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# $D$-irregularity Strength of a Graph

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ABSTRACT

We initiate to study a $D$-irregular labeling, which generalizes both non-inclusive and inclusive $d$-distance irregular labeling of graphs. Let $G = (V(G), E(G))$ be a graph, $D$ a set of distances, and $k$ a positive integer. A mapping $\varphi$ from $V(G)$ to the set of positive integers $\{1, 2, \ldots, k\}$ is called a $D$-irregular $k$-labeling of $G$ if every two distinct vertices have distinct weights, where the weight of a vertex $x$ is defined as the sum of labels of vertices whose distance from $x$ belongs to $D$. The least integer $k$ for which $G$ admits a $D$-irregular labeling is the $D$-irregularity strength of $G$ and denoted by $\mathrm{s}_D(G)$. In this paper, we establish several fundamental properties on $D$-irregularity strength for arbitrary graphs. We also determine this parameter exactly for families of graphs with small diameter or small maximum degree.

*Keywords:* Graph labeling, $D$-irregular $k$-labeling, $D$-irregularity strength, Distance

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## 1. Introduction

Throughout the paper, only undirected simple finite graphs are considered. For undefined terminology and notation, we follow [8]. Let $G$ be a graph. We will denote by $V(G)$ and $E(G)$ the *vertex* and *edge set* of $G$, respectively. The *distance* between two vertices $x, y \in V(G)$ will be denoted by $d(x, y)$.

Graph labeling has been an intriguing topic, extensively studied over the last six decades. To date, over 350 graph labelings techniques have been developed in over 3600 papers [11], one of which is a distance antimagic labeling, firstly established by Kamatchi and Arumugam [14]. A bijection $\varphi : V(G) \to \{1, 2, \ldots, |V(G)|\}$ is said to be a *distance antimagic labeling* of a graph $G$ if the *vertex weights*, given by

$$wt(x) = \sum_{z \in N(x)} \varphi(z),$$

are all distinct, where $N(x)$, the *open neighborhood* of $x$, is the set of all vertices adjacent to $x$. A *distance antimagic graph* is a graph that has a distance antimagic labeling.

Simanjuntak et al. [16] introduced a more generalized concept, called a *D-antimagic labeling*, where $D$ is a non-empty subset of the set $\{0, 1, \ldots, \mathrm{diam}(G)\}$. Here, the difference is that the *weight* of a vertex $x$ is now defined as

$$wt(x) = \sum_{z \in N_D(x)} \varphi(z),$$

where $N_D(x) = \{z : d(x, z) \in D\}$ is the *D-neighborhood* of $x$. A graph that admits a $D$-antimagic labeling is called *D-antimagic*. Notably, a $\{1\}$-antimagic graph is corresponds to a distance antimagic graph. Some graph families have been proved to be $D$-antimagic, among others, see [9, 12, 13, 14, 16, 17, 24].

Instead of looking at bijection type labelings, Chartrand et al. [7] considered an *edge k-labeling* $\varphi : E(G) \to \{1, 2, \ldots, k\}$ of a graph $G$, where distinct edges may receive an identical label, such that the *vertex weights*,

$$wt(x) = \sum_{xy \in E(G)} \varphi(xy),$$

are all distinct. They named such labelings *irregular assignments* and the *irregularity strength*, $\mathrm{s}(G)$, is the minimum positive integer $k$ such that $G$ has an irregular assignment using labels not exceeding $k$. This topic has gained significant interest, with numerous papers published; see [2, 3, 15] for some recent results.

Combining distance-based labelings and irregular assignments, Slamin [18] introduced a *non-inclusive distance irregular k-labeling* of a graph $G$ as a mapping $\varphi : V(G) \to \{1, 2, \ldots, k\}$ such that for every two distinct vertices $x, y \in V(G)$ there is $wt(x) \neq wt(y)$, where

$$wt(x) = \sum_{z \in N(x)} \varphi(z),$$

is the *weight* of a vertex $x$. The *non-inclusive distance irregularity strength*, $\mathrm{dis}(G)$, of $G$ is the smallest integer $k$ such that $G$ has a non-inclusive distance irregular labeling. There are not many graphs whose non-inclusive distance irregularity strengths are known. Among those that are known include paths, complete graphs, cycles, wheels, fan and book graphs [5, 18], disjoint union of paths, suns, helms and friendships [20].

Later on, Bača et al. [4] modified such a labeling and introduced an *inclusive distance irregular labeling* of graphs. In this case, the label of the vertex is also included in the computation of its vertex weight, while the non-inclusive is not. Furthermore, the two labelings were generalized by Bong et al. [6] to *non-inclusive* and *inclusive d-distance irregular labelings*, where $d$ is an integer arbitrarily taken from 1 up to the diameter of the graph. A (non-)inclusive 1-distance irregular labeling is also called a (non-)inclusive distance irregular labeling. For more information on these subjects one can see [19, 21, 22, 23].

In this paper, we propose a new invariant, called a $D$-irregular labeling, which generalizes the previous labelings mentioned in [4, 6, 18]. This generalization is analogous to $D$-antimagic labelings by Simanjuntak et al. [16]. The formal definition is provided below.

**Definition 1.1.** Let $G = (V(G), E(G))$ be a graph with diameter $\mathrm{diam}(G)$. For a non-empty set of distances $D \subseteq \{0, 1, \ldots, \mathrm{diam}(G)\}$ and a positive integer $k$, a function $\varphi : V(G) \to \{1, 2, \ldots, k\}$ is called a $D$-irregular $k$-labeling of $G$ if $wt(x) \neq wt(y)$ for every two distinct vertices $x, y \in V(G)$, where the weight of a vertex $x$ is defined as

$$wt(x) = \sum_{z \in N_D(x)} \varphi(z).$$

We set $wt(x) = 0$ if $N_D(x) = \emptyset$. The $D$-irregularity strength of $G$, denoted by $\mathrm{s}_D(G)$, is the minimum $k$ such that $G$ admits a $D$-irregular labeling. If such an integer $k$ does not exist then we define $\mathrm{s}_D(G) = \infty$.

In a case when $G$ is disconnected, we have that $\mathrm{diam}(G) = \infty$, and $D \subseteq \{0, 1, \ldots\}$. Suppose that $G_1, G_2, \ldots, G_m$, $m \geqslant 2$, be the components of $G$, $\max_{1 \leqslant i \leqslant m} \mathrm{diam}(G_i) = \partial$, and $D' \subseteq D \cap \{\partial + 1, \partial + 2, \ldots\}$. Then $\mathrm{s}_D(G) = \mathrm{s}_{D-D'}(G)$. Moreover, from Definition 1.1, it follows that

(a) A $\{1\}$-irregular labeling is a non-inclusive distance irregular labeling.

(b) A $\{0, 1\}$-irregular labeling is an inclusive distance irregular labeling.

(c) For $d \in \{1, 2, \ldots, \mathrm{diam}(G)\}$, a $\{1, 2, \ldots, d\}$-irregular labeling is a non-inclusive $d$-distance irregular labeling.

(d) For $d \in \{1, 2, \ldots, \mathrm{diam}(G)\}$, a $\{0, 1, \ldots, d\}$-irregular labeling is an inclusive $d$-distance irregular labeling.

The rest of the paper is organized as follows. In Section 2, we present fundamental properties of $D$-irregularity strength for arbitrary graphs. In Section 3, we compute the exact values on this parameter for certain graphs with small diameter. Section 4 addresses graphs with small maximum degree. Finally, Section 5 presents concluding remarks.

## 2. Basic Properties

We start with the following theorem which provides the necessary and sufficient conditions for a graph to possess finite $D$-irregularity strength.

**Theorem 2.1.** *Let $G$ be a graph. Then $\mathrm{s}_D(G) = \infty$ if and only if there exist two distinct vertices $x, y \in V(G)$ such that $N_D(x) = N_D(y)$. Moreover, if $\mathrm{s}_D(G) < \infty$ then $\mathrm{s}_D(G) \leqslant 2^{|V(G)|-1}$.*

**Proof.** Let $x, y$ be two vertices of $G$ with $N_D(x) = N_D(y)$. Then, in any vertex labeling $\varphi$ of $G$,

$$wt(x) = \sum_{z \in N_D(x)} \varphi(z) = \sum_{z \in N_D(y)} \varphi(z) = wt(y),$$

meaning that $\varphi$ can not be $D$-irregular.

Now suppose that $N_D(x) \neq N_D(y)$ for every two distinct vertices $x, y \in V(G)$. Let us denote the vertices of $G$ arbitrarily by the symbols $x_1, x_2, \ldots, x_{|V(G)|}$. Define a vertex $(2^{|V(G)|-1})$-labeling $\varphi$ of $G$

as follows:

$$\varphi(x_i) = 2^{i-1} \quad \text{for } i = 1, 2, \ldots, |V(G)|.$$

We define a labeling $\beta$ such that

$$\beta_{ij} = \begin{cases} 1 & \text{if } x_i \in N_D(x_j), \\ 0 & \text{if } x_i \notin N_D(x_j), \end{cases}$$

where $i = 1, 2, \ldots, |V(G)|$ and $j = 1, 2, \ldots, |V(G)|$. Then,

$$wt(x_j) = \sum_{x_i \in N_D(x_j)} \varphi(x_i) = \sum_{i:\, x_i \in N_D(x_j)} 2^{i-1} = \sum_{i=1}^{|V(G)|} \beta_{ij} 2^{i-1}. \tag{1}$$

To prove that all the vertex weights are distinct, it suffices to show that the sums $\sum_{i=1}^{|V(G)|} \beta_{ij} 2^{i-1}$ are all distinct for $j = 1, 2, \ldots, |V(G)|$. However, this is evident if we consider that the ordered $|V(G)|$-tuple $(\beta_{1j}, \beta_{2j}, \ldots, \beta_{|V(G)|j})$ corresponds to the binary code representation of the sum (1). Because $N_D(x) \neq N_D(y)$ for every two distinct vertices $x, y \in V(G)$ we can get that the $|V(G)|$-tuples are distinct for distinct vertices. Hence $s_D(G) \leqslant 2^{|V(G)|-1}$. $\qquad \square$

In a non-trivial graph $G$, it is evident that if $D = \{0, 1, \ldots, \text{diam}(G)\}$ then $N_D(x) = V(G) = N_D(y)$ for any two vertices $x, y \in V(G)$, and so by Theorem 2.1 $G$ has no $D$-irregular labeling. Hence, the next corollary holds.

**Corollary 2.2.** *Let $G$ be a non-trivial graph. Then $s_{\{0,1,\ldots,\text{diam}(G)\}}(G) = \infty$.*

For a graph $G$ and a set of distances $D$, the *complement* of $D$, denoted by $D^c$, is a set in which $D \cup D^c = \{0, 1, \ldots, \text{diam}(G)\}$ and $D \cap D^c = \emptyset$. In the next theorem, we give a relationship between $D$-irregularity strength and $D^c$-irregularity strength of graphs.

**Theorem 2.3.** *Let $G$ be a non-empty graph. Then*
*(a) $s_D(G) = \infty$ if and only if $s_{D^c}(G) = \infty$.*

*(b) If $G$ is connected and $s_D(G) < \infty$ then $s_{D^c}(G) = s_D(G)$.*

*(c) Let $G$ be a disconnected graph with components $G_1, G_2, \ldots, G_m$ for $m \geqslant 2$, and $s_D(G) < \infty$. If there exists a $D$-irregular $s_D(G)$-labeling $\varphi$ of $G$ with the property that $wt(x) - wt(y) \neq \sum_{z \in V(G_i)} \varphi(z) - \sum_{z \in V(G_j)} \varphi(z)$ for every two distinct vertices $x \in V(G_i)$ and $y \in V(G_j)$, $i \neq j$, then $s_{D^c}(G) \leqslant s_D(G)$.*

**Proof. (a)** If $s_D(G) = \infty$ then by Theorem 2.1 there exist two distinct vertices $x, y \in V(G)$ such that $N_D(x) = N_D(y)$. If $G$ is connected then

$$N_{D^c}(x) = N_{\{0,1,\ldots,\text{diam}(G)\}-D}(x)$$
$$= V(G) - N_D(x) = V(G) - N_D(y) = N_{\{0,1,\ldots,\text{diam}(G)\}-D}(y) = N_{D^c}(y),$$

and so $s_{D^c}(G) = \infty$ by Theorem 2.1.

Next, let $G$ be disconnected with components $G_1, G_2, \ldots, G_m$ for some $m \geqslant 2$. If $x, y \in V(G_i)$ for some $i$, then

$$
\begin{aligned}
N_{D^c}(x) = N_{\{0,1,\ldots\}-D}(x) &= N_{(\{0,1,\ldots,\mathrm{diam}(G_i)\}-D)\cup\{\mathrm{diam}(G_i)+1,\mathrm{diam}(G_i)+2,\ldots\}}(x) \\
&= N_{\{0,1,\ldots,\mathrm{diam}(G_i)\}-D}(x) = V(G_i) - N_D(x) = V(G_i) - N_D(y) \\
&= N_{\{0,1,\ldots,\mathrm{diam}(G_i)\}-D}(y) = N_{(\{0,1,\ldots,\mathrm{diam}(G_i)\}-D)\cup\{\mathrm{diam}(G_i)+1,\mathrm{diam}(G_i)+2,\ldots\}}(y) \\
&= N_{\{0,1,\ldots\}-D}(y) = N_{D^c}(y),
\end{aligned}
$$

thus $\mathrm{s}_{D^c}(G) = \infty$ by Theorem 2.1. If $x \in V(G_i)$ and $y \in V(G_j)$ with $\mathrm{diam}(G_i) \geqslant \mathrm{diam}(G_j)$ for $i \neq j$, then $N_D(x) = N_D(y) = \emptyset$, and so $D \subseteq \{\mathrm{diam}(G_i) + 1, \mathrm{diam}(G_i) + 2, \ldots\}$. As $G$ is a non-empty graph, we may assume that $G_i$ contains at least two vertices. Indeed, for every two distinct vertices $x, z \in V(G_i)$, $N_{D^c}(x) = N_{D^c}(z) = N_{\{0,1,\ldots\}-D}(z) = N_{(\{0,1,\ldots,\mathrm{diam}(G_i)\}-D)\cup\{\mathrm{diam}(G_i)+1,\mathrm{diam}(G_i)+2,\ldots\}}(z) = N_{\{0,1,\ldots,\mathrm{diam}(G_i)\}}(z) = V(G_i)$, hence $\mathrm{s}_{D^c}(G) = \infty$ by Theorem 2.1.

Conversely, if $\mathrm{s}_{D^c}(G) = \infty$, similar reasoning shows that $\mathrm{s}_D(G) = \infty$.

**(b)** Let $G$ be connected and $\mathrm{s}_D(G) < \infty$. Let $\varphi$ be a $D$-irregular $\mathrm{s}_D(G)$-labeling of $G$. We define a new vertex labeling $\varphi'$ of $G$ in such a way that

$$
\varphi'(x) = \varphi(x) \quad \text{for } x \in V(G).
$$

We show that $\varphi'$ is a $D^c$-irregular $\mathrm{s}_D(G)$-labeling of $G$.

Evidently $\varphi'(x) \leqslant \mathrm{s}_D(G)$ for all $x \in V(G)$. For the vertex weights under the labeling $\varphi'$ we have

$$
wt(x) = \sum_{z \in N_{D^c}(x)} \varphi'(z) = \sum_{z \in N_{\{0,1,\ldots,\mathrm{diam}(G)\}-D}(x)} \varphi'(z) = \sum_{z \in V(G_i)} \varphi(z) - \sum_{z \in N_D(x)} \varphi(z).
$$

Note that the sums $\sum_{z \in N_D(x)} \varphi(z)$ are distinct for each $x \in V(G)$ since $\varphi$ is $D$-irregular. This implies that $wt(x) \neq wt(y)$ for every two distinct vertices $x, y \in V(G)$, and so $\mathrm{s}_{D^c}(G) \leqslant \mathrm{s}_D(G)$.

Since $\mathrm{s}_D(G) < \infty$ we have $\mathrm{s}_{D^c}(G) < \infty$ according to (1). Then, by similar arguments, we can prove that $\mathrm{s}_D(G) \leqslant \mathrm{s}_{D^c}(G)$. Therefore $\mathrm{s}_{D^c}(G) = \mathrm{s}_D(G)$.

**(c)** Let $G$ be a disconnected graph with components $G_1, G_2, \ldots, G_m$ for $m \geqslant 2$, and $\mathrm{s}_D(G) < \infty$. Assume $\varphi$ be a $D$-irregular $\mathrm{s}_D(G)$-labeling of $G$ satisfying the stated condition. Define a new vertex labeling $\varphi'$ of $G$ such that

$$
\varphi'(x) = \varphi(x) \quad \text{for } x \in V(G).
$$

We shall show that $\varphi'$ is a $D^c$-irregular $\mathrm{s}_D(G)$-labeling of $G$.

Clearly $\varphi'(x) \leqslant \mathrm{s}_D(G)$ for all $x \in V(G)$. Let us consider the weight of any vertex $x$ in the component $G_i$, $i = 1, 2, \ldots, m$, under the labeling $\varphi'$. We have

$$
\begin{aligned}
wt(x) &= \sum_{z \in N_{D^c}(x)} \varphi'(z) = \sum_{z \in N_{\{0,1,\ldots\}-D}(x)} \varphi'(z) \\
&= \sum_{z \in N_{(\{0,1,\ldots,\mathrm{diam}(G_i)\}-D)\cup\{\mathrm{diam}(G_i)+1,\mathrm{diam}(G_i)+2,\ldots\}}(x)} \varphi'(z) \\
&= \sum_{z \in N_{\{0,1,\ldots,\mathrm{diam}(G_i)\}-D}(x)} \varphi'(z) \\
&= \sum_{z \in V(G_i)} \varphi(z) - \sum_{z \in N_D(x)} \varphi(z).
\end{aligned}
$$

Consider any two distinct vertices $x, y \in V(G)$. Evidently if $x$ and $y$ are in the same component then $wt(x) \neq wt(y)$ since $\sum_{z \in N_D(x)} \varphi(z) \neq \sum_{z \in N_D(y)} \varphi(z)$. Now, suppose that $x$ and $y$ are in the different components, say $x \in V(G_i)$ and $y \in V(G_j)$ for $i \neq j$. By the property in (3), it follows that $wt(x) \neq wt(y)$. So all the vertex weights are distinct, and hence $s_{D^c}(G) \leqslant s_D(G)$. □

It is evident that every $D$-antimagic labeling is also a $D$-irregular labeling. Consequently, we derive the following property, which provides a better upper bound for $s_D(G)$ in a case that $G$ is $D$-antimagic.

**Proposition 2.4.** *Let $G$ be a $D$-antimagic graph on $n$ vertices. Then $s_D(G) \leqslant n$.*

The next proposition, which is straightforward, shows that the upper bound in Proposition 2.4 is sharp.

**Proposition 2.5.** *Let $G$ be a graph on $n$ vertices. If one of the following conditions holds:*
(a) *$G$ is connected and $D \in \{\{0\}, \{1, 2, \ldots, \mathrm{diam}(G)\}\}$,*
(b) *$G \cong mK_2$, $m \geqslant 2$, and $D \in \{\{0\}, \{1\}\}$, or*
(c) *$G$ is disconnected, $G \ncong mK_2$, $m \geqslant 2$, and $D = \{0\}$,*
*then $G$ is $D$-antimagic and $s_D(G) = n$.*

In the next theorem, we provide lower bounds for $D$-irregularity strength of arbitrary graphs. For a vertex $x$ in a graph $G$, the *$D$-degree*, $d_D(x)$ of $x$ is the cardinality of the set $N_D(x)$.

**Theorem 2.6.** *Let $G$ be a graph with $s_D(G) < \infty$. Let $\delta_D(G)$ and $\Delta_D(G)$ be the minimum and the maximum $D$-degree of vertices of $G$, respectively. Let $n_{D,i}$ be the number of vertices in $G$ with $D$-degree $i$ for $i = \delta_D(G), \delta_D(G) + 1, \ldots, \Delta_D(G)$. Then,*

$$s_D(G) \geqslant \max_{\delta_D(G) \leqslant i \leqslant \Delta_D(G)} \left\{ \left\lceil \frac{\delta_D(G) + \sum_{j=\delta_D(G)}^{i} n_{D,j} - 1}{i} \right\rceil \right\}.$$

**Proof.** For some $t$, let

$$\left\lceil \frac{\delta_D(G) + \sum_{j=\delta_D(G)}^{t} n_{D,j} - 1}{t} \right\rceil = \max_{\delta_D(G) \leqslant i \leqslant \Delta_D(G)} \left\{ \left\lceil \frac{\delta_D(G) + \sum_{j=\delta_D(G)}^{i} n_{D,j} - 1}{i} \right\rceil \right\}.$$

In any $D$-irregular labeling of a graph $G$, the smallest weight of vertices with $D$-degrees $\delta_D(G), \delta_D(G)+1, \ldots, t$ is at least $\delta_D(G)$, and the largest among them must be at least $\delta_D(G) + \sum_{j=\delta_D(G)}^{t} n_{D,j} - 1$. Such largest weight is obtained from the sum of at most $t$ labels. Therefore

$$s_D(G) \geqslant \left\lceil \frac{\delta_D(G) + \sum_{j=\delta_D(G)}^{t} n_{D,j} - 1}{t} \right\rceil = \max_{\delta_D(G) \leqslant i \leqslant \Delta_D(G)} \left\{ \left\lceil \frac{\delta_D(G) + \sum_{j=\delta_D(G)}^{i} n_{D,j} - 1}{i} \right\rceil \right\}.$$

□

Notice that if $D = \{1\}$, $D = \{0, 1\}$, and $D = \{0, 1, \ldots, d\}$, $1 \leqslant d \leqslant \mathrm{diam}(G)$, then we acquire the lower bounds for $s_D(G)$ which were proved in [21, Theorem 2.1], [19, Theorem 2], and [23, Lemma 2.1], respectively.

The next two theorems demonstrate key properties that will be instrumental in determining the $D$-irregularity strength of a graph.

**Theorem 2.7.** *Let $G$ be a graph with $\mathrm{s}_D(G) < \infty$. Let $x, y$ be two distinct vertices of $G$ such that $N_D(x) - \{y\} = N_D(y) - \{x\}$. Then $x$ and $y$ must receive distinct labels in any $D$-irregular labeling of $G$. Furthermore, if $G$ is connected then $x$ and $y$ must receive distinct labels in any $D^c$-irregular labeling of $G$.*

**Proof.** Let $x, y$ be two distinct vertices of $G$ with $N_D(x) - \{y\} = N_D(y) - \{x\}$. As $\mathrm{s}_D(G) < \infty$, $x \in N_D(y)$ and $y \in N_D(x)$. Next, in any $D$-irregular labeling $\varphi$ of $G$,

$$wt(x) = \sum_{z \in N_D(x)} \varphi(z) = \varphi(y) + \sum_{z \in N_D(x) - \{y\}} \varphi(z),$$

and

$$wt(y) = \sum_{z \in N_D(y)} \varphi(z) = \varphi(x) + \sum_{z \in N_D(y) - \{x\}} \varphi(z).$$

Since $wt(x) \neq wt(y)$ and $\sum_{z \in N_D(x) - \{y\}} \varphi(z) = \sum_{z \in N_D(y) - \{x\}} \varphi(z)$ then $\varphi(x) \neq \varphi(y)$.

Now let $G$ be connected. Then, in any $D^c$-irregular labeling $\varphi$ of $G$,

$$wt(x) = \sum_{z \in N_{D^c}(x)} \varphi(z) = \sum_{z \in N_{\{0,1,\ldots,\mathrm{diam}(G)\} - D}(x)} \varphi(z) = \sum_{z \in V(G)} \varphi(z) - \sum_{z \in N_D(x)} \varphi(z)$$

$$= \sum_{z \in V(G)} \varphi(z) - \varphi(y) - \sum_{z \in N_D(x) - \{y\}} \varphi(z),$$

and

$$wt(y) = \sum_{z \in N_{D^c}(y)} \varphi(z) = \sum_{z \in N_{\{0,1,\ldots,\mathrm{diam}(G)\} - D}(y)} \varphi(z) = \sum_{z \in V(G)} \varphi(z) - \sum_{z \in N_D(y)} \varphi(z)$$

$$= \sum_{z \in V(G)} \varphi(z) - \varphi(x) - \sum_{z \in N_D(y) - \{x\}} \varphi(z).$$

Since $wt(x) \neq wt(y)$ and $\sum_{z \in N_D(x) - \{y\}} \varphi(z) = \sum_{z \in N_D(y) - \{x\}} \varphi(z)$ then $\varphi(x) \neq \varphi(y)$. $\square$

**Theorem 2.8.** *Let $G$ be a graph, $D = D' \cup D''$, $D' \cap D'' = \emptyset$, and $\mathrm{s}_D(G) < \infty$. Let $x, y$ be two distinct vertices of $G$ such that $x \notin N_D(y)$ and $y \notin N_D(x)$. If $N_{D'}(x) = N_{D'}(y)$, $N_{D''}(x) \neq N_{D''}(y)$ and $|N_{D''}(x)| = |N_{D''}(y)| = 1$ then vertices in $N_{D''}(x)$ and $N_{D''}(y)$ must receive distinct labels in any $D$-irregular labeling of $G$.*

**Proof.** Let $x, y$ be two distinct vertices of $G$ with $x \notin N_D(y)$ and $y \notin N_D(x)$ such that $N_{D'}(x) = N_{D'}(y)$, $N_{D''}(x) \neq N_{D''}(y)$ and $|N_{D''}(x)| = |N_{D''}(y)| = 1$. Let $N_{D''}(x) = \{x'\}$ and $N_{D''}(y) = \{y'\}$, where $x' \neq y'$. Then, in any $D$-irregular labeling $\varphi$ of $G$,

$$wt(x) = \sum_{z \in N_D(x)} \varphi(z) = \sum_{z \in N_{D' \cup D''}(x)} \varphi(z) = \sum_{z \in N_{D'}(x)} \varphi(z) + \sum_{z \in N_{D''}(x)} \varphi(z)$$

$$= \sum_{z \in N_{D'}(x)} \varphi(z) + \varphi(x'),$$

and

$$wt(y) = \sum_{z \in N_D(y)} \varphi(z) = \sum_{z \in N_{D' \cup D''}(y)} \varphi(z) = \sum_{z \in N_{D'}(y)} \varphi(z) + \sum_{z \in N_{D''}(y)} \varphi(z)$$

$$= \sum_{z \in N_{D'}(y)} \varphi(z) + \varphi(y').$$

Since $wt(x) \neq wt(y)$ and $\sum_{z \in N_{D'}(x)} \varphi(z) = \sum_{z \in N_{D'}(y)} \varphi(z)$ then $\varphi(x') \neq \varphi(y')$.      □

Notice that when $D = \{1\}$, Theorem 2.7 yields the result mentioned in [18, Observation 2], and also when $D = \{0, 1\}$, $D' = \{1\}$, and $D'' = \{0\}$, Theorem 2.8 provides the property proved in [4, Lemma 3.1].

In [16], Simanjuntak et al. defined a *distance-D graph* of a graph $G$, denoted by $G_D$, as a simple graph with the same vertices as $G$, where two vertices are adjacent if their distance in $G$ is an element of $D$. Note that, since $G_D$ is a simple graph, any loops that would arise when $0 \in D$ are excluded from the graph.

**Theorem 2.9.** *Let $G$ be a graph.*

  *(a) If $D$ does not contain $0$ then $\mathrm{s}_{\{1\}}(G_D) = \mathrm{s}_D(G)$.*

  *(b) If $D$ contains $0$ then $\mathrm{s}_{\{0,1\}}(G_D) = \mathrm{s}_D(G)$.*

**Proof.** Let $\varphi$ be a $D$-irregular $\mathrm{s}_D(G)$-labeling of a graph $G$. We define a vertex labeling $\varphi'$ of $G_D$ in the following way.

$$\varphi'(x') = \varphi(x),$$

for any vertex $x' \in V(G_D)$ corresponding to a vertex $x \in V(G)$.

If $0 \notin D$, then we have

$$\sum_{y \in N(x')} \varphi'(y) = \sum_{y \in N_D(x)} \varphi(y),$$

which implies that $\mathrm{s}_{\{1\}}(G_D) \leqslant \mathrm{s}_D(G)$. Using similar arguments, we can also show that $\mathrm{s}_D(G) \leqslant \mathrm{s}_{\{1\}}(G_D)$, so $\mathrm{s}_{\{1\}}(G_D) = \mathrm{s}_D(G)$. This proves (1).

Furthermore, if $0 \in D$ then

$$\varphi'(x) + \sum_{y \in N(x')} \varphi'(y) = \sum_{y \in N_D(x)} \varphi(y),$$

which leads to $\mathrm{s}_{\{0,1\}}(G_D) \leqslant \mathrm{s}_D(G)$. Again, using similar reasoning, we can demonstrate that $\mathrm{s}_D(G) \leqslant \mathrm{s}_{\{0,1\}}(G_D)$, thus $\mathrm{s}_{\{0,1\}}(G_D) = \mathrm{s}_D(G)$. This completes the proof of (2).      □

## 3.   Graphs with Small Diameter

This section showcases the $D$-irregularity strength for families of graphs with diameter at most three, including complete graphs, the join graph $G + K_1$, complete bipartite graphs, and double stars, for all possible sets $D$.

A *complete* graph $K_n$ is the graph with $n$ vertices in which any two of them are joined by an edge. Evidently, the graph $K_1$ is the only graph of diameter zero, and the graph $K_n$, $n \geqslant 2$, is the only graph with diameter one.

A *complete bipartite* graph $K_{m,n}$ is the graph whose vertices can be partitioned into two subsets $X$ and $Y$ with $|X| = m$ and $|Y| = n$, such that $xy$ is an edge if and only if $x \in X$ and $y \in Y$. The graph $K_{m,n}$, $1 \leqslant m \leqslant n$, $(m, n) \neq (1, 1)$, has diameter two. The complete bipartite graph $K_{1,n}$ is called a *star*, the only tree with diameter two.

The *join product* between two given graphs $G$ and $H$, denoted by $G + H$, is the graph obtained from $G$ and $H$ by joining an edge from every vertex of $G$ to every vertex of $H$. The graph $G + H$

has diameter one or two, with diameter one if and only if $G$ and $H$ are complete graphs. Therefore, the graph $G + K_1$ has diameter two if and only if $G$ is not a complete graph.

A *double star* $S_{m,n}$ is the graph obtained from two stars $K_{1,m}$ and $K_{1,n}$ by connecting the two vertices of degrees $m$ and $n$ in $K_{1,m}$ and $K_{1,n}$, respectively. The graph $S_{m,n}$, $1 \leqslant m \leqslant n$, is the only tree with diameter three.

We first examine complete graphs. It is evident that $s_D(K_1) = 1$. We shall show that $K_1$ is the only graph with $D$-irregularity strength one. Beforehand, let us prove in the next proposition a generalization of the well-known property that a simple graph on $n \geqslant 2$ vertices whose all the vertex degrees are distinct does not exist.

**Proposition 3.1.** *There is no simple graph on $n \geqslant 2$ vertices whose vertex $D$-degrees are all distinct.*

**Proof.** Assume that such a graph $G$ exists. Note that the $D$-degree of a vertex is at least 0 and at most $n$. Thus, $n$ numbers in the set $\{0, 1, \ldots, n\}$ have to be the $D$-degree of vertices of $G$.

We will show that $n$ cannot be a vertex $D$-degree. Assume otherwise, that $d_D(x) = n$ for some vertex $x \in V(G)$. This means that $N_D(x) = V(G)$, which forces $G$ to be connected and $D = \{0, 1, \ldots, \text{diam}(G)\}$. But then, this implies that $N_D(y) = V(G)$ for any vertex $y \neq x \in V(G)$, and so $N_D(x) = N_D(y)$, a contradiction. Therefore the vertex $D$-degrees of $G$ must be $0, 1, \ldots, n-1$.

Consider two distinct vertices $x, y \in V(G)$ with $d_D(x) = 0$ and $d_D(y) = n - 1$. Since $d_D(x) = 0$, we get $0 \notin D$. This implies that $N_D(y) = V(G) - \{y\}$, and so $x \in N_D(y)$. However, this is impossible since $d_D(x) = 0$. Hence, the statement holds. $\qquad\qquad\square$

From Proposition 3.1 it follows that for any graph $G$ of order $n \geqslant 2$, labeling all vertices of $G$ with 1 will always result in at least two vertices with the same weight. This brings us to the characterization of graphs with $D$-irregularity strength one.

**Theorem 3.2.** *Let $G$ be a graph. Then $s_D(G) = 1$ if and only if $G \cong K_1$.*

**Question 3.3.** *Characterize all graphs with $D$-irregularity strength two.*

For $n \geqslant 2$, the $D$-irregularity strength of the complete graph $K_n$ which can be directly derived from Proposition 2.5 and Corollary 2.2, is presented in the following theorem. Note that the cases when $D = \{1\}$ and $D = \{0, 1\}$ were also proved in [18] and [4], respectively.

**Theorem 3.4** ([4, 18]). *Let $K_n$ be a complete graph for $n \geqslant 2$. Then*

$$s_D(K_n) = \begin{cases} n & \text{if } D \in \{\{0\}, \{1\}\}, \\ \infty & \text{if } D = \{0, 1\}. \end{cases}$$

Next, we deal with the join graph $G + K_1$. To begin, let us restate the results on $s_D(G + K_1)$ for $D = \{1\}$ and $D = \{0, 1\}$.

**Theorem 3.5** ([22]). *Let $G$ be a graph. If $s_{\{1\}}(G) = \infty$ then $s_{\{1\}}(G + K_1) = \infty$. Moreover, if*

$s_{\{1\}}(G) < \infty$ *then*

$$s_{\{1\}}(G + K_1) = \begin{cases} s_{\{1\}}(G) & \textit{if there exists a } \{1\}\textit{-irregular } s_{\{1\}}(G)\textit{-labeling } \varphi \textit{ of } G \\ & \textit{such that } \sum_{x \in V(G)} \varphi(x) > \max\{wt(x) : x \in V(G)\} + 1, \\ s_{\{1\}}(G) + 1 & \textit{otherwise.} \end{cases}$$

**Theorem 3.6** ([4]). *Let $G$ be a graph with maximum degree $\Delta(G)$. Then*

$$s_{\{0,1\}}(G + K_1) = \begin{cases} s_{\{0,1\}}(G) & \textit{if } \Delta(G) < |V(G)| - 1, \\ \infty & \textit{if } \Delta(G) = |V(G)| - 1. \end{cases}$$

Corollary 2.2, Proposition 2.5, and Theorems 2.3, 3.5 and 3.6 allow us to completely obtained exact values on $D$-irregularity strength for the graph $G + K_1$.

**Theorem 3.7.** *Let $G$ be a not complete graph on $n \geqslant 2$ vertices with maximum degree $\Delta(G)$. Then*
  (a) *If $D \in \{\{0\}, \{1, 2\}\}$ then $s_D(G + K_1) = n + 1$.*
  (b) *If one of the following conditions is satisfied:*
     • *$D \in \{\{2\}, \{0, 1\}\}$ and $\Delta(G) < |V(G)| - 1$, or*
     • *$D \in \{\{1\}, \{0, 2\}\}$ and there exists a $\{1\}$-irregular $s_{\{1\}}(G)$-labeling $\varphi$ of $G$ such that $\sum_{x \in V(G)} \varphi(x) > \max\{wt(x) : x \in V(G)\} + 1$,*
     *then $s_D(G + K_1) = s_D(G)$.*
  (c) *If $D \in \{\{1\}, \{0, 2\}\}$ and $\sum_{x \in V(G)} \varphi(x) = \max\{wt(x) : x \in V(G)\} + 1$ for every $\{1\}$-irregular $s_{\{1\}}(G)$-labeling $\varphi$ of $G$, then $s_D(G + K_1) = s_D(G) + 1$.*
  (d) *If one of the following conditions is satisfied:*
     • *$D \in \{\{1\}, \{0, 2\}\}$ and $s_D(G) = \infty$,*
     • *$D \in \{\{2\}, \{0, 1\}\}$ and $\Delta(G) = |V(G)| - 1$, or*
     • *$D = \{0, 1, 2\}$,*
     *then $s_D(G + K_1) = \infty$.*

Now, consider the complete bipartite graph $K_{m,n}$ for $1 \leqslant m \leqslant n$, $(m, n) \neq (1, 1)$. By Proposition 2.5, it is clear that $s_{\{0\}}(K_{m,n}) = s_{\{1,2\}}(K_{m,n}) = m + n$. From [4, Theorem 3.1] and Theorem 2.3, we know that $s_{\{0,1\}}(K_{m,n}) = s_{\{2\}}(K_{m,n}) = n$ when $m < n$, and $s_{\{0,1\}}(K_{m,n}) = s_{\{2\}}(K_{m,n}) = n + 2$ when $m = n \geqslant 2$. Moreover, $s_D(K_{m,n}) = \infty$ whenever $D \in \{\{1\}, \{0, 2\}, \{0, 1, 2\}\}$, which follows immediately from [18, Observation 1], Theorem 2.3 and Corollary 2.2. Thus, we arrive at the following theorem.

**Theorem 3.8** ([4, 18]). *Let $K_{m,n}$ be a complete bipartite graph for $1 \leqslant m \leqslant n$, $(m, n) \neq (1, 1)$. Then*

$$s_D(K_{m,n}) = \begin{cases} m + n & \textit{if } D \in \{\{0\}, \{1, 2\}\}, \\ n & \textit{if } D \in \{\{2\}, \{0, 1\}\} \textit{ and } m < n, \\ n + 2 & \textit{if } D \in \{\{2\}, \{0, 1\}\} \textit{ and } m = n \geqslant 2, \\ \infty & \textit{if } D \in \{\{1\}, \{0, 2\}, \{0, 1, 2\}\}. \end{cases}$$

**Corollary 3.9.** *Let $K_{1,n}$ be a star for $n \geqslant 2$. Then*

$$\mathrm{s}_D(K_{1,n}) = \begin{cases} n+1 & \text{if } D \in \{\{0\}, \{1,2\}\}, \\ n & \text{if } D \in \{\{2\}, \{0,1\}\}, \\ \infty & \text{if } D \in \{\{1\}, \{0,2\}, \{0,1,2\}\}. \end{cases}$$

We now focus on the double star. Let us denote the vertex and edge set of the double star $S_{m,n}$ respectively by $V(S_{m,n}) = \{x, x_i : i = 1, 2, \ldots, m\} \cup \{y, y_j : j = 1, 2, \ldots, n\}$ and $E(S_{m,n}) = \{xy\} \cup \{xx_i : i = 1, 2, \ldots, m\} \cup \{yy_j : j = 1, 2, \ldots, n\}$.

**Theorem 3.10** ([6]). *Let $S_{m,n}$ be a double star for $1 \leqslant m \leqslant n$. Then*

$$\mathrm{s}_{\{0,1\}}(S_{m,n}) = \begin{cases} n & \text{if } m < n, \\ n+1 & \text{otherwise.} \end{cases}$$

**Lemma 3.11.** *If $D \in \{\{1\}, \{0,2,3\}\}$ then $\mathrm{s}_D(S_{1,1}) = 2$.*

**Proof.** Obviously, $\mathrm{s}_{\{1\}}(S_{1,1}) = 2$. Combining this with Theorem 2.3, $\mathrm{s}_{\{0,2,3\}}(S_{1,1}) = 2$. $\qquad\square$

**Lemma 3.12.** *Let $S_{m,n}$ be a double star for $1 \leqslant m < n$. If $D \in \{\{0,1\}, \{0,3\}, \{1,2\}, \{2,3\}\}$ then $\mathrm{s}_D(S_{m,n}) = n$.*

**Proof.** From Theorems 2.3 and 3.10, we get $\mathrm{s}_{\{0,1\}}(S_{m,n}) = \mathrm{s}_{\{2,3\}}(S_{m,n}) = n$. Let $D = \{0,3\}$. By Theorem 2.8, $y_i$ and $y_j$, $i \neq j$, must receive distinct labels, thus $\mathrm{s}_{\{0,3\}}(S_{m,n}) \geqslant n$. Define a labeling $\varphi : V(S_{m,n}) \to \{1, 2, \ldots, n\}$ such that

$$\begin{aligned} \varphi(x) &= 1, \\ \varphi(y) &= 2, \\ \varphi(x_i) &= i \quad \text{for } i = 1, 2, \ldots, m, \\ \varphi(y_j) &= j \quad \text{for } j = 1, 2, \ldots, n. \end{aligned}$$

It is easy to see that all vertex labels are at most $n$. For the vertex weights we have

$$\begin{aligned} wt(x) &= 1, \\ wt(y) &= 2, \\ wt(x_i) &= \frac{n(n+1)}{2} + i \quad \text{for } i = 1, 2, \ldots, m, \\ wt(y_j) &= \frac{m(m+1)}{2} + j \quad \text{for } j = 1, 2, \ldots, n. \end{aligned}$$

As $m < n$, we get that $\dfrac{m(m+1)}{2} + n < \dfrac{n(n+1)}{2} + 1$, which implies that the vertex weights are all distinct. Therefore $\mathrm{s}_{\{0,3\}}(S_{m,n}) = n$. By combining this with Theorem 2.3 we also have $\mathrm{s}_{\{1,2\}}(S_{m,n}) = n$. $\qquad\square$

**Lemma 3.13.** *Let $S_{m,n}$ be a double star for $1 \leqslant m \leqslant n$. If one of the following conditions holds:*

(a) $D \in \{\{2\}, \{0, 1, 3\}\}$ and $m < n$, or

(b) $D \in \{\{0, 1\}, \{2, 3\}\}$ and $m = n$,

then $s_D(S_{m,n}) = n + 1$.

**Proof.** Let $D = \{2\}$ and $m < n$. Suppose that $\varphi$ is a $\{2\}$-irregular labeling of $S_{m,n}$. By Theorem 2.7, $\varphi(y_i) \neq \varphi(y_j)$ for $i \neq j$, so $s_{\{2\}}(S_{m,n}) \geqslant n$.

Assume that $s_{\{2\}}(S_{m,n}) = n$. Then $\{\varphi(y_j) : j = 1, 2, \ldots, n\} = \{1, 2, \ldots, n\}$, which leads to $wt(x) = \dfrac{n(n+1)}{2}$ and $\{wt(y_j) : j = 1, 2, \ldots, n\} = \{\dfrac{n(n+1)}{2} - n + \varphi(x), \dfrac{n(n+1)}{2} - n + 1 + \varphi(x), \ldots, \dfrac{n(n+1)}{2} - 1 + \varphi(x)\}$. As $1 \leqslant \varphi(x) \leqslant n$, the vertex weights are not distinct. Therefore, $s_{\{2\}}(S_{m,n}) \geqslant n + 1$.

Let $\varphi$ be a vertex labeling of $S_{m,n}$ defined such that

$$\varphi(x) = n + 1,$$
$$\varphi(y) = n,$$
$$\varphi(x_i) = i \quad \text{for } i = 1, 2, \ldots, m,$$
$$\varphi(y_j) = j \quad \text{for } j = 1, 2, \ldots, n.$$

Then we obtain the vertex weights as follows.

$$wt(x) = \frac{n(n+1)}{2},$$
$$wt(y) = \frac{m(m+1)}{2},$$
$$wt(x_i) = \frac{m(m+1)}{2} + n - i \quad \text{for } i = 1, 2, \ldots, m,$$
$$wt(y_j) = \frac{n(n+1)}{2} + n + 1 - j \quad \text{for } j = 1, 2, \ldots, n.$$

It is not difficult to verify that the vertex weights are distinct and the labels used in the labeling $\varphi$ are not greater than $n + 1$, which means that $\varphi$ is a $\{2\}$-irregular $(n + 1)$-labeling of $S_{m,n}$ for $m < n$. Hence $s_{\{2\}}(S_{m,n}) = n + 1$. Furthermore, combining this with Theorem 2.3 we also have $s_{\{0,1,3\}}(S_{m,n}) = n + 1$ for $m < n$.

Next, we have $s_{\{0,1\}}(S_{n,n}) = s_{\{2,3\}}(S_{n,n}) = n + 1$ by Theorems 2.3 and 3.10. $\qquad \square$

**Lemma 3.14.** Let $S_{n,n}$ be a double star for $n \geqslant 2$. If $D \in \{\{0, 3\}, \{1, 2\}\}$ then $s_D(S_{n,n}) = n + 2$.

**Proof.** Let $D = \{0, 3\}$. Define a vertex labeling $\varphi : V(S_{n,n}) \to \{1, 2, \ldots, n + 2\}$ of $S_{n,n}$ in the following way:

$$\varphi(x) = 1,$$
$$\varphi(y) = 2,$$
$$\varphi(x_i) = i \quad \text{for } i = 1, 2, \ldots, n,$$
$$\varphi(y_i) = i + 2 \quad \text{for } i = 1, 2, \ldots, n.$$

Obviously, the largest label that appears on the vertices is $n + 2$. Moreover, we have the following vertex weights.

$$
\begin{aligned}
wt(x) &= 1, \\
wt(y) &= 2, \\
wt(x_i) &= \frac{n(n + 5)}{2} + i \quad \text{for } i = 1, 2, \ldots, n, \\
wt(y_i) &= \frac{n(n + 1)}{2} + i + 2 \quad \text{for } i = 1, 2, \ldots, n.
\end{aligned}
$$

It is easy to see that the vertex weights are distinct. Therefore $s_{\{0,3\}}(S_{n,n}) \leqslant n + 2$.

Next, we shall show that $s_{\{0,3\}}(S_{n,n}) \geqslant n + 2$. For a contradiction, assume that there is a $\{0, 3\}$-irregular $(n + 1)$-labeling $\varphi$ of $S_{n,n}$ for $n \geqslant 2$. Then by Theorem 2.8, the following properties hold:

* $\varphi(x_i) \neq \varphi(x_j)$ for $i \neq j$, and

* $\varphi(y_i) \neq \varphi(y_j)$ for $i \neq j$.

We may assume, without loss of generality, that $\{\varphi(x_i) : i = 1, 2, \ldots, n\} = \{1, 2, \ldots, n\}$ and $\{\varphi(y_i) : i = 1, 2, \ldots, n\} = \{1, 2, \ldots, n + 1\} - \{r\}$, where $1 \leqslant r \leqslant n$. Then

$$
\begin{aligned}
\{wt(x_i) : i = 1, 2, \ldots, n\} = \bigg\{ &\frac{n(n + 1)}{2} + n + 2 - r, \frac{n(n + 1)}{2} + n + 3 - r, \ldots, \frac{n(n + 1)}{2} \\
&+ 2n + 1 - r \bigg\},
\end{aligned}
$$

and

$$
\begin{aligned}
\{wt(y_i) : i = 1, 2, \ldots, n\} = \bigg\{ &\frac{n(n + 1)}{2} + 1, \frac{n(n + 1)}{2} + 2, \ldots, \frac{n(n + 1)}{2} + n + 1 \bigg\} \\
&- \bigg\{ \frac{n(n + 1)}{2} + r \bigg\}.
\end{aligned}
$$

Since these sets are not disjoint, the vertex weights are not distinct, a contradiction. Thus $s_{\{0,3\}}(S_{n,n}) \geqslant n + 2$.

As $n + 2 \leqslant s_{\{0,3\}}(S_{n,n})$ and $s_{\{0,3\}}(S_{n,n}) \leqslant n + 2$ we conclude that $s_{\{0,3\}}(S_{n,n}) = n + 2$. Furthermore, combining this with Theorem 2.3 we get $s_{\{1,2\}}(S_{n,n}) = n + 2$. $\square$

**Lemma 3.15.** *Let $S_{n,n}$ be a double star for $n \geqslant 1$. If $D \in \{\{2\}, \{0, 1, 3\}\}$ then $s_D(S_{n,n}) = n + 3$.*

**Proof.** Let $D = \{2\}$. Define a vertex labeling $\varphi : V(S_{n,n}) \to \{1, 2, \ldots, n + 3\}$ of $S_{n,n}$ in the following way:

$$
\begin{aligned}
\varphi(x) &= n + 3, \\
\varphi(y) &= n + 1, \\
\varphi(x_i) &= i \quad \text{for } i = 1, 2, \ldots, n, \\
\varphi(y_i) &= i + 2 \quad \text{for } i = 1, 2, \ldots, n.
\end{aligned}
$$

Then we get the following vertex weights.

$$wt(x) = \frac{n(n+5)}{2},$$
$$wt(y) = \frac{n(n+1)}{2},$$
$$wt(x_i) = \frac{(n+1)(n+2)}{2} - i \quad \text{for } i = 1, 2, \ldots, n,$$
$$wt(y_i) = \frac{n(n+7)}{2} + 1 - i \quad \text{for } i = 1, 2, \ldots, n.$$

It is not hard to show that the vertex weights are distinct and the labels used in the labeling $\varphi$ are at most $n + 3$. This means that $\varphi$ is a $\{2\}$-irregular $(n+3)$-labeling of $S_{n,n}$ for $n \geqslant 1$. Therefore $\mathrm{s}_{\{2\}}(S_{n,n}) \leqslant n + 3$.

Next, we shall show that $\mathrm{s}_{\{2\}}(S_{n,n}) \geqslant n + 3$. For a contradiction, assume that there exists a $\{2\}$-irregular $(n+2)$-labeling $\varphi$ of $S_{n,n}$ for $n \geqslant 1$. Using Theorem 2.7, the following properties must be hold:

- $\varphi(y) \neq \varphi(x_i) \neq \varphi(x_j)$ for $i \neq j$, and

- $\varphi(x) \neq \varphi(y_i) \neq \varphi(y_j)$ for $i \neq j$.

Without loss of generality we may assume that $\{\varphi(y), \varphi(x_i) : i = 1, 2, \ldots, n\} = \{1, 2, \ldots, n+1\}$ and $\{\varphi(x), \varphi(y_i) : i = 1, 2, \ldots, n\} = \{1, 2, \ldots, n+2\} - \{r\}$, where $1 \leqslant r \leqslant n+1$. Then

$$\{wt(y), wt(x_i) : i = 1, 2, \ldots, n\} = \left\{ \frac{(n+1)(n+2)}{2} - n - 1, \frac{(n+1)(n+2)}{2} - n, \ldots, \right.$$
$$\left. \frac{(n+1)(n+2)}{2} - 1 \right\}$$

and

$$\{wt(x), wt(y_i) : i = 1, 2, \ldots, n\} = \left\{ \frac{(n+1)(n+2)}{2} - r, \frac{(n+1)(n+2)}{2} - r + 1, \ldots, \right.$$
$$\left. \frac{(n+1)(n+2)}{2} - r + n + 1 \right\} \left\{ \frac{(n+1)(n+2)}{2} + n \right.$$
$$\left. + 2 - 2r \right\},$$

which implies that the vertex weights are not distinct, a contradiction. Therefore $\mathrm{s}_{\{2\}}(S_{n,n}) \geqslant n + 3$.

Since $n + 3 \leqslant \mathrm{s}_{\{2\}}(S_{n,n}) \leqslant n + 3$, we get $\mathrm{s}_{\{2\}}(S_{n,n}) = n + 3$ for $n \geqslant 1$. Moreover, combining this with Theorem 2.3 we also obtain $\mathrm{s}_{\{0,1,3\}}(S_{n,n}) = n + 3$ for $n \geqslant 1$. $\qquad\square$

In the next theorem, we present $D$-irregularity strength for the double stars.

**Theorem 3.16.** *Let $S_{m,n}$ be a double star for $1 \leqslant m \leqslant n$. Then*
*(a) If $D \in \{\{1\}, \{0, 2, 3\}\}$ then $\mathrm{s}_D(S_{1,1}) = 2$.*

*(b) If $D \in \{\{0, 1\}, \{0, 3\}, \{1, 2\}, \{2, 3\}\}$ and $m < n$ then $\mathrm{s}_D(S_{m,n}) = n$.*

*(c) If one of the following conditions is satisfied:*

- $D \in \{\{2\}, \{0, 1, 3\}\}$ *and* $m < n$, *or*
- $D \in \{\{0, 1\}, \{2, 3\}\}$ *and* $m = n$,

then $s_D(S_{m,n}) = n + 1$.

(d) If $D \in \{\{0,3\},\{1,2\}\}$ and $m = n \geqslant 2$ then $s_D(S_{m,n}) = n + 2$.

(e) If $D \in \{\{2\},\{0,1,3\}\}$ and $m = n$ then $s_D(S_{m,n}) = n + 3$.

(f) If $D \in \{\{0\},\{1,2,3\}\}$ then $s_D(S_{m,n}) = m + n + 2$.

(g) If one of the following conditions is satisfied:

- $D \in \{\{3\},\{0,2\},\{1,3\},\{0,1,2\},\{0,1,2,3\}\}$,
- $D \in \{\{1\},\{0,2,3\}\}$ and $(m,n) \neq (1,1)$, or
- $D \in \{\{0,3\},\{1,2\}\}$ and $m = n = 1$,

then $s_D(S_{m,n}) = \infty$.

**Proof.** (1)-(5) follow from Lemmas 3.11, 3.12, 3.13, 3.14 and 3.15, respectively. Similarly, (6)-(7) are derived by applying Proposition 2.5, and Theorems 2.1 and 2.3. $\qquad\square$

It is worth noting that, from Theorems 3.4 and 3.16, along with Corollary 3.9, we have fully characterized the $D$-irregularity strength for all non-trivial trees with diameter at most three, for all possible sets $D$.

**Question 3.17.** *Find the D-irregularity strength of all trees with diameter four for all possible Ds.*

## 4. Graphs with Small Maximum Degree

This section explores the $\{d\}$-irregularity strength of certain graphs with maximum degree two, focusing particularly on the disjoint union of paths and 2-regular graphs, for a positive integer $d$.

We begin by examining the $\{d\}$-irregularity strength of the disjoint union of paths $mP_n$ for $m \geqslant 1$ and $n \geqslant 2$. For $d = 1$, the following theorem, established in [18, 20], provides the result.

**Theorem 4.1** ([18, 20]). *Let $mP_n$ be an $m$ disjoint union of a path on $n$ vertices for $m \geqslant 1$ and $n \geqslant 2$. Then*

$$s_{\{1\}}(mP_n) = \begin{cases} 2m & \text{if } n = 2, \\ \infty & \text{if } n = 3, \\ \lceil \frac{mn}{2} \rceil & \text{if } n \geqslant 4. \end{cases}$$

In the next theorem, we determine $\{d\}$-irregularity strength of $mP_n$ for certain $d \geqslant 2$.

**Theorem 4.2.** *Let $mP_n$ be an $m$ disjoint union of a path on $n$ vertices for $m \geqslant 1$ and $n \geqslant 2$, and let $2 \leqslant d \leqslant n - 1$. Then*

(a) $s_{\{\frac{n}{2}\}}(mP_n) = mn$.

(b) For $d \in \{\frac{n}{k} : 4 \leqslant k \leqslant \frac{n}{2}\}$, $s_{\{d\}}(mP_n) = \lceil \frac{mn}{2} \rceil$.

(c) $s_{\{d\}}(mP_n) = \infty$ if and only if

$$d = \begin{cases} \frac{n}{3} & \text{if } n \equiv 0 \pmod{d}, \\ \frac{n-t}{3} \text{ or } \frac{n-t}{2} & \text{if } n \equiv t \pmod{d}, \ 0 < t < d, \\ n - t & \text{if } n \equiv t \pmod{d}, \ 0 < t < d, \text{ and } m(n - 2t) \geqslant 2. \end{cases}$$

**Proof.** (1) Evidently, $(mP_n)_{\{\frac{n}{2}\}} \cong \frac{mn}{2}K_2$. Then by Theorem 2.9, we have $\mathrm{s}_{\{\frac{n}{2}\}}(mP_n) = \mathrm{s}_{\{1\}}(\frac{mn}{2}K_2) = mn$.

(2) If $d \in \{\frac{n}{k} : 4 \leqslant k \leqslant \frac{n}{2}\}$ then $(mP_n)_{\{d\}} \cong mdP_{\frac{n}{d}} \ncong \frac{mn}{2}P_2 \ncong \frac{mn}{3}P_3$. By Theorems 2.9 and 4.1,

$$\mathrm{s}_{\{d\}}(mP_n) = \mathrm{s}_{\{1\}}(mdP_{\frac{n}{d}}) = \left\lceil \frac{mn}{2} \right\rceil.$$

(3) From Theorem 2.9 we know that $\mathrm{s}_{\{d\}}(mP_n) = \infty$ if and only if $\mathrm{s}_{\{1\}}((mP_n)_{\{d\}}) = \infty$. Furthermore, by Theorem 2.1 we have that $\mathrm{s}_{\{1\}}((mP_n)_{\{d\}}) = \infty$ if and only if $(mP_n)_{\{d\}}$ contains at least one component isomorphic to $P_3$ or at least two components isomorphic to $P_1$. Thus, $\mathrm{s}_{\{d\}}(mP_n) = \infty$ if and only if $(mP_n)_{\{d\}}$ has at least one component isomorphic to $P_3$ or at least two components isomorphic to $P_1$.

We have that

$$(mP_n)_{\{d\}} = \begin{cases} mdP_{\frac{n}{d}} & \text{if } n \equiv 0 \pmod{d}, \\ m(d-t)P_{\frac{n-t}{d}} \cup mtP_{\frac{n-t+d}{d}} & \text{if } n \equiv t \pmod{d}, \ 0 < t < d. \end{cases}$$

Clearly, $(mP_n)_{\{d\}}$ contains at least one component isomorphic to $P_3$ if and only if

$$d = \begin{cases} \frac{n}{3} & \text{if } n \equiv 0 \pmod{d}, \\ \frac{n-t}{3} \text{ or } \frac{n-t}{2} & \text{if } n \equiv t \pmod{d}, \ 0 < t < d, \end{cases}$$

and $(mP_n)_{\{d\}}$ has at least two components isomorphic to $P_1$ if and only if $n \equiv t \pmod{d}$, $0 < t < d$, $d = n-t$ and $m(n-2t) \geqslant 2$. Therefore, combining both conditions, we conclude that $\mathrm{s}_{\{d\}}(mP_n) = \infty$ if and only if

$$d = \begin{cases} \frac{n}{3} & \text{if } n \equiv 0 \pmod{d}, \\ \frac{n-t}{3} \text{ or } \frac{n-t}{2} & \text{if } n \equiv t \pmod{d}, \ 0 < t < d, \\ n-t & \text{if } n \equiv t \pmod{d}, \ 0 < t < d, \text{ and } m(n-2t) \geqslant 2. \end{cases}$$

$\square$

**Question 4.3.** *Find $\{d\}$-irregularity strength of $mP_n$ for other values of $d \geqslant 2$ not covered in Theorem 4.2.*

Next, we address the $\{d\}$-irregularity strength for 2-regular graphs, i.e., the disjoint union of cycles. To achieve this, we first employ the irregularity strength and edge irregularity strength of 2-regular graphs to obtain $\{1\}$-irregularity strength of 2-regular graphs. Subsequently, we use Theorem 2.9 to derive the $\{d\}$-irregularity strength of 2-regular graphs. Recall that the *edge irregularity strength* of a graph $G$, denoted by $\mathrm{es}(G)$, is the minimum $k$ such that an assignment of integers $1, 2, \ldots, k$ to the vertices of $G$ implies that distinct edges have distinct weights, where the *edge weight* is the sum of labels of its two end vertices [1].

We begin by stating the theorem of Faudree et al. [10], which determines the irregularity strength of 2-regular graphs.

**Theorem 4.4** ([10]). *Let $G$ be a 2-regular graph on $n$ vertices. Then*

$$\left\lfloor \frac{n}{2} \right\rfloor \leqslant \mathrm{s}(G) \leqslant \left\lfloor \frac{n}{2} \right\rfloor + 2.$$

*Moreover,*

(a) *If G is the disjoint union of triangles $\frac{n}{3}C_3$ then*

$$s(\frac{n}{3}C_3) = \begin{cases} \lceil \frac{n}{2} \rceil + 1 & \text{if } n \equiv 3 \pmod{4}, \\ \lceil \frac{n}{2} \rceil + 2 & \text{otherwise}. \end{cases}$$

(b) *If G has no triangle then*

$$s(G) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 1 \pmod{4}, \\ \lceil \frac{n}{2} \rceil + 1 & \text{otherwise}. \end{cases}$$

In the next theorem, we prove that the irregularity strength and the edge irregularity strength of 2-regular graphs are equivalent.

**Theorem 4.5.** *Let G be a 2-regular graph on n vertices. Then* $es(G) = s(G)$. *Consequently,*

$$\lfloor \frac{n}{2} \rfloor \leqslant es(G) \leqslant \lfloor \frac{n}{2} \rfloor + 2.$$

*Moreover,*
(a) *If G is the disjoint union of triangles $\frac{n}{3}C_3$ then*

$$es(\frac{n}{3}C_3) = \begin{cases} \lceil \frac{n}{2} \rceil + 1 & \text{if } n \equiv 3 \pmod{4}, \\ \lceil \frac{n}{2} \rceil + 2 & \text{otherwise}. \end{cases}$$

(b) *If G has no triangle then*

$$es(G) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 1 \pmod{4}, \\ \lceil \frac{n}{2} \rceil + 1 & \text{otherwise}. \end{cases}$$

**Proof.** Let $G$ be a 2-regular graph on $n$ vertices. In any edge irregular labeling of $G$, if we shift the label of each vertex to the edge incident to it in the clockwise direction, we obtain an irregular labeling of $G$. This demonstrates that $es(G) \geqslant s(G)$. Conversely, in any irregular labeling of $G$, shifting the label of each edge to the vertex incident to it in the clockwise direction results in an edge irregular labeling of $G$, implying that $es(G) \leqslant s(G)$. Consequently, $es(G) = s(G)$. The rest of the theorem follows immediately from Theorem 4.4. □

Let $G \cong mC_3 \cup (G - mC_3)$ be a 2-regular graph on $n$ vertices containing $m$ triangles and no square. Evidently, $mC_3 \cup (G - mC_3)_{\{2\}}$ is a 2-regular graph with $n$ vertices. Moreover, it is not hard to know that every $\{1\}$-irregular labeling of $G$ induces an edge irregular labeling of $mC_3 \cup (G - mC_3)_{\{2\}}$, and every edge irregular labeling of $mC_3 \cup (G - mC_3)_{\{2\}}$ induces a $\{1\}$-irregular labeling of $G$, meaning that $\{1\}$-irregularity strength of $G$ and edge irregularity strength of $mC_3 \cup (G - mC_3)_{\{2\}}$ are equivalent.

Further, if $G \cong mC_3 \cup \frac{n-3m}{6}C_6$ then $mC_3 \cup (\frac{n-3m}{6}C_6)_{\{2\}}$ is the disjoint union of triangles $\frac{n}{3}C_3$, and if $m = 0$ and $G$ has no hexagon then $G_{\{2\}}$ is a 2-regular graph without any triangle.

With the above facts in hand along with Theorem 4.5 we obtain the following.

**Theorem 4.6.** *Let $G \cong mC_3 \cup (G - mC_3)$ be a 2-regular graph on $n$ vertices containing $m$ triangles and no square for $n \geqslant 3$ and $0 \leqslant m \leqslant \frac{n}{3}$. Then $\mathrm{s}_{\{1\}}(G) = \mathrm{es}(mC_3 \cup (G - mC_3)_{\{2\}})$. Consequently,*

$$\left\lfloor \frac{n}{2} \right\rfloor \leqslant \mathrm{s}_{\{1\}}(G) \leqslant \left\lfloor \frac{n}{2} \right\rfloor + 2.$$

*Moreover,*
(a) *If $G = mC_3 \cup \frac{n-3m}{6} C_6$ then*

$$\mathrm{s}_{\{1\}}(G) = \begin{cases} \lceil \frac{n}{2} \rceil + 1 & \text{if } n \equiv 3 \pmod 4, \\ \lceil \frac{n}{2} \rceil + 2 & \text{otherwise.} \end{cases}$$

(b) *If $m = 0$ and $G$ has no hexagon then*

$$\mathrm{s}_{\{1\}}(G) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 1 \pmod 4, \\ \lceil \frac{n}{2} \rceil + 1 & \text{otherwise.} \end{cases}$$

The next theorem provides $\{d\}$-irregularity strength of the isomorphic copies of cycles $mC_n$ for $1 \leqslant d \leqslant \lfloor \frac{n}{2} \rfloor$.

**Theorem 4.7.** *Let $mC_n$ be an $m$ disjoint union of a cycle on $n$ vertices for $m \geqslant 1$ and $n \geqslant 3$. Then*
(a) $\mathrm{s}_{\{\frac{n}{4}\}}(mC_n) = \infty.$
(b) $\mathrm{s}_{\{\frac{n}{2}\}}(mC_n) = mn.$
(c) *For $d \in \{\frac{n}{6}, \frac{n}{3}\}$,*

$$\mathrm{s}_{\{d\}}(mC_n) = \begin{cases} \lceil \frac{mn}{2} \rceil + 1 & \text{if } mn \equiv 3 \pmod 4, \\ \lceil \frac{mn}{2} \rceil + 2 & \text{otherwise.} \end{cases}$$

(d) *For $d \in (\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\} - \{\frac{n}{6}, \frac{n}{4}, \frac{n}{3}, \frac{n}{2}\})$,*

$$\mathrm{s}_{\{d\}}(mC_n) = \begin{cases} \lceil \frac{mn}{2} \rceil & \text{if } mn \equiv 1 \pmod 4, \\ \lceil \frac{mn}{2} \rceil + 1 & \text{otherwise.} \end{cases}$$

**Proof.** By simple observation, $(mC_n)_{\{\frac{n}{4}\}} = \frac{mn}{4} C_4$ and $(mC_n)_{\{\frac{n}{2}\}} = \frac{mn}{2} K_2$. Then by Theorem 2.9, we find that $\mathrm{s}_{\{\frac{n}{4}\}}(mC_n) = \mathrm{s}_{\{1\}}(\frac{mn}{4} C_4) = \infty$ and $\mathrm{s}_{\{\frac{n}{2}\}}(mC_n) = \mathrm{s}_{\{1\}}(\frac{mn}{2} K_2) = mn$. This proves (1) and (2).

Next, let $d \in \{\frac{n}{6}, \frac{n}{3}\}$. If $d = \frac{n}{6}$ then $(mC_n)_{\{\frac{n}{6}\}} = \frac{mn}{6} C_6$. By Theorems 2.9 and 4.6, we obtain

$$\mathrm{s}_{\{\frac{n}{6}\}}(mC_n) = \mathrm{s}_{\{1\}}(\frac{mn}{6} C_6) = \begin{cases} \lceil \frac{mn}{2} \rceil + 1 & \text{if } mn \equiv 3 \pmod 4, \\ \lceil \frac{mn}{2} \rceil + 2 & \text{otherwise.} \end{cases}$$

If $d = \frac{n}{3}$ then $(mC_n)_{\{\frac{n}{3}\}} = \frac{mn}{3} C_3$. Again, by Theorems 2.9 and 4.6, we have

$$\mathrm{s}_{\{\frac{n}{3}\}}(mC_n) = \mathrm{s}_{\{1\}}(\frac{mn}{3} C_3) = \begin{cases} \lceil \frac{mn}{2} \rceil + 1 & \text{if } mn \equiv 3 \pmod 4, \\ \lceil \frac{mn}{2} \rceil + 2 & \text{otherwise.} \end{cases}$$

This proves (3).

Now, for $d \in (\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\} - \{\frac{n}{6}, \frac{n}{4}, \frac{n}{3}, \frac{n}{2}\})$, we observe that $(mC_n)_{\{d\}}$ is a 2-regular graph without any cycle $C_t$ for $t = 3, 4, 6$. By Theorems 2.9 and 4.6, we conclude that

$$s_{\{d\}}(mC_n) = s_{\{1\}}((mC_n)_{\{d\}}) = \begin{cases} \lceil \frac{mn}{2} \rceil & \text{if } mn \equiv 1 \ (\text{mod } 4), \\ \lceil \frac{mn}{2} \rceil + 1 & \text{otherwise.} \end{cases}$$

Hence (4) is proved. $\qquad\square$

Under certain conditions, Theorem 4.7 can be extended to 2-regular graphs that contain non-isomorphic cycle components.

**Theorem 4.8.** *Let* $G = \bigcup_{i=1}^m C_{n_i}$ *be a 2-regular graph on* $n = n_1 + n_2 + \cdots + n_m$ *vertices with* $m \geqslant 2$ *components such that* $3 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_m$, *and* $G \not\cong mC_p$ *for some* $p \geqslant 3$. *Then*
  *(a)* *For* $d \in (\{\frac{n_i}{4} \leqslant \lfloor \frac{n_1}{2} \rfloor : i = 1, 2, \ldots, m\} \cup \{\lfloor \frac{n_1}{2} \rfloor + 1, \lfloor \frac{n_1}{2} \rfloor + 2, \ldots, \lfloor \frac{n_m}{2} \rfloor\})$, $s_{\{d\}}(G) = \infty$.
  *(b)* *For* $d \in (\{\frac{n_i}{6}, \frac{n_i}{3} \leqslant \lfloor \frac{n_1}{2} \rfloor : i = 1, 2, \ldots, m\} - \{\frac{n_j}{4}, \frac{n_j}{2} : j = 1, 2, \ldots, m\})$,

$$\left\lfloor \frac{n}{2} \right\rfloor \leqslant s_{\{d\}}(G) \leqslant \left\lfloor \frac{n}{2} \right\rfloor + 2.$$

  *In particular, if* $d \in \{\frac{n_i}{6}, \frac{n_i}{3}\}$ *for some* $i$, *and* $\frac{n_j}{\gcd(n_j, d)} \in \{3, 6\}$ *for every* $j = 1, 2, \ldots, m$, $j \neq i$, *then*

$$s_{\{d\}}(G) = \begin{cases} \lceil \frac{n}{2} \rceil + 1 & \text{if } n \equiv 3 \ (\text{mod } 4), \\ \lceil \frac{n}{2} \rceil + 2 & \text{otherwise.} \end{cases}$$

  *(c)* *For* $d \in (\{1, 2, \ldots, \lfloor \frac{n_1}{2} \rfloor\} - \{\frac{n_j}{6}, \frac{n_j}{4}, \frac{n_j}{3}, \frac{n_j}{2} : j = 1, 2, \ldots, m\})$,

$$s_{\{d\}}(G) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 1 \ (\text{mod } 4), \\ \lceil \frac{n}{2} \rceil + 1 & \text{otherwise.} \end{cases}$$

**Proof.** (1) If $d \in \{\frac{n_i}{4} \leqslant \lfloor \frac{n_1}{2} \rfloor : i = 1, 2, \ldots, m\}$ then $G_{\{d\}}$ contains a square, and so $s_{\{d\}}(G) = s_{\{1\}}(G_{\{d\}}) = \infty$. If $d \in \{\lfloor \frac{n_1}{2} \rfloor + 1, \lfloor \frac{n_1}{2} \rfloor + 2, \ldots, \lfloor \frac{n_m}{2} \rfloor\}$ then $N_{\{d\}}(x) = N_{\{d\}}(y) = \emptyset$ for any two distinct vertices $x$ and $y$ in the component $C_{n_1}$, thus $s_{\{d\}}(G) = \infty$.

(2) If $d \in (\{\frac{n_i}{6}, \frac{n_i}{3} \leqslant \lfloor \frac{n_1}{2} \rfloor : i = 1, 2, \ldots, m\} - \{\frac{n_j}{4}, \frac{n_j}{2} : j = 1, 2, \ldots, m\})$ then $G_{\{d\}} \cong tC_3 \cup (G_{\{d\}} - tC_3)$ is a 2-regular graph of order $n$ containing $t$ triangles and no square for some $t \geqslant 0$. Therefore, by Theorems 2.9 and 4.6,

$$\left\lfloor \frac{n}{2} \right\rfloor \leqslant s_{\{d\}}(G) = s_{\{1\}}(G_{\{d\}}) \leqslant \left\lfloor \frac{n}{2} \right\rfloor + 2.$$

For some $i$, let $d \in \{\frac{n_i}{6}, \frac{n_i}{3}\}$, and for every $j = 1, 2, \ldots, m$, $j \neq i$, let $\frac{n_j}{\gcd(n_j, d)} \in \{3, 6\}$. Then $G_{\{d\}} \cong m_1 C_3 \cup m_2 C_6$, where $3m_1 + 6m_2 = n$. Thus, by Theorems 2.9 and 4.6,

$$s_{\{d\}}(G) = s_{\{1\}}(G_{\{d\}}) = \begin{cases} \lceil \frac{n}{2} \rceil + 1 & \text{if } n \equiv 3 \ (\text{mod } 4), \\ \lceil \frac{n}{2} \rceil + 2 & \text{otherwise.} \end{cases}$$

(3) If $d \in (\{1, 2, \ldots, \lfloor \frac{n_1}{2} \rfloor\} - \{\frac{n_j}{6}, \frac{n_j}{4}, \frac{n_j}{3}, \frac{n_j}{2} : j = 1, 2, \ldots, m\})$ then $G_{\{d\}}$ is a 2-regular graphs with $n$ vertices containing no cycle $C_t$, $t = 3, 4, 6$. By Theorems 2.9 and 4.6,

$$s_{\{d\}}(G) = s_{\{1\}}(G_{\{d\}}) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 1 \ (\text{mod } 4), \\ \lceil \frac{n}{2} \rceil + 1 & \text{otherwise.} \end{cases}$$

$\qquad\square$

**Question 4.9.** *Find the $\{d\}$-irregularity strength of arbitrary 2-regular graphs for other values of $d$ not mentioned in Theorems 4.7 and 4.8.*

## 5. Final Remarks

In this work, we introduced a $D$-irregular labeling as a generalization of non-inclusive and inclusive $d$-distance irregular labeling of graphs. We derived fundamental properties on $D$-irregularity strength for arbitrary graphs, and presented exact values on this parameter for specific families of graphs with small diameter or small maximum degree.

The authors of [16] posed a conjecture below.

**Conjecture 5.1** ([16]). *A graph $G$ is $D$-antimagic if and only if $N_D(x) \neq N_D(y)$ for every two distinct vertices $x, y \in V(G)$.*

Note that if Conjecture 5.1 holds, then for a graph $G$ with finite $D$-irregularity strength, the inequality $s_D(G) \leqslant |V(G)|$ holds as well. Motivated by this observation and the results presented in Proposition 2.5, we believe that the following conjecture is true.

**Conjecture 5.2.** *Let $G$ be a graph on $n$ vertices with $s_D(G) < \infty$. Then $s_D(G) \leqslant n$ with the equality if and only if one of the following conditions holds:*
   *(a) $G$ is connected and $D \in \{\{0\}, \{1, 2, \ldots, \mathrm{diam}(G)\}\}$,*

   *(b) $G \cong mK_2$, $m \geqslant 2$, and $D = \{\{0\}, \{1\}\}$, or*

   *(c) $G$ is disconnected, $G \not\cong mK_2$, $m \geqslant 2$, and $D = \{0\}$.*

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