\}$ -irregular  $(n + 1)$ -labeling  $\varphi$  of  $S_{n,n}$  for  $n \geq 2$ . Then by Theorem 2.8, the following properties hold:

- $\varphi(x_i) \neq \varphi(x_j)$  for  $i \neq j$ , and
- $\varphi(y_i) \neq \varphi(y_j)$  for  $i \neq j$ .

We may assume, without loss of generality, that  $\{\varphi(x_i) : i = 1, 2, \dots, n\} = \{1, 2, \dots, n\}$  and  $\{\varphi(y_i) : i = 1, 2, \dots, n\} = \{1, 2, \dots, n + 1\} - \{r\}$ , where  $1 \leq r \leq n$ . Then

$$\begin{aligned} \{wt(x_i) : i = 1, 2, \dots, n\} &= \left\{ \frac{n(n+1)}{2} + n + 2 - r, \frac{n(n+1)}{2} + n + 3 - r, \dots, \frac{n(n+1)}{2} \right. \\ &\quad \left. + 2n + 1 - r \right\}, \end{aligned}$$

and

$$\begin{aligned} \{wt(y_i) : i = 1, 2, \dots, n\} &= \left\{ \frac{n(n+1)}{2} + 1, \frac{n(n+1)}{2} + 2, \dots, \frac{n(n+1)}{2} + n + 1 \right\} \\ &\quad - \left\{ \frac{n(n+1)}{2} + r \right\}. \end{aligned}$$

Since these sets are not disjoint, the vertex weights are not distinct, a contradiction. Thus  $s_{\{0,3\}}(S_{n,n}) \geq n + 2$ .

As  $n + 2 \leq s_{\{0,3\}}(S_{n,n})$  and  $s_{\{0,3\}}(S_{n,n}) \leq n + 2$  we conclude that  $s_{\{0,3\}}(S_{n,n}) = n + 2$ . Furthermore, combining this with Theorem 2.3 we get  $s_{\{1,2\}}(S_{n,n}) = n + 2$ .  $\square$

**Lemma 3.15.** *Let  $S_{n,n}$  be a double star for  $n \geq 1$ . If  $D \in \{\{2\}, \{0, 1, 3\}\}$  then  $s_D(S_{n,n}) = n + 3$ .*

**Proof.** Let  $D = \{2\}$ . Define a vertex labeling  $\varphi : V(S_{n,n}) \rightarrow \{1, 2, \dots, n + 3\}$  of  $S_{n,n}$  in the following way:

$$\begin{aligned} \varphi(x) &= n + 3, \\ \varphi(y) &= n + 1, \\ \varphi(x_i) &= i \quad \text{for } i = 1, 2, \dots, n, \\ \varphi(y_i) &= i + 2 \quad \text{for } i = 1, 2, \dots, n. \end{aligned}$$

Then we get the following vertex weights.

$$\begin{aligned} wt(x) &= \frac{n(n+5)}{2}, \\ wt(y) &= \frac{n(n+1)}{2}, \\ wt(x_i) &= \frac{(n+1)(n+2)}{2} - i \quad \text{for } i = 1, 2, \dots, n, \\ wt(y_i) &= \frac{n(n+7)}{2} + 1 - i \quad \text{for } i = 1, 2, \dots, n. \end{aligned}$$

It is not hard to show that the vertex weights are distinct and the labels used in the labeling  $\varphi$  are at most  $n+3$ . This means that  $\varphi$  is a  $\{2\}$ -irregular  $(n+3)$ -labeling of  $S_{n,n}$  for  $n \geq 1$ . Therefore  $s_{\{2\}}(S_{n,n}) \leq n+3$ .

Next, we shall show that  $s_{\{2\}}(S_{n,n}) \geq n+3$ . For a contradiction, assume that there exists a  $\{2\}$ -irregular  $(n+2)$ -labeling  $\varphi$  of  $S_{n,n}$  for  $n \geq 1$ . Using Theorem 2.7, the following properties must be hold:

- $\varphi(y) \neq \varphi(x_i) \neq \varphi(x_j)$  for  $i \neq j$ , and
- $\varphi(x) \neq \varphi(y_i) \neq \varphi(y_j)$  for  $i \neq j$ .

Without loss of generality we may assume that  $\{\varphi(y), \varphi(x_i) : i = 1, 2, \dots, n\} = \{1, 2, \dots, n+1\}$  and  $\{\varphi(x), \varphi(y_i) : i = 1, 2, \dots, n\} = \{1, 2, \dots, n+2\} - \{r\}$ , where  $1 \leq r \leq n+1$ . Then

$$\{wt(y), wt(x_i) : i = 1, 2, \dots, n\} = \left\{ \frac{(n+1)(n+2)}{2} - n - 1, \frac{(n+1)(n+2)}{2} - n, \dots, \frac{(n+1)(n+2)}{2} - 1 \right\}$$

and

$$\{wt(x), wt(y_i) : i = 1, 2, \dots, n\} = \left\{ \frac{(n+1)(n+2)}{2} - r, \frac{(n+1)(n+2)}{2} - r + 1, \dots, \frac{(n+1)(n+2)}{2} - r + n + 1 \right\} \left\{ \frac{(n+1)(n+2)}{2} + n + 2 - 2r \right\},$$

which implies that the vertex weights are not distinct, a contradiction. Therefore  $s_{\{2\}}(S_{n,n}) \geq n+3$ .

Since  $n+3 \leq s_{\{2\}}(S_{n,n}) \leq n+3$ , we get  $s_{\{2\}}(S_{n,n}) = n+3$  for  $n \geq 1$ . Moreover, combining this with Theorem 2.3 we also obtain  $s_{\{0,1,3\}}(S_{n,n}) = n+3$  for  $n \geq 1$ .  $\square$

In the next theorem, we present  $D$ -irregularity strength for the double stars.

**Theorem 3.16.** *Let  $S_{m,n}$  be a double star for  $1 \leq m \leq n$ . Then*

- (a) *If  $D \in \{\{1\}, \{0, 2, 3\}\}$  then  $s_D(S_{1,1}) = 2$ .*
- (b) *If  $D \in \{\{0, 1\}, \{0, 3\}, \{1, 2\}, \{2, 3\}\}$  and  $m < n$  then  $s_D(S_{m,n}) = n$ .*
- (c) *If one of the following conditions is satisfied:*
  - $D \in \{\{2\}, \{0, 1, 3\}\}$  and  $m < n$ , or
  - $D \in \{\{0, 1\}, \{2, 3\}\}$  and  $m = n$ ,

then  $s_D(S_{m,n}) = n + 1$ .

(d) If  $D \in \{\{0, 3\}, \{1, 2\}\}$  and  $m = n \geq 2$  then  $s_D(S_{m,n}) = n + 2$ .

(e) If  $D \in \{\{2\}, \{0, 1, 3\}\}$  and  $m = n$  then  $s_D(S_{m,n}) = n + 3$ .

(f) If  $D \in \{\{0\}, \{1, 2, 3\}\}$  then  $s_D(S_{m,n}) = m + n + 2$ .

(g) If one of the following conditions is satisfied:

- $D \in \{\{3\}, \{0, 2\}, \{1, 3\}, \{0, 1, 2\}, \{0, 1, 2, 3\}\}$ ,
- $D \in \{\{1\}, \{0, 2, 3\}\}$  and  $(m, n) \neq (1, 1)$ , or
- $D \in \{\{0, 3\}, \{1, 2\}\}$  and  $m = n = 1$ ,

then  $s_D(S_{m,n}) = \infty$ .

**Proof.** (1)-(5) follow from Lemmas 3.11, 3.12, 3.13, 3.14 and 3.15, respectively. Similarly, (6)-(7) are derived by applying Proposition 2.5, and Theorems 2.1 and 2.3.  $\square$

It is worth noting that, from Theorems 3.4 and 3.16, along with Corollary 3.9, we have fully characterized the  $D$ -irregularity strength for all non-trivial trees with diameter at most three, for all possible sets  $D$ .

**Question 3.17.** Find the  $D$ -irregularity strength of all trees with diameter four for all possible  $D$ s.

#### 4. Graphs with small maximum degree

This section explores the  $\{d\}$ -irregularity strength of certain graphs with maximum degree two, focusing particularly on the disjoint union of paths and 2-regular graphs, for a positive integer  $d$ .

We begin by examining the  $\{d\}$ -irregularity strength of the disjoint union of paths  $mP_n$  for  $m \geq 1$  and  $n \geq 2$ . For  $d = 1$ , the following theorem, established in [18, 20], provides the result.

**Theorem 4.1** ([18, 20]). Let  $mP_n$  be an  $m$  disjoint union of a path on  $n$  vertices for  $m \geq 1$  and  $n \geq 2$ . Then

$$s_{\{1\}}(mP_n) = \begin{cases} 2m & \text{if } n = 2, \\ \infty & \text{if } n = 3, \\ \lceil \frac{mn}{2} \rceil & \text{if } n \geq 4. \end{cases}$$

In the next theorem, we determine  $\{d\}$ -irregularity strength of  $mP_n$  for certain  $d \geq 2$ .

**Theorem 4.2.** Let  $mP_n$  be an  $m$  disjoint union of a path on  $n$  vertices for  $m \geq 1$  and  $n \geq 2$ , and let  $2 \leq d \leq n - 1$ . Then

- (a)  $s_{\{\frac{n}{2}\}}(mP_n) = mn$ .
- (b) For  $d \in \{\frac{n}{k} : 4 \leq k \leq \frac{n}{2}\}$ ,  $s_{\{d\}}(mP_n) = \lceil \frac{mn}{2} \rceil$ .
- (c)  $s_{\{d\}}(mP_n) = \infty$  if and only if

$$d = \begin{cases} \frac{n}{3} & \text{if } n \equiv 0 \pmod{d}, \\ \frac{n-t}{3} \text{ or } \frac{n-t}{2} & \text{if } n \equiv t \pmod{d}, 0 < t < d, \\ n - t & \text{if } n \equiv t \pmod{d}, 0 < t < d, \text{ and } m(n - 2t) \geq 2. \end{cases}$$

**Proof.** (1) Evidently,  $(mP_n)_{\{\frac{n}{2}\}} \cong \frac{mn}{2}K_2$ . Then by Theorem 2.9, we have  $s_{\{\frac{n}{2}\}}(mP_n) = s_{\{1\}}(\frac{mn}{2}K_2) = mn$ .

(2) If  $d \in \{\frac{n}{k} : 4 \leq k \leq \frac{n}{2}\}$  then  $(mP_n)_{\{d\}} \cong mdP_{\frac{n}{d}} \not\cong \frac{mn}{2}P_2 \not\cong \frac{mn}{3}P_3$ . By Theorems 2.9 and 4.1,

$$s_{\{d\}}(mP_n) = s_{\{1\}}(mdP_{\frac{n}{d}}) = \left\lceil \frac{mn}{2} \right\rceil.$$

(3) From Theorem 2.9 we know that  $s_{\{d\}}(mP_n) = \infty$  if and only if  $s_{\{1\}}((mP_n)_{\{d\}}) = \infty$ . Furthermore, by Theorem 2.1 we have that  $s_{\{1\}}((mP_n)_{\{d\}}) = \infty$  if and only if  $(mP_n)_{\{d\}}$  contains at least one component isomorphic to  $P_3$  or at least two components isomorphic to  $P_1$ . Thus,  $s_{\{d\}}(mP_n) = \infty$  if and only if  $(mP_n)_{\{d\}}$  has at least one component isomorphic to  $P_3$  or at least two components isomorphic to  $P_1$ .

We have that

$$(mP_n)_{\{d\}} = \begin{cases} mdP_{\frac{n}{d}} & \text{if } n \equiv 0 \pmod{d}, \\ m(d-t)P_{\frac{n-t}{d}} \cup mtP_{\frac{n-t+d}{d}} & \text{if } n \equiv t \pmod{d}, 0 < t < d. \end{cases}$$

Clearly,  $(mP_n)_{\{d\}}$  contains at least one component isomorphic to  $P_3$  if and only if

$$d = \begin{cases} \frac{n}{3} & \text{if } n \equiv 0 \pmod{d}, \\ \frac{n-t}{3} \text{ or } \frac{n-t}{2} & \text{if } n \equiv t \pmod{d}, 0 < t < d, \end{cases}$$

and  $(mP_n)_{\{d\}}$  has at least two components isomorphic to  $P_1$  if and only if  $n \equiv t \pmod{d}$ ,  $0 < t < d$ ,  $d = n - t$  and  $m(n - 2t) \geq 2$ . Therefore, combining both conditions, we conclude that  $s_{\{d\}}(mP_n) = \infty$  if and only if

$$d = \begin{cases} \frac{n}{3} & \text{if } n \equiv 0 \pmod{d}, \\ \frac{n-t}{3} \text{ or } \frac{n-t}{2} & \text{if } n \equiv t \pmod{d}, 0 < t < d, \\ n - t & \text{if } n \equiv t \pmod{d}, 0 < t < d, \text{ and } m(n - 2t) \geq 2. \end{cases}$$

□

**Question 4.3.** Find  $\{d\}$ -irregularity strength of  $mP_n$  for other values of  $d \geq 2$  not covered in Theorem 4.2.

Next, we address the  $\{d\}$ -irregularity strength for 2-regular graphs, i.e., the disjoint union of cycles. To achieve this, we first employ the irregularity strength and edge irregularity strength of 2-regular graphs to obtain  $\{1\}$ -irregularity strength of 2-regular graphs. Subsequently, we use Theorem 2.9 to derive the  $\{d\}$ -irregularity strength of 2-regular graphs. Recall that the *edge irregularity strength* of a graph  $G$ , denoted by  $es(G)$ , is the minimum  $k$  such that an assignment of integers  $1, 2, \dots, k$  to the vertices of  $G$  implies that distinct edges have distinct weights, where the *edge weight* is the sum of labels of its two end vertices [1].

We begin by stating the theorem of Faudree et al. [10], which determines the irregularity strength of 2-regular graphs.

**Theorem 4.4** ([10]). *Let  $G$  be a 2-regular graph on  $n$  vertices. Then*

$$\lfloor \frac{n}{2} \rfloor \leq s(G) \leq \lfloor \frac{n}{2} \rfloor + 2.$$

Moreover,

(a) If  $G$  is the disjoint union of triangles  $\frac{n}{3}C_3$  then

$$s\left(\frac{n}{3}C_3\right) = \begin{cases} \lceil \frac{n}{2} \rceil + 1 & \text{if } n \equiv 3 \pmod{4}, \\ \lceil \frac{n}{2} \rceil + 2 & \text{otherwise.} \end{cases}$$

(b) If  $G$  has no triangle then

$$s(G) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 1 \pmod{4}, \\ \lceil \frac{n}{2} \rceil + 1 & \text{otherwise.} \end{cases}$$

In the next theorem, we prove that the irregularity strength and the edge irregularity strength of 2-regular graphs are equivalent.

**Theorem 4.5.** *Let  $G$  be a 2-regular graph on  $n$  vertices. Then  $es(G) = s(G)$ . Consequently,*

$$\lfloor \frac{n}{2} \rfloor \leq es(G) \leq \lfloor \frac{n}{2} \rfloor + 2.$$

Moreover,

(a) If  $G$  is the disjoint union of triangles  $\frac{n}{3}C_3$  then

$$es\left(\frac{n}{3}C_3\right) = \begin{cases} \lceil \frac{n}{2} \rceil + 1 & \text{if } n \equiv 3 \pmod{4}, \\ \lceil \frac{n}{2} \rceil + 2 & \text{otherwise.} \end{cases}$$

(b) If  $G$  has no triangle then

$$es(G) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 1 \pmod{4}, \\ \lceil \frac{n}{2} \rceil + 1 & \text{otherwise.} \end{cases}$$

**Proof.** Let  $G$  be a 2-regular graph on  $n$  vertices. In any edge irregular labeling of  $G$ , if we shift the label of each vertex to the edge incident to it in the clockwise direction, we obtain an irregular labeling of  $G$ . This demonstrates that  $es(G) \geq s(G)$ . Conversely, in any irregular labeling of  $G$ , shifting the label of each edge to the vertex incident to it in the clockwise direction results in an edge irregular labeling of  $G$ , implying that  $es(G) \leq s(G)$ . Consequently,  $es(G) = s(G)$ . The rest of the theorem follows immediately from Theorem 4.4.  $\square$

Let  $G \cong mC_3 \cup (G - mC_3)$  be a 2-regular graph on  $n$  vertices containing  $m$  triangles and no square. Evidently,  $mC_3 \cup (G - mC_3)_{\{2\}}$  is a 2-regular graph with  $n$  vertices. Moreover, it is not hard to know that every  $\{1\}$ -irregular labeling of  $G$  induces an edge irregular labeling of  $mC_3 \cup (G - mC_3)_{\{2\}}$ , and every edge irregular labeling of  $mC_3 \cup (G - mC_3)_{\{2\}}$  induces a  $\{1\}$ -irregular labeling of  $G$ , meaning that  $\{1\}$ -irregularity strength of  $G$  and edge irregularity strength of  $mC_3 \cup (G - mC_3)_{\{2\}}$  are equivalent.

Further, if  $G \cong mC_3 \cup \frac{n-3m}{6}C_6$  then  $mC_3 \cup (\frac{n-3m}{6}C_6)_{\{2\}}$  is the disjoint union of triangles  $\frac{n}{3}C_3$ , and if  $m = 0$  and  $G$  has no hexagon then  $G_{\{2\}}$  is a 2-regular graph without any triangle.

With the above facts in hand along with Theorem 4.5 we obtain the following.

**Theorem 4.6.** *Let  $G \cong mC_3 \cup (G - mC_3)$  be a 2-regular graph on  $n$  vertices containing  $m$  triangles and no square for  $n \geq 3$  and  $0 \leq m \leq \frac{n}{3}$ . Then  $s_{\{1\}}(G) = \text{es}(mC_3 \cup (G - mC_3)_{\{2\}})$ . Consequently,*

$$\lfloor \frac{n}{2} \rfloor \leq s_{\{1\}}(G) \leq \lfloor \frac{n}{2} \rfloor + 2.$$

Moreover,

(a) *If  $G = mC_3 \cup \frac{n-3m}{6}C_6$  then*

$$s_{\{1\}}(G) = \begin{cases} \lfloor \frac{n}{2} \rfloor + 1 & \text{if } n \equiv 3 \pmod{4}, \\ \lfloor \frac{n}{2} \rfloor + 2 & \text{otherwise.} \end{cases}$$

(b) *If  $m = 0$  and  $G$  has no hexagon then*

$$s_{\{1\}}(G) = \begin{cases} \lfloor \frac{n}{2} \rfloor & \text{if } n \equiv 1 \pmod{4}, \\ \lfloor \frac{n}{2} \rfloor + 1 & \text{otherwise.} \end{cases}$$

The next theorem provides  $\{d\}$ -irregularity strength of the isomorphic copies of cycles  $mC_n$  for  $1 \leq d \leq \lfloor \frac{n}{2} \rfloor$ .

**Theorem 4.7.** *Let  $mC_n$  be an  $m$  disjoint union of a cycle on  $n$  vertices for  $m \geq 1$  and  $n \geq 3$ . Then*

(a)  $s_{\{\frac{n}{4}\}}(mC_n) = \infty$ .

(b)  $s_{\{\frac{n}{2}\}}(mC_n) = mn$ .

(c) *For  $d \in \{\frac{n}{6}, \frac{n}{3}\}$ ,*

$$s_{\{d\}}(mC_n) = \begin{cases} \lfloor \frac{mn}{2} \rfloor + 1 & \text{if } mn \equiv 3 \pmod{4}, \\ \lfloor \frac{mn}{2} \rfloor + 2 & \text{otherwise.} \end{cases}$$

(d) *For  $d \in (\{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\} - \{\frac{n}{6}, \frac{n}{4}, \frac{n}{3}, \frac{n}{2}\})$ ,*

$$s_{\{d\}}(mC_n) = \begin{cases} \lfloor \frac{mn}{2} \rfloor & \text{if } mn \equiv 1 \pmod{4}, \\ \lfloor \frac{mn}{2} \rfloor + 1 & \text{otherwise.} \end{cases}$$

**Proof.** By simple observation,  $(mC_n)_{\{\frac{n}{4}\}} = \frac{mn}{4}C_4$  and  $(mC_n)_{\{\frac{n}{2}\}} = \frac{mn}{2}K_2$ . Then by Theorem 2.9, we find that  $s_{\{\frac{n}{4}\}}(mC_n) = s_{\{1\}}(\frac{mn}{4}C_4) = \infty$  and  $s_{\{\frac{n}{2}\}}(mC_n) = s_{\{1\}}(\frac{mn}{2}K_2) = mn$ . This proves (1) and (2).

Next, let  $d \in \{\frac{n}{6}, \frac{n}{3}\}$ . If  $d = \frac{n}{6}$  then  $(mC_n)_{\{\frac{n}{6}\}} = \frac{mn}{6}C_6$ . By Theorems 2.9 and 4.6, we obtain

$$s_{\{\frac{n}{6}\}}(mC_n) = s_{\{1\}}(\frac{mn}{6}C_6) = \begin{cases} \lfloor \frac{mn}{2} \rfloor + 1 & \text{if } mn \equiv 3 \pmod{4}, \\ \lfloor \frac{mn}{2} \rfloor + 2 & \text{otherwise.} \end{cases}$$

If  $d = \frac{n}{3}$  then  $(mC_n)_{\{\frac{n}{3}\}} = \frac{mn}{3}C_3$ . Again, by Theorems 2.9 and 4.6, we have

$$s_{\{\frac{n}{3}\}}(mC_n) = s_{\{1\}}(\frac{mn}{3}C_3) = \begin{cases} \lfloor \frac{mn}{2} \rfloor + 1 & \text{if } mn \equiv 3 \pmod{4}, \\ \lfloor \frac{mn}{2} \rfloor + 2 & \text{otherwise.} \end{cases}$$

This proves (3).

Now, for  $d \in (\{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\} - \{\frac{n}{6}, \frac{n}{4}, \frac{n}{3}, \frac{n}{2}\})$ , we observe that  $(mC_n)_{\{d\}}$  is a 2-regular graph without any cycle  $C_t$  for  $t = 3, 4, 6$ . By Theorems 2.9 and 4.6, we conclude that

$$s_{\{d\}}(mC_n) = s_{\{1\}}((mC_n)_{\{d\}}) = \begin{cases} \lceil \frac{mn}{2} \rceil & \text{if } mn \equiv 1 \pmod{4}, \\ \lceil \frac{mn}{2} \rceil + 1 & \text{otherwise.} \end{cases}$$

Hence (4) is proved.  $\square$

Under certain conditions, Theorem 4.7 can be extended to 2-regular graphs that contain non-isomorphic cycle components.

**Theorem 4.8.** *Let  $G = \bigcup_{i=1}^m C_{n_i}$  be a 2-regular graph on  $n = n_1 + n_2 + \dots + n_m$  vertices with  $m \geq 2$  components such that  $3 \leq n_1 \leq n_2 \leq \dots \leq n_m$ , and  $G \not\cong mC_p$  for some  $p \geq 3$ . Then*

(a) *For  $d \in (\{\frac{n_i}{4} \leq \lfloor \frac{n_1}{2} \rfloor : i = 1, 2, \dots, m\} \cup \{\lfloor \frac{n_1}{2} \rfloor + 1, \lfloor \frac{n_1}{2} \rfloor + 2, \dots, \lfloor \frac{n_m}{2} \rfloor\})$ ,  $s_{\{d\}}(G) = \infty$ .*

(b) *For  $d \in (\{\frac{n_i}{6}, \frac{n_i}{3} \leq \lfloor \frac{n_1}{2} \rfloor : i = 1, 2, \dots, m\} - \{\frac{n_j}{4}, \frac{n_j}{2} : j = 1, 2, \dots, m\})$ ,*

$$\lfloor \frac{n}{2} \rfloor \leq s_{\{d\}}(G) \leq \lfloor \frac{n}{2} \rfloor + 2.$$

*In particular, if  $d \in \{\frac{n_i}{6}, \frac{n_i}{3}\}$  for some  $i$ , and  $\frac{n_j}{\gcd(n_j, d)} \in \{3, 6\}$  for every  $j = 1, 2, \dots, m$ ,  $j \neq i$ , then*

$$s_{\{d\}}(G) = \begin{cases} \lceil \frac{n}{2} \rceil + 1 & \text{if } n \equiv 3 \pmod{4}, \\ \lceil \frac{n}{2} \rceil + 2 & \text{otherwise.} \end{cases}$$

(c) *For  $d \in (\{1, 2, \dots, \lfloor \frac{n_1}{2} \rfloor\} - \{\frac{n_j}{6}, \frac{n_j}{4}, \frac{n_j}{3}, \frac{n_j}{2} : j = 1, 2, \dots, m\})$ ,*

$$s_{\{d\}}(G) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 1 \pmod{4}, \\ \lceil \frac{n}{2} \rceil + 1 & \text{otherwise.} \end{cases}$$

**Proof.** (1) If  $d \in \{\frac{n_i}{4} \leq \lfloor \frac{n_1}{2} \rfloor : i = 1, 2, \dots, m\}$  then  $G_{\{d\}}$  contains a square, and so  $s_{\{d\}}(G) = s_{\{1\}}(G_{\{d\}}) = \infty$ . If  $d \in \{\lfloor \frac{n_1}{2} \rfloor + 1, \lfloor \frac{n_1}{2} \rfloor + 2, \dots, \lfloor \frac{n_m}{2} \rfloor\}$  then  $N_{\{d\}}(x) = N_{\{d\}}(y) = \emptyset$  for any two distinct vertices  $x$  and  $y$  in the component  $C_{n_1}$ , thus  $s_{\{d\}}(G) = \infty$ .

(2) If  $d \in (\{\frac{n_i}{6}, \frac{n_i}{3} \leq \lfloor \frac{n_1}{2} \rfloor : i = 1, 2, \dots, m\} - \{\frac{n_j}{4}, \frac{n_j}{2} : j = 1, 2, \dots, m\})$  then  $G_{\{d\}} \cong tC_3 \cup (G_{\{d\}} - tC_3)$  is a 2-regular graph of order  $n$  containing  $t$  triangles and no square for some  $t \geq 0$ . Therefore, by Theorems 2.9 and 4.6,

$$\lfloor \frac{n}{2} \rfloor \leq s_{\{d\}}(G) = s_{\{1\}}(G_{\{d\}}) \leq \lfloor \frac{n}{2} \rfloor + 2.$$

For some  $i$ , let  $d \in \{\frac{n_i}{6}, \frac{n_i}{3}\}$ , and for every  $j = 1, 2, \dots, m$ ,  $j \neq i$ , let  $\frac{n_j}{\gcd(n_j, d)} \in \{3, 6\}$ . Then  $G_{\{d\}} \cong m_1C_3 \cup m_2C_6$ , where  $3m_1 + 6m_2 = n$ . Thus, by Theorems 2.9 and 4.6,

$$s_{\{d\}}(G) = s_{\{1\}}(G_{\{d\}}) = \begin{cases} \lceil \frac{n}{2} \rceil + 1 & \text{if } n \equiv 3 \pmod{4}, \\ \lceil \frac{n}{2} \rceil + 2 & \text{otherwise.} \end{cases}$$

(3) If  $d \in (\{1, 2, \dots, \lfloor \frac{n_1}{2} \rfloor\} - \{\frac{n_j}{6}, \frac{n_j}{4}, \frac{n_j}{3}, \frac{n_j}{2} : j = 1, 2, \dots, m\})$  then  $G_{\{d\}}$  is a 2-regular graphs with  $n$  vertices containing no cycle  $C_t$ ,  $t = 3, 4, 6$ . By Theorems 2.9 and 4.6,

$$s_{\{d\}}(G) = s_{\{1\}}(G_{\{d\}}) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 1 \pmod{4}, \\ \lceil \frac{n}{2} \rceil + 1 & \text{otherwise.} \end{cases}$$

$\square$

**Question 4.9.** Find the  $\{d\}$ -irregularity strength of arbitrary 2-regular graphs for other values of  $d$  not mentioned in Theorems 4.7 and 4.8.

## 5. Final remarks

In this work, we introduced a  $D$ -irregular labeling as a generalization of non-inclusive and inclusive  $d$ -distance irregular labeling of graphs. We derived fundamental properties on  $D$ -irregularity strength for arbitrary graphs, and presented exact values on this parameter for specific families of graphs with small diameter or small maximum degree.

The authors of [16] posed a conjecture below.

**Conjecture 5.1** ([16]). *A graph  $G$  is  $D$ -antimagic if and only if  $N_D(x) \neq N_D(y)$  for every two distinct vertices  $x, y \in V(G)$ .*

Note that if Conjecture 5.1 holds, then for a graph  $G$  with finite  $D$ -irregularity strength, the inequality  $s_D(G) \leq |V(G)|$  holds as well. Motivated by this observation and the results presented in Proposition 2.5, we believe that the following conjecture is true.

**Conjecture 5.2.** *Let  $G$  be a graph on  $n$  vertices with  $s_D(G) < \infty$ . Then  $s_D(G) \leq n$  with the equality if and only if one of the following conditions holds:*

- (a)  $G$  is connected and  $D \in \{\{0\}, \{1, 2, \dots, \text{diam}(G)\}\}$ ,
- (b)  $G \cong mK_2$ ,  $m \geq 2$ , and  $D = \{\{0\}, \{1\}\}$ , or
- (c)  $G$  is disconnected,  $G \not\cong mK_2$ ,  $m \geq 2$ , and  $D = \{0\}$ .

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