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Introducing Soft Directed Hypergraphs: A Fusion of Soft Set Theory and Directed Hypergraphs

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ABSTRACT

Directed hypergraphs represent a natural extension of directed graphs, while soft set theory provides a method for addressing vagueness and uncertainty. This paper introduces the notion of soft directed hypergraphs by integrating soft set principles into directed hypergraphs. Through parameterization, soft directed hypergraphs yield a sequence of relation descriptions derived from a directed hypergraph. Additionally, we present several operations for soft directed hypergraphs, including extended union, restricted union, extended intersection, and restricted intersection, and explore their characteristics.

Keywords: Directed Hypergraph, Soft Directed Hypergraph

2020 Mathematics Subject Classification: 05C12; 05C35

1. Introduction

Directed hypergraphs[7, 8, 10] serve as a powerful modelling tool in both Operations Research and Computer Science, offering a broader representation than traditional directed graphs. They allow for the representation of relationships among multiple entities, making them particularly useful in scenarios where complex interactions need to be captured.

The concept of soft sets, pioneered by Molodtsov [27] in 1999, revolutionized mathematical approaches to dealing with uncertainties that evade conventional methods. Soft sets provide a flexible

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framework for handling imprecise or uncertain information, enabling more robust decision-making processes. Subsequent research by Maji, Roy, and Biswas [25, 26] has delved deeper into the theory of soft sets, exploring their applications in various decision-making contexts.

Thumbakara and George [31] introduced soft graphs and some of their properties [32, 33, 34], which have since been refined and extended by researchers like Akram and Nawas [2, 3]. Their work has led to the development of fuzzy soft graphs and fuzzy soft trees, further expanding the applicability of soft graph theory [4, 5, 6]. Contributions from Thenge, Jain, and Reddy [30, 28, 29] have also significantly advanced the field of soft graphs.

George, Thumbakara, and Jose have made substantial contributions to the domain by introducing concepts such as soft hypergraphs [11] and soft directed graphs [23, 22]. These extensions have broadened the scope of soft graph theory, enabling the modelling of more complex systems and phenomena. Moreover, they have investigated various product operations on soft graphs, including modular products and homomorphic products, which have implications for graph analysis and manipulation [12, 14, 15, 19, 21, 20, 24].

Baghernejad and Borzooei [9] have demonstrated the practical utility of soft graphs and soft multigraphs in managing urban traffic flows, showcasing the real-world applicability of these theoretical constructs. Additionally, innovations like soft semigraphs [13, 18, 17] and soft disemigraphs [16] have further enriched the field, providing new avenues for exploration and application.

In this paper, the authors introduce the concept of soft directed hypergraphs, which represent a further extension of soft graph theory. They also investigate various operations on soft directed hypergraphs, aiming to elucidate their properties.

2. Preliminaries

For basic concepts of a directed hypergraph, we refer [1, 4, 8]. "A *directed hypergraph* $\Delta^* = (\Gamma, \Xi)$ consists of a vertex set Γ and a set of directed hyperedges or hyperarcs $\Xi = \{e = (T(e), H(e)) \mid T(e) \subseteq \Gamma$ and $H(e) \subseteq \Gamma\}$, where $T(e) \neq \phi$ and $H(e) \neq \phi$. The sets $T(e)$ and $H(e)$ are called *tail* and *head* of the hyperarc e, respectively. A directed hypergraph is called *k-uniform* if $|T(e)|= |H(e)|= k$ for all $e \in \Xi$. Two hyperarcs e and e' are said to be *parallel* if $T(e) = T(e')$ and $H(e) = H(e')$. A hyperarc e is said to be a *loop* if $T(e) = H(e)$. A directed hypergraph $\Delta^* = (\Gamma, \Xi)$ is called *simple* if it has no parallel hyperarcs and loops. A directed hypergraph is called *trivial* if $|\Gamma|= 1$ and $\Xi = \phi$. If the two vertices u and v of Δ^* are such that $u \in T(e)$ and $v \in H(e)$ then we say that v is *adjacent from* u or u is *adjacent to* v. The *indegree* of a vertex v in Δ^*, denoted by $d^-(v)$ is the number of hyperarcs that contain v in their head. The *outdegree* of a vertex v in Δ^*, denoted by $d^+(v)$ is the number of hyperarcs that contain v in their tail. A directed hypergraph $\Delta' = (\Gamma', \Xi')$ is a *weak subhypergraph* of the directed hypergraph $\Delta^* = (\Gamma, \Xi)$ if $\Gamma' \subseteq \Gamma$ and Ξ' consists of hyperarcs e' with $T(e') = \{v|v \in T(e) \cap \Gamma'\}$ and $H(e') = \{v|v \in H(e) \cap \Gamma'\}$ for some $e \in \Xi$. A directed hypergraph $\Delta' = (\Gamma', \Xi')$ is a *weak induced subhypergraph* of the directed hypergraph $\Delta^* = (\Gamma, \Xi)$ if $\Gamma' \subseteq \Gamma$ and hyperarc set $\Xi' = \{(T(e) \cap \Gamma', H(e) \cap \Gamma') \mid e \in \Xi$ and $T(e) \cap \Gamma' \neq \phi$ and $H(e) \cap \Gamma' \neq \phi\}$."

In 1999 Molodtsov [11] initiated the concept of soft sets. Let U be an initial universe set and let Π be a set of parameters. A pair (F, Π) is called a *soft set* (over U) if and only F is a mapping of Π into the set of all subsets of the set U. That is, $F : \Pi \to \mathcal{P}(U)$.

3. Soft Directed Hypergraphs

Definition 3.1. Let $\Delta^* = (\Gamma, \Xi)$ be a simple directed hypergraph with vertex set Γ and directed hyperedge(hyperarc) set Ξ. Then a $e' = (T(e'), H(e'))$ where $T(e')$ and $H(e')$ are nonempty subsets of Γ, is said to be a *subhyperarc* of Δ if there exists a hyperarc e in Δ^* such that $T(e') \subseteq T(e)$ and $H(e') \subseteq H(e)$. We also say that e' is a subhyperarc of e. Clearly, a hyperarc is a subhyperarc of itself. e' is said to be a *proper subhyperarc* of e if either $T(e') \subset T(e)$ or $H(e') \subset H(e)$.

Definition 3.2. Consider $\Delta^* = (\Gamma, \Xi)$ as a simple directed hypergraph comprising a vertex set Γ and a set of directed hyperedges (or hyperarcs) Ξ, and let Π denote any nonempty set. Denote Ξ_s as the collection of all subhyperarcs of Δ^*. Let R represent an arbitrary relation between elements from Π and those from Γ. That is $R \subseteq \Pi \times \Gamma$. A mapping $\Omega : \Pi \to \mathcal{P}(\Gamma)$ can be defined as $\Omega(\pi) = \{v \in \Gamma : \pi R v\}$ where $\mathcal{P}(\Gamma)$ denotes the powerset of Γ . Also define a mapping $Y : \Pi \to \mathcal{P}(\Xi_s)$ by $\Psi(\pi) = \{(T(e) \cap \Omega(\pi), H(e) \cap \Omega(\pi)) \mid e \in \Xi \text{ and } T(e) \cap \Omega(\pi) \neq \phi \text{ and } H(e) \cap \Omega(\pi) \neq \phi\}$ where $\mathcal{P}(\Xi_s)$ denotes the powerset of Ξ_s. The pair (Ω, Π) is a soft set over Γ and the pair (Ψ, Π) is a soft set over Ξ_s. Then the 4 -tuple $\Delta = (\Delta^*, \Omega, \Psi, \Pi)$ is called a *soft directed hypergraph* if it satisfies the following conditions:

(a) $\Delta^* = (\Gamma, \Xi)$ is a simple directed hypergraph having vertex set Γ and hyperarc set Ξ,

(b) Π is a nonempty set of parameters,

(c) (Ω, Π) is a soft set over Γ,

(d) (Ψ, Π) is a soft set over Ξ_s,

(e) $(\Omega(\pi), \Psi(\pi))$ is a weak induced subhypergraph of Δ^* for all $\pi \in \Pi$.

If we represent $(\Omega(\pi), \Psi(\pi))$ by $Z(\pi)$, then the soft directed hypergraph Δ is also given by $\{Z(\pi) : \pi \in \Pi\}$. Then $Z(\pi)$ corresponding to a parameter π in Π is called a *directed hyperpart* or simply *dh-part* of the soft directed hypergraph Δ.

Example 3.3. Consider a directed hypergraph $\Delta^* = (\Gamma, \Xi)$ given in Figure 1.

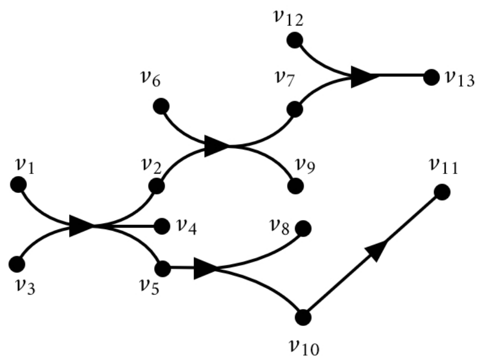

Fig. 1. Directed hypergraph $\Delta^* = (\Gamma, \Xi)$

Let $\Pi = \{v_2, v_{10}\} \subseteq \Gamma$ be a parameter set. Define a function $X : \Pi \to \mathcal{P}(\Gamma)$ defined by $\Omega(\pi) = \{v \in \Gamma : \pi R v \Leftrightarrow v = \pi \text{ or } v \text{ is adjacent from } \pi \text{ or } v \text{ is adjacent to } \pi\}$ for all $\pi \in P$. That is, $\Omega(v_2) = \{v_1, v_2, v_3, v_7, v_9\}$ and $\Omega(v_{10}) = \{v_5, v_{10}, v_{11}\}$. Then (Ω, Π) is a soft set over Γ. Define another function $\Psi : \Pi \to \mathcal{P}(\Xi_s)$ defined by $\Psi(\pi) = \{(T(e) \cap \Omega(\pi), H(e) \cap \Omega(\pi)) \mid e \in \Xi \text{ and } T(e) \cap \Omega(\pi) \neq \phi \text{ and } H(e) \cap \Omega(\pi) \neq \phi\}$. That is, $\Psi(v_2) = \{(\{v_1, v_3\}, \{v_2\}), (\{v_2\}, \{v_7, v_9\})\}$ and $v_{10}) =$

$\{(\{v_5\},\{v_{10}\}),(\{v_{10}\},\{v_{11}\})\}$. Then (Ψ,Π) is a soft set over Ξ_s. Also $Z(v_2)=(\Omega(v_2),\Psi(v_2))$ and $Z(v_{10})=(\Omega(v_{10}),\Psi(v_{10}))$ are weak induced subhypergraphs of Δ^* as shown in Figure 2. Hence $\Delta=\{Z(v_2),Z(v_{10})\}$ is a soft directed hypergraph of Δ^*.

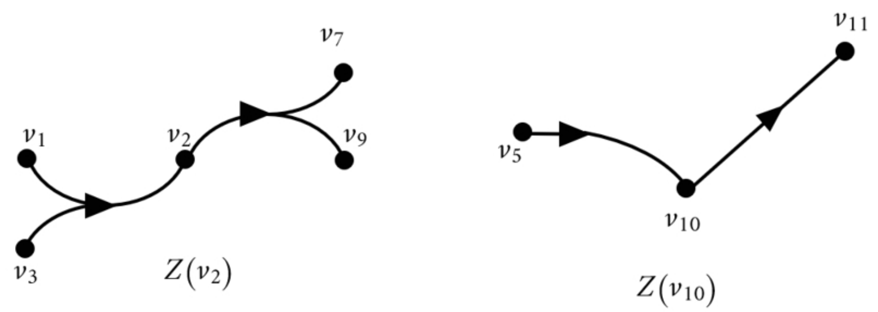

Fig. 2. Soft directed hypergraph $\Delta=\{Z(v_2),Z(v_{10})\}$

Definition 3.4. Let $\Delta^*=(\Gamma,\Xi)$ be a simple hypergraph having vertex set Γ and hyperarc set Ξ. Also let $\Delta_1=(\Delta^*,\Omega_1,\Psi_1,\Pi_1)$ and $\Delta_2=(\Delta^*,\Omega_2,\Psi_2,\Pi_2)$ be two soft directed hypergraphs of Δ^*. Then Δ_2 is a *soft weak induced subhypergraph* of Δ_1 if

(a) $\Pi_2\subseteq\Pi_1$,

(b) $Z_2(\pi)=(\Omega_2(\pi),\Psi_2(\pi))$ is a weak induced subhypergraph of $Z_1(\pi)=(\Omega_1(\pi),\Psi_1(\pi))$ for all $\pi\in\Pi_2$.

Example 3.5. Consider a directed hypergraph $\Delta^*=(\Gamma,\Xi)$ given in Figure 3.

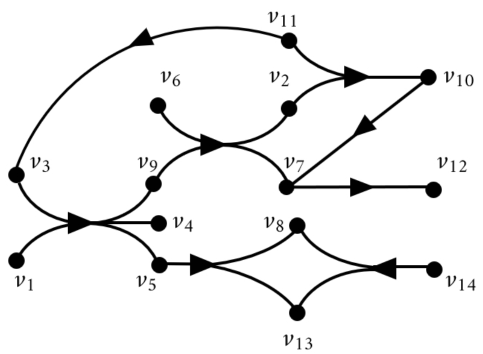

Fig. 3. Directed hypergraph $\Delta^*=(\Gamma,\Xi)$

Let $\Pi_1=\{v_3,v_7\}\subseteq\Gamma$ be a parameter set. Define a function $\Omega_1:\Pi_1\to\mathcal{P}(\Gamma)$ defined by $\Omega_1(\pi)=\{v\in\Gamma:\pi Rv\Leftrightarrow v=\pi$ or v is adjacent from π or v is adjacent to $\pi\}$ for all $\pi\in\Pi_1$. That is, $\Omega_1(v_3)=\{v_3,v_4,v_5,v_9,v_{11}\}$ and $\Omega_1(v_7)=\{v_6,v_7,v_9,v_{10},v_{12}\}$.

Then (Ω_1,Π_1) is a soft set over Γ. Define another function $\Psi_1:\Pi_1\to\mathcal{P}(\Xi_s)$ defined by $\Psi_1(\pi)=\{(T(e)\cap\Omega_1(\pi),H(e)\cap\Omega_1(\pi))\mid e\in\Xi$ and $T(e)\cap\Omega_1(\pi)\neq\phi$ and $H(e)\cap\Omega_1(\pi)\neq\phi\}$. That is, $\Psi_1(v_3)=\{(\{v_{11}\},\{v_3\}),(\{v_3\},\{v_4,v_5,v_9\})\}$ and $\Psi_1(v_7)=\{(\{v_6,v_9\},\{v_7\}),(\{v_{10}\},\{v_7\}),(\{v_7\},\{v_{12}\})\}$. Then (Ψ_1,Π_1) is a soft set over Ξ_s. Also $Z_1(v_3)=(\Omega_1(v_3),\Psi_1(v_3))$ and $Z_1(v_7)=(\Omega_1(v_7),\Psi_1(v_7))$ are weak induced subhypergraphs of Δ^* as shown in Figure 4. Hence $\Delta_1=\{Z_1(v_3),Z_1(v_7)\}$ is a soft directed hypergraph of Δ^*.

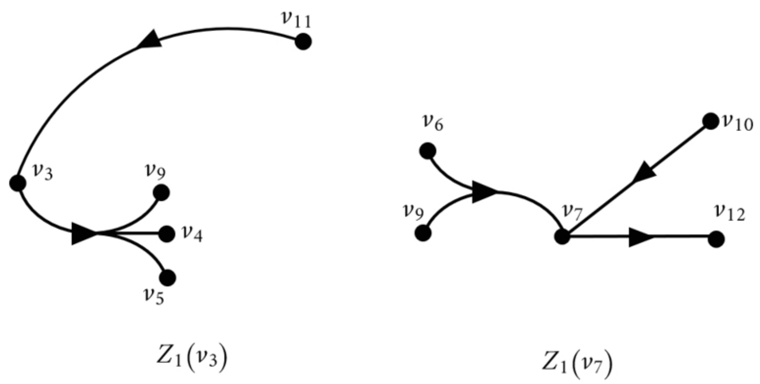

Fig. 4. Soft Directed hypergraph $\Delta_1 = \{Z_1(v_3), Z_1(v_7)\}$

Let $\Pi_2 = \{v_7\} \subseteq \Gamma$ be another parameter set. Define a function $\Omega_2 : \Pi_2 \to \mathcal{P}(\Gamma)$ defined by $\Omega_2(\pi) = \{v \in \Gamma : \pi R v \Leftrightarrow v = \pi$ or v is adjacent to $\pi\}$ for all $\pi \in \Pi_2$. That is, $\Omega_2(v_7) = \{v_6, v_7, v_9, v_{10}\}$. Then (Ω_2, Π_2) is a soft set over Γ. Define another function $\Psi_2 : \Pi_2 \to \mathcal{P}(\Xi_s)$ defined by $\Psi_2(\pi) = \{(T(e) \cap \Omega_2(\pi), H(e) \cap \Omega_2(\pi)) \mid e \in \Xi$ and $T(e) \cap \Omega_2(\pi) \neq \phi$ and $H(e) \cap \Omega_2(\pi) \neq \phi\}$. That is, $\Psi_2(v_7) = \{(\{v_6, v_9\}, \{v_7\}), (\{v_{10}\}, \{v_7\})\}$. Then (Ψ_2, Π_2) is a soft set over Ξ_s. Also $Z_2(v_7) = (\Omega_2(v_7), \Psi_2(v_7))$ is a weak induced subhypergraph of Δ^* as shown in Figure 5. Hence $\Delta_2 = \{Z_2(v_7)\}$ is a soft directed hypergraph of Δ^*.

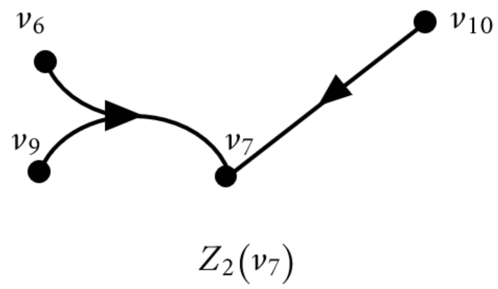

Fig. 5. Soft Directed hypergraph $\Delta_2 = \{Z_2(v_7)\}$

Here Δ_2 is a soft weak induced subhypergraph of Δ_1 since

(a) $\Pi_2 \subseteq \Pi_1$,

(b) $Z_2(v_7) = (\Omega_2(v_7), \Psi_2(v_7))$ is a weak induced subhypergraph of $Z_1(v_7) = (\Omega_1(v_7), \Psi_1(v_7))$.

4. Extended Union of Two Soft Directed Hypergraphs

Definition 4.1. Let $\Delta^* = (\Gamma, \Xi)$ be a simple directed hypergraph having vertex set Γ and hyperarc set Ξ. Also let $\Delta_1 = (\Delta^*, \Omega_1, \Psi_1, \Pi_1)$ and $\Delta_2 = (\Delta^*, \Omega_2, \Psi_2, \Pi_2)$ be two soft directed hypergraphs of Δ^* . Then the *extended union* of Δ_1 and Δ_2 denoted by $\Delta_1 \cup_E \Delta_2$ is defined as $\Delta_1 \cup_E \Delta_2 = \Delta = (\Delta^*, \Omega, \Psi, \Pi)$, where $\Pi = \Pi_1 \cup \Pi_2$ and for all $\pi \in \Pi$,

$$\Omega(\pi) = \begin{cases} \Omega_1(\pi), & \text{if } \pi \in \Pi_1 - \Pi_2, \\ \Omega_2(\pi), & \text{if } \pi \in \Pi_2 - \Pi_1, \\ \Omega_1(\pi) \cup \Omega_2(\pi), & \text{if } \pi \in \Pi_1 \cap \Pi_2. \end{cases}$$

and

$$\Psi(\pi) = \begin{cases} \Psi_1(\pi), & \text{if } \pi \in \Pi_1 - \Pi_2, \\ \Psi_2(\pi), & \text{if } \pi \in \Pi_2 - \Pi_1, \\ \{(T(e) \cap \Omega(\pi), H(e) \cap \Omega(\pi)) \mid e \in \Xi \text{ and } T(e) \cap \Omega(\pi) \neq \phi & \\ \text{ and } H(e) \cap \Omega(\pi) \neq \phi\}, & \text{if } \pi \in \Pi_1 \cap \Pi_2. \end{cases}$$

If $Z(\pi) = (\Omega(\pi), \Psi(\pi)), \forall \pi \in \Pi$, then $\Delta_1 \cup_E \Delta_2 = \{Z(\pi) : \pi \in \Pi\}$.

Example 4.2. Consider a simple directed hypergraph $\Delta^* = (\Gamma, \Xi)$ given in Figure 6. Let $\Pi_1 =$

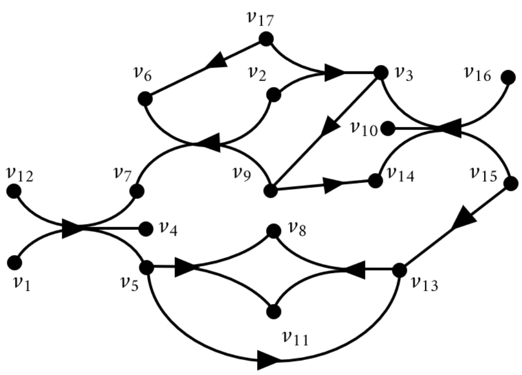

Fig. 6. Directed hypergraph $\Delta^* = (\Gamma, \Xi)$

$\{v_7, v_9\} \subseteq \Gamma$ be a parameter set. Define a function $\Omega_1 : \Pi_1 \to \mathcal{P}(\Gamma)$ defined by $\Omega_1(\pi) = \{v \in \Gamma : \pi R v \Leftrightarrow v = \pi$ or v is adjacent from π or v is adjacent to $\pi\}$, for all $\pi \in \Pi_1$. That is, $\Omega_1(v_7) = \{v_1, v_2, v_7, v_9, v_{12}\}$ and $\Omega_1(v_9) = \{v_3, v_6, v_7, v_9, v_{14}\}$. Then (Ω_1, Π_1) is a soft set over Γ. Define another function $\Psi_1 : \Pi_1 \to \mathcal{P}(\Xi_s)$ defined by $\Psi_1(\pi) = \{(T(e) \cap \Omega_1(\pi), H(e) \cap \Omega_1(\pi)) \mid e \in \Xi$ and $T(e) \cap \Omega_1(\pi) \neq \phi$ and $H(e) \cap \Omega_1(\pi) \neq \phi\}$. That is, $\Psi_1(v_7) = \{(\{v_1, v_{12}\}, \{v_7\}), (\{v_2, v_9\}, \{v_7\})\}$ and $\Psi_1(v_9) = \{(\{v_9\}, \{v_6, v_7\}), (\{v_3\}, \{v_9\}), (\{v_9\}, \{v_{14}\})\}$. Then (Ψ_1, Π_1) is a soft set over Ξ_s. Also $Z_1(v_7) = (\Omega_1(v_7), \Psi_1(v_7))$ and $Z_1(v_9) = (\Omega_1(v_9), \Psi_1(v_9))$ are weak induced subhypergraphs of Δ^* as shown in Figure 7.

Hence $\Delta_1 = \{Z_1(v_7), Z_1(v_9)\}$ is a soft directed hypergraph of Δ^*.

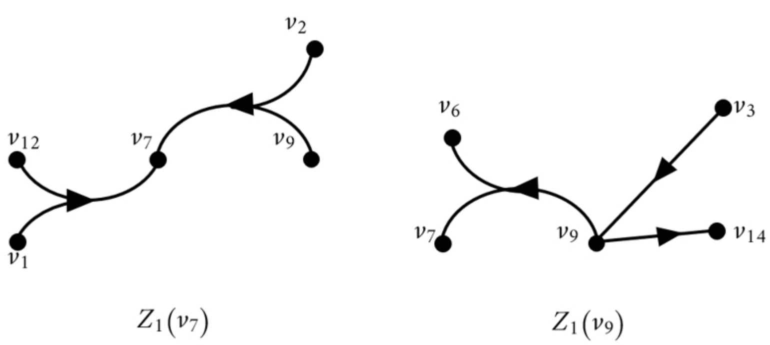

Fig. 7. Soft Directed hypergraph $\Delta_1 = \{Z_1(v_7), Z_1(v_9)\}$

Let $\Pi_2 = \{v_9, v_{15}\} \subseteq \Gamma$ be another parameter set. Define a function $\Omega_2 : \Pi_2 \to \mathcal{P}(\Gamma)$ defined by $\Omega_2(\pi) = \{v \in \Gamma : \pi R v \Leftrightarrow v = \pi$ or v is adjacent from $\pi\}$, for all $\pi \in \Pi_2$. That is,

$\Omega_2(v_9) = \{v_6, v_7, v_9, v_{14}\}$ and $\Omega_2(v_{15}) = \{v_3, v_{10}, v_{13}, v_{14}, v_{15}\}$. Then (Ω_2, Π_2) is a soft set over Γ. Define another function $\Psi_2 : \Pi_2 \to \mathcal{P}(\Xi_s)$ defined by $\Psi_2(\pi) = \{(T(e) \cap \Omega_2(\pi), H(e) \cap \Omega_2(\pi)) \mid e \in \Xi$ and $T(e) \cap \Omega_2(\pi) \neq \phi$ and $H(e) \cap \Omega_2(\pi) \neq \phi\}$. That is, $\Psi_2(v_9) = \{(\{v_9\}, \{v_6, v_7\}), (\{v_9\}, \{v_{14}\})\}$ and $\Psi_2(v_{15}) = \{(\{v_{15}\}, \{v_3, v_{10}, v_{14}\}), (\{v_{15}\}, \{v_{13}\})\}$. Then (Ψ_2, Π_2) is a soft set over Ξ_s. Also $Z_2(v_9) = (\Omega_2(v_9), \Psi_2(v_9))$ and $Z_2(v_{15}) = (\Omega_2(v_{15}), \Psi_2(v_{15}))$ are weak induced subhypergraphs of Δ^* as shown in Figure 8.

Hence $\Delta_2 = \{Z_2(v_9), Z_2(v_{15})\}$ is a soft directed hypergraph of Δ^*.

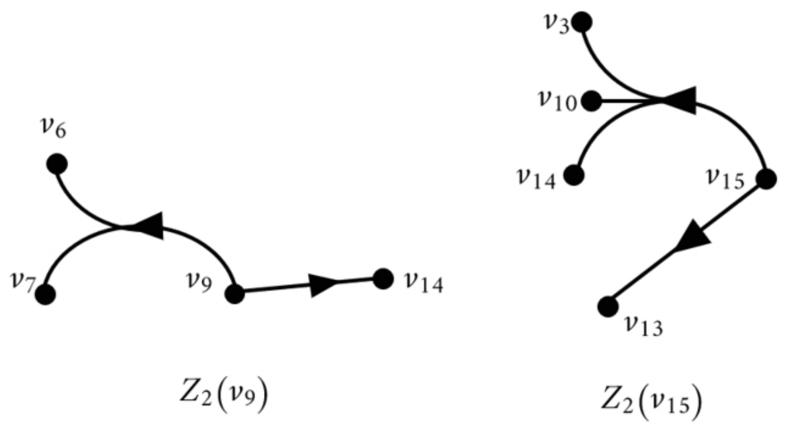

Fig. 8. Soft Directed Hypergraph $\Delta_2 = \{Z_2(v_9), Z_2(v_{15})\}$

The extended union of two soft directed hypergraphs Δ_1 and Δ_2 is $\Delta = \Delta_1 \cup_E \Delta_2 = (\Delta^*, \Omega, \Psi, \Pi)$ where $\Pi = \Pi_1 \cup \Pi_2 = \{v_7, v_9, v_{15}\}$. Also $\Omega(v_7) = \Omega_1(v_7) = \{v_1, v_2, v_7, v_9, v_{12}\}$, $\Psi(v_7) = \Psi_1(v_7) = \{(\{v_1, v_{12}\}, \{v_7\}), (\{v_2, v_9\}, \{v_7\})\}$, $\Omega(v_9) = \Omega_1(v_9) \cup \Omega_2(v_9) = \Omega_1(v_9) = \{v_3, v_6, v_7, v_9, v_{14}\}$, $\Psi(v_9) = \{(T(e) \cap \Omega(v_9), H(e) \cap \Omega(v_9)) \mid e \in \Xi$ and $T(e) \cap \Omega(v_9) \neq \phi$ and $H(e) \cap \Omega(v_9) \neq \phi\} = \{(\{v_9\}, \{v_6, v_7\}), (\{v_3\}, \{v_9\}), (\{v_9\}, \{v_{14}\})\}$, $\Omega(v_{15}) = \Omega_2(v_{15}) = \{v_3, v_{10}, v_{13}, v_{14}, v_{15}\}$ and $\Psi(v_{15}) = \Psi_2(v_{15}) = \{(\{v_{15}\}, \{v_3, v_{10}, v_{14}\}), (\{v_{15}\}, \{v_{13}\})\}$. Here (Ω, Π) is a soft set over Γ and (Ψ, Π) is a soft set over Ξ_s. Also $Z(v_7) = (\Omega(v_7), \Psi(v_7)), Z(v_9) = (\Omega(v_9), \Psi(v_9))$ and $Z(v_{15}) = (\Omega(v_{15}), \Psi(v_{15}))$ are weak induced subhypergraphs of Δ^*. Hence $\Delta_1 \cup_E \Delta_2 = \{Z(v_7), Z(v_9), Z(v_{15})\}$ is a soft directed hypergrah of Δ^* and is given in Figure 9.

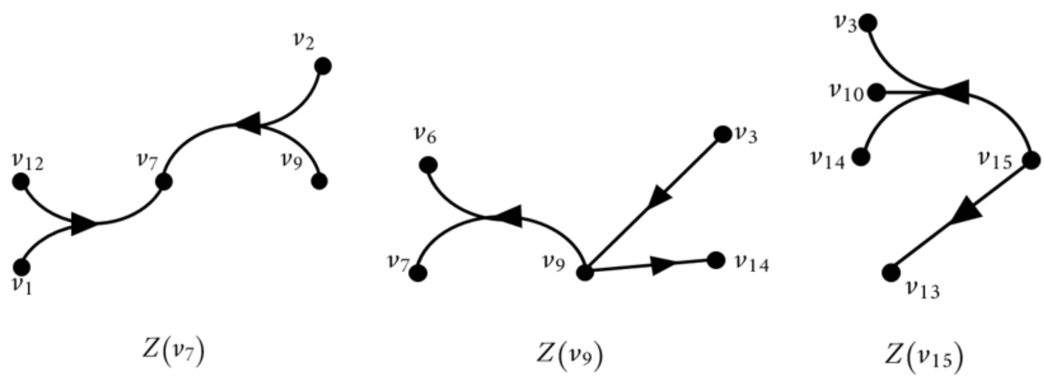

Fig. 9. $\Delta_1 \cup_E \Delta_2 = \{Z(v_7), Z(v_9), Z(v_{15})\}$

Theorem 4.3. *Let $\Delta^* = (\Gamma, \Xi)$ be a simple directed hypergraph having vertex set Γ and hyperarc set*

Ξ. Also let $\Delta_1 = (\Delta^*, \Omega_1, \Psi_1, \Pi_1)$ and $\Delta_2 = (\Delta^*, \Omega_2, \Psi_2, \Pi_2)$ be two soft directed hypergraphs of Δ^*. Then their extended union $\Delta_1 \cup_E \Delta_2$ is also a soft directed hypergraph of Δ^*.

Proof. The extended union of Δ_1 and Δ_2 is given by $\Delta_1 \cup_E \Delta_2 = \Delta = (\Delta^*, \Omega, \Psi, \Pi)$, where $\Pi = \Pi_1 \cup \Pi_2$ and for all $\pi \in \Pi$,

$$\Omega(\pi) = \begin{cases} \Omega_1(\pi), & \text{if } \pi \in \Pi_1 - \Pi_2, \\ \Omega_2(\pi), & \text{if } \pi \in \Pi_2 - \Pi_1, \\ \Omega_1(\pi) \cup \Omega_2(\pi), & \text{if } \pi \in \Pi_1 \cap \Pi_2. \end{cases}$$

and

$$\Psi(\pi) = \begin{cases} \Psi_1(\pi), & \text{if } \pi \in \Pi_1 - \Pi_2, \\ \Psi_2(\pi), & \text{if } \pi \in \Pi_2 - \Pi_1, \\ \{(T(e) \cap \Omega(\pi), H(e) \cap \Omega(\pi)) \mid e \in \Xi \text{ and } T(e) \cap \Omega(\pi) \neq \phi \text{ and} & \\ H(e) \cap \Omega(\pi) \neq \phi\}, & \text{if } \pi \in \Pi_1 \cap \Pi_2. \end{cases}.$$

That is, in $\Delta_1 \cup_E \Delta_2$, $\Pi = \Pi_1 \cup \Pi_2$ is a parameter set, Ω is a mapping from Π to $\mathcal{P}(\Gamma)$ and Ψ is a mapping from Π to $\mathcal{P}(\Xi_s)$. Here (Ω, Π) is a soft set over Γ and (Ψ, Π) is a soft set over Ξ_s. When $\pi \in \Pi_1 - \Pi_2$, the corresponding dh-part $Z(\pi)$ of $\Delta_1 \cup_E \Delta_2$ is $Z(\pi) = (\Omega_1(\pi), \Psi_1(\pi))$. This is a weak induced subhypergraph of Δ^* since Δ_1 is a soft directed hypergraph of Δ^*. When $\pi \in \Pi_2 - \Pi_1$, the corresponding dh-part of $\Delta_1 \cup_E \Delta_2$ is $Z(\pi) = (\Omega_2(\pi), \Psi_2(\pi))$. This is a weak induced subhypergraph of Δ^* since Δ_2 is a soft directed hypergraph of Δ^*. When $\pi \in \Pi_1 \cap \Pi_2$, the corresponding dh-part of $\Delta_1 \cup_E \Delta_2$ is $Z(\pi) = (\Omega(\pi), \Psi(\pi))$ where $\Omega(\pi) = \Omega_1(\pi) \cup \Omega_2(\pi)$ and $\Psi(\pi) = \{(T(e) \cap \Omega(\pi), H(e) \cap \Omega(\pi)) \mid e \in \Xi \text{ and } T(e) \cap \Omega(\pi) \neq \phi \text{ and } H(e) \cap \Omega(\pi) \neq \phi\}$. We have $\Omega_1(\pi) \cup \Omega_2(\pi) \subseteq \Gamma$ and each hyperarc in $\Psi(\pi)$ is a subhyperarc of Δ^* for all $\pi \in \Pi_1 \cap \Pi_2$. So $Z(\pi)$ is a weak induced subhypergraph of Δ^* for all $\pi \in \Pi_1 \cap \Pi_2$. That is, $Z(\pi) = (\Omega(\pi), \Psi(\pi))$ is a weak induced subhypergraph of Δ^* for all $\pi \in \Pi = \Pi_1 \cup \Pi_2$. That is, $\Delta_1 \cup_E \Delta_2 = (\Delta^*, \Omega, \Psi, \Pi)$ is a soft directed hypergraph of Δ^* since the following conditions are satisfied:

(a) $\Delta^* = (\Gamma, \Xi)$ is a simple directed hypergraph,

(b) $\Pi = \Pi_1 \cup \Pi_2$ is a nonempty set of parameters,

(c) (Ω, Π) is a soft set over Γ,

(d) (Ψ, Π) is a soft set over Ξ_s,

(e) $(\Omega(\pi), \Psi(\pi))$ is a weak induced subhypergraph of Δ^* for all $\pi \in \Pi = \Pi_1 \cup \Pi_2$.

□

5. Extended Intersection of Two Soft Directed Hypergraphs

Definition 5.1. Consider $\Delta^* = (\Gamma, \Xi)$ as a simple directed hypergraph with a vertex set Γ and a set of hyperarcs Ξ. Let $\Delta_1 = (\Delta^*, \Omega_1, \Psi_1, \Pi_1)$ and $\Delta_2 = (\Delta^*, \Omega_2, \Psi_2, \Pi_2)$ represent two soft directed hypergraphs derived from Δ^*, with the condition that $\Omega_1(\pi) \cap \Omega_2(\pi) \neq \emptyset$ for all $\pi \in \Pi_1 \cap \Pi_2$. Then, the extended intersection of Δ_1 and Δ_2, denoted by $\Delta_1 \cap_E \Delta_2$, is defined as $\Delta_1 \cap_E \Delta_2 = \Delta = (\Delta^*, \Omega, \Psi, \Pi)$, where $\Pi = \Pi_1 \cup \Pi_2$ and for all $\pi \in \Pi$,

$$\Omega(\pi) = \begin{cases} \Omega_1(\pi), & \text{if } \pi \in \Pi_1 - \Pi_2, \\ \Omega_2(\pi), & \text{if } \pi \in \Pi_2 - \Pi_1, \\ \Omega_1(\pi) \cap \Omega_2(\pi), & \text{if } \pi \in \Pi_1 \cap \Pi_2. \end{cases}$$

and

$$\Psi(\pi) = \begin{cases} \Psi_1(\pi), & \text{if } \pi \in \Pi_1 - \Pi_2, \\ \Psi_2(\pi), & \text{if } \pi \in \Pi_2 - \Pi_1, \\ \{(T(e) \cap \Omega(\pi), H(e) \cap \Omega(\pi)) \mid e \in \Xi \text{ and } T(e) \cap \Omega(\pi) \neq \phi \text{ and} & \\ H(e) \cap \Omega(\pi) \neq \phi\}, & \text{if } \pi \in \Pi_1 \cap \Pi_2. \end{cases}.$$

If $Z(\pi) = (\Omega(\pi), \Psi(\pi)), \forall \pi \in \Pi$, then $\Delta_1 \cap_E \Delta_2 = \{Z(\pi) : \pi \in \Pi\}$.

Example 5.2. Examine a simple directed hypergraph $\Delta^* = (\Gamma, \Xi)$ depicted in Figure 6, along with its corresponding soft directed hypergraphs Δ_1 illustrated in Figure 7 and Δ_2 depicted in Figure 8. The extended intersection of these two soft directed hypergraphs Δ_1 and Δ_2 is $\Delta = \Delta_1 \cap_E \Delta_2 = (\Delta^*, \Omega, \Psi, \Pi)$ where $\Pi = \Pi_1 \cup \Pi_2 = \{v_7, v_9, v_{15}\}$. Also $\Omega(v_7) = \Omega_1(v_7) = \{v_1, v_2, v_7, v_9, v_{12}\}$, $\Psi(v_7) = \Psi_1(v_7) = \{(\{v_1, v_{12}\}, \{v_7\}), (\{v_2, v_9\}, \{v_7\})\}$, $\Omega(v_9) = \Omega_1(v_9) \cap \Omega_2(v_9) = \Omega_2(v_9) = \{v_6, v_7, v_9, v_{14}\}$, $\Psi(v_9) = \{(T(e) \cap \Omega(v_9), H(e) \cap \Omega(v_9)) \mid e \in \Xi \text{ and } T(e) \cap \Omega(v_9) \neq \phi \text{ and } H(e) \cap \Omega(v_9) \neq \phi\} = \{(\{v_9\}, \{v_6, v_7\}), (\{v_9\}, \{v_{14}\})\}$, $\Omega(v_{15}) = \Omega_2(v_{15}) = \{v_3, v_{10}, v_{13}, v_{14}, v_{15}\}$ and $\Psi(v_{15}) = \Psi_2(v_{15}) = \{(\{v_{15}\}, \{v_3, v_{10}, v_{14}\}), (\{v_{15}\}, \{v_{13}\})\}$. Here (Ω, Π) is a soft set over Γ and (Ψ, Π) is a soft set over Ξ_s. Also $Z(v_7) = (\Omega(v_7), \Psi(v_7)), Z(v_9) = (\Omega(v_9), \Psi(v_9))$ and $Z(v_{15}) = (\Omega(v_{15}), \Psi(v_{15}))$ are weak induced subhypergraphs of Δ^*. Hence $\Delta_1 \cap_E \Delta_2 = \{Z(v_7), Z(v_9), Z(v_{15})\}$ is a soft directed hypergrah of Δ^* and is given in Figure 10.

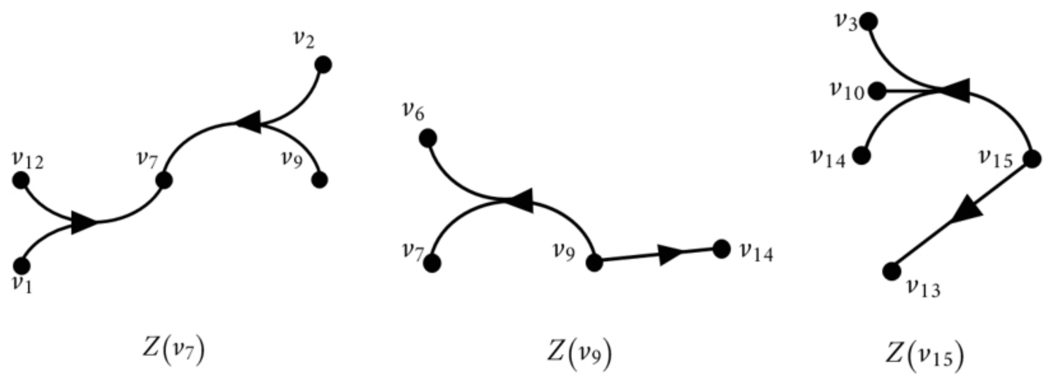

Fig. 10. $\Delta_1 \cap_E \Delta_2 = \{Z(v_7), Z(v_9), Z(v_{15})\}$

Theorem 5.3. *Let $\Delta^* = (\Gamma, \Xi)$ be a simple directed hypergraph having vertex set Γ and hyperarc set Ξ. Also let $\Delta_1 = (\Delta^*, \Omega_1, \Psi_1, \Pi_1)$ and $\Delta_2 = (\Delta^*, \Omega_2, \Psi_2, \Pi_2)$ be two soft hypergraphs of Δ^* such that $\Omega_1(\pi) \cap \Omega_2(\pi) \neq \phi$ for all $\pi \in \Pi_1 \cap \Pi_2$. Then their extended intersection $\Delta_1 \cap_E \Delta_2$ is also a soft directed hypergraph of Δ^*.*

Proof. The extended intersection of Δ_1 and Δ_2 is given by $\Delta_1 \cap_E \Delta_2 = \Delta = (\Delta^*, \Omega, \Psi, \Pi)$, where

$\Pi = \Pi_1 \cup \Pi_2$ and for all $\pi \in \Pi$,

$$\Omega(\pi) = \begin{cases} \Omega_1(\pi), & \text{if } \pi \in \Pi_1 - \Pi_2, \\ \Omega_2(\pi), & \text{if } \pi \in \Pi_2 - \Pi_1, \\ \Omega_1(\pi) \cap \Omega_2(\pi), & \text{if } \pi \in \Pi_1 \cap \Pi_2. \end{cases}$$

and

$$\Psi(\pi) = \begin{cases} \Psi_1(\pi), & \text{if } \pi \in \Pi_1 - \Pi_2, \\ \Psi_2(\pi), & \text{if } \pi \in \Pi_2 - \Pi_1, \\ \{(T(e) \cap \Omega(\pi), H(e) \cap \Omega(\pi)) \mid e \in \Xi \text{ and } T(e) \cap \Omega(\pi) \neq \phi \text{ and} & \\ H(e) \cap \Omega(\pi) \neq \phi\}, & \text{if } \pi \in \Pi_1 \cap \Pi_2. \end{cases}.$$

That is, in $\Delta_1 \cap_E \Delta_2$, $\Pi = \Pi_1 \cup \Pi_2$ is a parameter set, Ω is a mapping from Π to $\mathcal{P}(\Gamma)$ and Ψ is a mapping from Π to $\mathcal{P}(\Xi_s)$. Here (Ω, Π) is a soft set over Γ and (Ψ, Π) is a soft set over Ξ_s. When $\pi \in \Pi_1 - \Pi_2$, the corresponding dh-part $Z(\pi)$ of $\Delta_1 \cap_E \Delta_2$ is $Z(\pi) = (\Omega_1(\pi), \Psi_1(\pi))$. This is a weak induced subhypergraph of Δ^* since Δ_1 is a soft directed hypergraph of Δ^*. When $\pi \in \Pi_2 - \Pi_1$, the corresponding dh-part of $\Delta_1 \cap_E \Delta_2$ is $Z(\pi) = (\Omega_2(\pi), \Psi_2(\pi))$. This is a weak induced subhypergraph of Δ^* since Δ_2 is a soft directed hypergraph of Δ^*. When $\pi \in \Pi_1 \cap \Pi_2$, the corresponding dh-part of $\Delta_1 \cap_E \Delta_2$ is $Z(\pi) = (\Omega(\pi), \Psi(\pi))$ where $\Omega(\pi) = \Omega_1(\pi) \cap \Omega_2(\pi)$ and $\Psi(\pi) = \{(T(e) \cap \Omega(\pi), H(e) \cap \Omega(\pi)) \mid e \in \Xi \text{ and } T(e) \cap \Omega(\pi) \neq \phi \text{ and } H(e) \cap \Omega(\pi) \neq \phi\}$. We have $\Omega_1(\pi) \cap \Omega_2(\pi) \subseteq \Gamma$ and each hyperarc in $\Psi(\pi)$ is a subhyperarc of Δ^* for all $\pi \in \Pi_1 \cap \Pi_2$. So $Z(\pi)$ is a weak induced subhypergraph of Δ^* for all $\pi \in \Pi_1 \cap \Pi_2$. That is, $Z(\pi) = (\Omega(\pi), \Psi(\pi))$ is a weak induced subhypergraph of Δ^*, for all $\pi \in \Pi = \Pi_1 \cup \Pi_2$. That is, $\Delta_1 \cap_E \Delta_2 = (\Delta^*, \Omega, \Psi, \Pi)$ is a soft directed hypergraph of Δ^* since the following conditions are satisfied:

(a) $\Delta^* = (\Gamma, \Xi)$ is a simple directed hypergraph,

(b) $\Pi = \Pi_1 \cup \Pi_2$ is a nonempty set of parameters,

(c) (Ω, Π) is a soft set over Γ,

(d) (Ψ, Π) is a soft set over Ξ_s,

(e) $(\Omega(\pi), \Psi(\pi))$ is a weak induced subhypergraph of Δ^* for all $\pi \in \Pi$.

□

Theorem 5.4. *Let $\Delta^* = (\Gamma, \Xi)$ be a simple directed hypergraph and $\Delta_1 = (\Delta^*, \Omega_1, \Psi_1, \Pi_1)$ and $\Delta_2 = (\Delta^*, \Omega_2, \Psi_2, \Pi_2)$ be two soft directed hypergraphs of Δ^* such that $\Omega_1(\pi) \cap \Omega_2(\pi) \neq \phi$ for all $\pi \in \Pi_1 \cap \Pi_2$. Then $\Delta_1 \cap_E \Delta_2$ is a soft weak induced subhypergraph of $\Delta_1 \cup_E \Delta_2$.*

Proof. By Theorems 4.3 and 5.3, we have $\Delta_1 \cup_E \Delta_2$ and $\Delta_1 \cap_E \Delta_2$ are soft directed hypergraphs of Δ^*. Assume that $\Delta_1 \cup_E \Delta_2 = \Delta_{EU} = (\Delta^*, \Omega_{EU}, \Psi_{EU}, \Pi_{EU})$ and $\Delta_1 \cap_E \Delta_2 = \Delta_{EI} = (\Delta^*, \Omega_{EI}, \Psi_{EI}, \Pi_{EI})$. By the definitions of extended union and the extended intersection of two soft directed hypergraphs, $\Pi_{EU} = \Pi_{EI} = \Pi_1 \cup \Pi_2$. Therefore we have $\Pi_{EI} \subseteq \Pi_{EU}$.

We divide the parameter set $\Pi_{EI} = \Pi_1 \cup \Pi_2$ into three parts: (i) $\Pi_1 - \Pi_2$ (ii) $\Pi_2 - \Pi_1$ (iii) $\Pi_1 \cap \Pi_2$. We consider the three cases one by one.

(i) If $\pi \in \Pi_1 - \Pi_2$, the corresponding dh-parts $Z_{EI}(\pi) = (\Omega_{EI}(\pi), \Psi_{EI}(\pi))$ and $Z_{EU}(\pi) = (\Omega_{EU}(\pi), \Psi_{EU}(\pi))$ of Δ_{EI} and Δ_{EU} respectively are equal to $Z_1(\pi) = (\Omega_1(\pi), \Psi_1(\pi))$. That is, $Z_{EI}(\pi)$ is a weak induced subhypergraph of $Z_{EU}(\pi), \forall \pi \in \Pi_1 - \Pi_2$, since both dh-parts are identical.

(ii) If $\pi \in \Pi_2 - \Pi_1$, the corresponding dh-parts $Z_{EI}(\pi) = (\Omega_{EI}(\pi), \Psi_{EI}(\pi))$ and $Z_{EU}(\pi) = (\Omega_{EU}(\pi), \Psi_{EU}(\pi))$ of Δ_{EI} and Δ_{EU} respectively are equal to $Z_2(\pi) = (\Omega_2(\pi), \Psi_2(\pi))$. That is, $Z_{EI}(\pi)$ is a weak induced subhypergraph of $Z_{EU}(\pi), \forall \pi \in \Pi_2 - \Pi_1$, since both dh-parts are identical.

(iii) If $\pi \in \Pi_1 \cap \Pi_2$, $Z_{EI}(\pi) = (\Omega_{EI}(\pi), \Psi_{EI}(\pi))$, where $\Omega_{EI}(\pi) = \Omega_1(\pi) \cap \Omega_2(\pi)$ and $\Psi_{EI}(\pi) = \{(T(e) \cap \Omega_{EI}(\pi), H(e) \cap \Omega_{EI}(\pi)) \mid e \in \Xi \text{ and } T(e) \cap \Omega_{EI}(\pi) \neq \phi \text{ and } H(e) \cap \Omega_{EI}(\pi) \neq \phi\}$ and $Z_{EU}(\pi) = (\Omega_{EU}(\pi), \Psi_{EU}(\pi))$, where $\Omega_{EU}(\pi) = \Omega_1(\pi) \cup \Omega_2(\pi)$ and $\Psi_{EU}(\pi) = \{(T(e) \cap \Omega_{EU}(\pi), H(e) \cap \Omega_{EU}(\pi)) \mid e \in \Xi \text{ and } T(e) \cap \Omega_{EU}(\pi) \neq \phi \text{ and } H(e) \cap \Omega_{EU}(\pi) \neq \phi\}$. Clearly $\Omega_{EI}(\pi) \subseteq \Omega_{EU}(\pi)$ and each hyperarc present in $\Psi_{EI}(\pi)$ is a subhyperarc of a hyperarc present in $\Psi_{EU}(\pi)$. So $Z_{EI}(\pi)$ is a weak induced subhypergraph of $Z_{EU}(\pi), \forall \pi \in \Pi_2 \cap \Pi_1$.

That is, we have

(a) $\Pi_{EI} \subseteq \Pi_{EU}$,

(b) For all $\pi \in \Pi_{EI}$, $Z_{EI}(\pi) = (\Omega_{EI}(\pi), \Psi_{EI}(\pi))$ is a weak induced subhypergraph of $Z_{EU}(\pi) = (\Omega_{EU}(\pi), \Psi_{EU}(\pi))$.

Hence $\Delta_1 \cap_E \Delta_2$ is a soft weak induced subhypergraph of $\Delta_1 \cup_E \Delta_2$. □

6. Restricted Union of Two Soft Directed Hypergraphs

Definition 6.1. Consider $\Delta^* = (\Gamma, \Xi)$ as a simple directed hypergraph with a vertex set Γ and a set of hyperarcs Ξ. Let $\Delta_1 = (\Delta^*, \Omega_1, \Psi_1, \Pi_1)$ and $\Delta_2 = (\Delta^*, \Omega_2, \Psi_2, \Pi_2)$ represent two soft directed hypergraphs derived from Δ^* such that $\Pi_1 \cap \Pi_2 \neq \emptyset$. Then, the restricted union of Δ_1 and Δ_2, denoted by $\Delta_1 \cup_R \Delta_2$, is defined as $\Delta_1 \cup_R \Delta_2 = \Delta = (\Delta^*, \Omega, \Psi, \Pi)$, where $\Pi = \Pi_1 \cap \Pi_2$ and for all $\pi \in \Pi$, $\Omega(\pi) = \Omega_1(\pi) \cup \Omega_2(\pi)$ and $\Psi(\pi) = \{(T(e) \cap \Omega(\pi), H(e) \cap \Omega(\pi)) \mid e \in \Xi \text{ and } T(e) \cap \Omega(\pi) \neq \phi \text{ and } H(e) \cap \Omega(\pi) \neq \phi\}$. If $Z(\pi) = (\Omega(\pi), \Psi(\pi)), \forall \pi \in \Pi$, then $\Delta_1 \cup_R \Delta_2 = \{Z(\pi) : \pi \in \Pi\}$.

Example 6.2. Examine a simple directed hypergraph $\Delta^* = (\Gamma, \Xi)$ as illustrated in Figure 6, along with its soft directed hypergraphs Δ_1 depicted in Figure 7 and Δ_2 shown in Figure 8. The restricted union of these two soft directed hypergraphs Δ_1 and Δ_2 is $\Delta = \Delta_1 \cup_R \Delta_2 = (\Delta^*, \Omega, \Psi, \Pi)$ where $\Pi = \Pi_1 \cap \Pi_2 = \{v_9\}$. Also $\Omega(v_9) = \Omega_1(v_9) \cup \Omega_2(v_9) = \{v_3, v_6, v_7, v_9, v_{14}\}$, $\Psi(v_9) = \{(T(e) \cap \Omega(v_9), H(e) \cap \Omega(v_9)) \mid e \in \Xi \text{ and } T(e) \cap \Omega(v_9) \neq \phi \text{ and } H(e) \cap \Omega(v_9) \neq \phi\} = \{(\{v_9\}, \{v_6, v_7\}), (\{v_3\}, \{v_9\}), (\{v_9\}, \{v_{14}\})\}$ Here (Ω, Π) is a soft set over Γ and (Ψ, Π) is a soft set over Ξ_s. Also $Z(v_9) = (\Omega(v_9), \Psi(v_9))$ is a weak induced subhypergraph of Δ^*. Hence $\Delta_1 \cup_R \Delta_2 = \{Z(v_9)\}$ is a soft directed hypergrah of Δ^* and is given in Figure 11.

Theorem 6.3. *Let $\Delta^* = (\Gamma, \Xi)$ be a simple directed hypergraph and $\Delta_1 = (\Delta^*, \Omega_1, \Psi_1, \Pi_1)$ and $\Delta_2 = (\Delta^*, \Omega_2, \Psi_2, \Pi_2)$ be two soft directed hypergraphs of Δ^* such that $\Pi_1 \cap \Pi_2 \neq \phi$. Then their restricted union $\Delta_1 \cup_R \Delta_2$ is also a soft directed hypergraph of Δ^*.*

Proof. The restricted union $\Delta_1 \cup_R \Delta_2$ is defined as $\Delta_1 \cup_R \Delta_2 = \Delta = (\Delta^*, \Omega, \Psi, \Pi)$, where $\Pi = \Pi_1 \cap \Pi_2 \neq \phi$ is the parameter set and for all $\pi \in \Pi$, $\Omega(\pi) = \Omega_1(\pi) \cup \Omega_2(\pi)$ and $\Psi(\pi) = \{(T(e) \cap \Omega(\pi), H(e) \cap \Omega(\pi)) \mid e \in \Xi \text{ and } T(e) \cap \Omega(\pi) \neq \phi \text{ and } H(e) \cap \Omega(\pi) \neq \phi\}$. Here Ω is a mapping from Π to $\mathcal{P}(\Gamma)$ and Ψ is a mapping from Π to $\mathcal{P}(\Xi_s)$. Also (Ω, Π) is a soft set over Γ and (Ψ, Π) is a soft

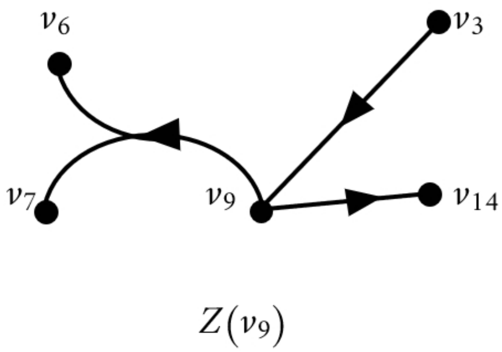

Fig. 11. $\Delta_1 \cup_R \Delta_2 = \{Z(v_9)\}$

set over Ξ_s. When $\pi \in \Pi = \Pi_1 \cap \Pi_2$, the corresponding dh-part of $\Delta_1 \cup_R \Delta_2$ is $Z(\pi) = (\Omega(\pi), \Psi(\pi))$ where $\Omega(\pi) = \Omega_1(\pi) \cup \Omega_2(\pi)$ and $\Psi(\pi) = \{(T(e) \cap \Omega(\pi), H(e) \cap \Omega(\pi)) \mid e \in \Xi \text{ and } T(e) \cap \Omega(\pi) \neq \phi \text{ and } H(e) \cap \Omega(\pi) \neq \phi\}$. We have $\Omega_1(\pi) \cup \Omega_2(\pi) \subseteq \Gamma$ and each hyperarc in $\Psi(\pi)$ is a subhyperarc of Δ^* for all $\pi \in \Pi = \Pi_1 \cap \Pi_2$. So $Z(\pi) = (\Omega(\pi), \Psi(\pi))$ is a weak induced subhypergraph of Δ^* for all $\pi \in \Pi = \Pi_1 \cup \Pi_2$. That is, $\Delta_1 \cup_R \Delta_2 = (\Delta^*, \Omega, \Psi, \Pi)$ is a soft directed hypergraph of Δ^* since all the conditions for a soft directed hypergraph are satisfied. □

Theorem 6.4. *Let $\Delta^* = (\Gamma, \Xi)$ be a simple directed hypergraph and $\Delta_1 = (\Delta^*, \Omega_1, \Psi_1, \Pi_1)$ and $\Delta_2 = (\Delta^*, \Omega_2, \Psi_2, \Pi_2)$ be two soft directed hypergraphs of Δ^* such that $\Pi_1 \cap \Pi_2 \neq \phi$. Then $\Delta_1 \cup_R \Delta_2$ is a soft weak induced subhypergraph of $\Delta_1 \cup_E \Delta_2$.*

Proof. By Theorems 4.3 and 6.3, we have $\Delta_1 \cup_E \Delta_2$ and $\Delta_1 \cup_R \Delta_2$ are soft directed hypergraphs of Δ^*. Assume that $\Delta_1 \cup_E \Delta_2 = \Delta_{EU} = (\Delta^*, \Omega_{EU}, \Psi_{EU}, \Pi_{EU})$ and $\Delta_1 \cup_R \Delta_2 = \Delta_{RU} = (\Delta^*, \Omega_{RU}, \Psi_{RU}, \Pi_{RU})$. By the definitions of extended union and restricted union of two soft directed hypergraphs, the parameter set Π_{EU} of Δ_{EU} is $\Pi_1 \cup \Pi_2$ and the parameter set Π_{RU} of Δ_{RU} is $\Pi_1 \cap \Pi_2$. Clearly we have $\Pi_{RU} \subseteq \Pi_{EU}$ since $\Pi_1 \cap \Pi_2 \subseteq \Pi_1 \cup \Pi_2$. If $\pi \in \Pi_{RU} = \Pi_1 \cap \Pi_2$, $Z_{RU}(\pi) = (\Omega_{RU}(\pi), \Psi_{RU}(\pi))$, where $\Omega_{RU}(\pi) = \Omega_1(\pi) \cup \Omega_2(\pi)$ and $\Psi_{RU}(\pi) = \{(T(e) \cap \Omega_{RU}(\pi), H(e) \cap \Omega_{RU}(\pi)) \mid e \in \Xi \text{ and } T(e) \cap \Omega_{RU}(\pi) \neq \phi \text{ and } H(e) \cap \Omega_{RU}(\pi) \neq \phi\}$ and $Z_{EU}(\pi) = (\Omega_{EU}(\pi), \Psi_{EU}(\pi))$, where $\Omega_{EU}(\pi) = \Omega_1(\pi) \cup \Omega_2(\pi)$ and $\Psi_{EU}(\pi) = \{(T(e) \cap \Omega_{EU}(\pi), H(e) \cap \Omega_{EU}(\pi)) \mid e \in \Xi \text{ and } T(e) \cap \Omega_{EU}(\pi) \neq \phi \text{ and } H(e) \cap \Omega_{EU}(\pi) \neq \phi\}$. Clearly $Z_{RU}(\pi)$ is a weak induced subhypergraph of $Z_{EU}(\pi), \forall \pi \in \Pi_{RU} = \Pi_1 \cap \Pi_2$, since both dh-parts are identical. That is, we have

(a) $\Pi_{RU} \subseteq \Pi_{EU}$,

(b) For all $\pi \in \Pi_{RU}$, $Z_{RU}(\pi) = (\Omega_{RU}(\pi), \Psi_{RU}(\pi))$ is a weak induced subhypergraph of $Z_{EU}(\pi) = (\Omega_{EU}(\pi), \Psi_{EU}(\pi))$.

Hence $\Delta_1 \cup_R \Delta_2$ is a soft weak induced subhypergraph of $\Delta_1 \cup_E \Delta_2$. □

7. Restricted Intersection of Two Soft Directed Hypergraphs

Definition 7.1. Let $\Delta^* = (\Gamma, \Xi)$ be a simple directed hypergraph having vertex set Γ and hyperarc set Ξ. Also let $\Delta_1 = (\Delta^*, \Omega_1, \Psi_1, \Pi_1)$ and $\Delta_2 = (\Delta^*, \Omega_2, \Psi_2, \Pi_2)$ be two soft directed hypergraphs of Δ^* such that $\Pi_1 \cap \Pi_2 \neq \phi$ and $\Omega_1(\pi) \cap \Omega_2(\pi) \neq \phi$ for all $\pi \in \Pi_1 \cap \Pi_2$. Then the *restricted intersection* of Δ_1 and Δ_2 denoted by $\Delta_1 \cap_R \Delta_2$ is defined as $\Delta_1 \cap_R \Delta_2 = \Delta = (\Delta^*, \Omega, \Psi, \Pi)$, where $\Pi = \Pi_1 \cap \Pi_2$ and for all $\pi \in \Pi$, $\Omega(\pi) = \Omega_1(\pi) \cap \Omega_2(\pi)$ and $\Psi(\pi) = \{(T(e) \cap \Omega(\pi), H(e) \cap \Omega(\pi)) \mid e \in \Xi \text{ and } T(e) \cap \Omega(\pi) \neq \phi$ and $H(e) \cap \Omega(\pi) \neq \phi\}$. If $Z(\pi) = (\Omega(\pi), \Psi(\pi)), \forall \pi \in \Pi$, then $\Delta_1 \cap_R \Delta_2 = \{Z(\pi) : \pi \in \Pi\}$.

Example 7.2. Take into account a simple directed hypergraph $\Delta^* = (\Gamma, \Xi)$ as depicted in Figure 6, along with its soft directed hypergraphs Δ_1 presented in Figure 7 and Δ_2 illustrated in Figure 8, correspondingly. The restricted intersection of these two soft directed hypergraphs Δ_1 and Δ_2 is $\Delta = \Delta_1 \cap_R \Delta_2 = (\Delta^*, \Omega, \Psi, \Pi)$ where $\Pi = \Pi_1 \cap \Pi_2 = \{v_9\}$. Also $\Omega(v_9) = \Omega_1(v_9) \cap \Omega_2(v_9) = \{v_6, v_7, v_9, v_{14}\}$, $\Psi(v_9) = \{(T(e) \cap \Omega(v_9), H(e) \cap \Omega(v_9)) \mid e \in \Xi \text{ and } T(e) \cap \Omega(v_9) \neq \phi \text{ and } H(e) \cap \Omega(v_9) \neq \phi\} = \{(\{v_9\}, \{v_6, v_7\}), (\{v_9\}, \{v_{14}\})\}$. Here (Ω, Π) is a soft set over Γ and (Ψ, Π) is a soft set over Ξ_s. Also $Z(v_9) = (\Omega(v_9), \Psi(v_9))$ is a weak induced subhypergraph of Δ^*. Hence $\Delta_1 \cap_R \Delta_2 = \{Z(v_9)\}$ is a soft directed hypergraph of Δ^* and is given in Figure 12.

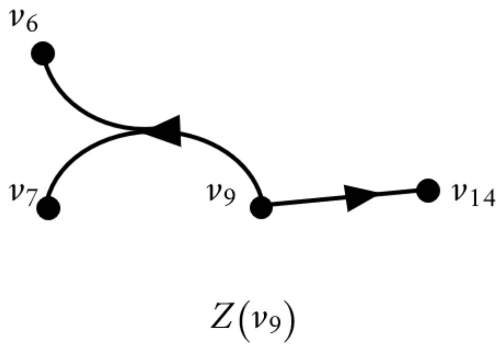

Fig. 12. $\Delta_1 \cap_R \Delta_2 = \{Z(v_9)\}$

Theorem 7.3. *Let $\Delta^* = (\Gamma, \Xi)$ be a simple directed hypergraph and $\Delta_1 = (\Delta^*, \Omega_1, \Psi_1, \Pi_1)$ and $\Delta_2 = (\Delta^*, \Omega_2, \Psi_2, \Pi_2)$ be two soft directed hypergraphs of Δ^* such that $\Pi_1 \cap \Pi_2 \neq \phi$ and $\Omega_1(\pi) \cap \Omega_2(\pi) \neq \phi$ for all $\pi \in \Pi_1 \cap \Pi_2$. Then their restricted intersection $\Delta_1 \cap_R \Delta_2$ is also a soft directed hypergraph of Δ^*.*

Proof. The restricted intersection $\Delta_1 \cap_R \Delta_2$ is defined as $\Delta_1 \cap_R \Delta_2 = \Delta = (\Delta^*, \Omega, \Psi, \Pi)$, where $\Pi = \Pi_1 \cap \Pi_2 \neq \phi$ is the parameter set and for all $\pi \in \Pi$, $\Omega(\pi) = \Omega_1(\pi) \cap \Omega_2(\pi)$ and $\Psi(\pi) = \{(T(e) \cap \Omega(\pi), H(e) \cap \Omega(\pi)) \mid e \in \Xi \text{ and } T(e) \cap \Omega(\pi) \neq \phi \text{ and } H(e) \cap \Omega(\pi) \neq \phi\}$. Here Ω is a mapping from Π to $\mathcal{P}(\Gamma)$ and Ψ is a mapping from Π to $\mathcal{P}(\Xi_s)$. Also (Ω, Π) is a soft set over Γ and (Ψ, Π) is a soft set over Ξ_s. When $\pi \in \Pi = \Pi_1 \cap \Pi_2$, the corresponding dh-part of $\Delta_1 \cap_R \Delta_2$ is $Z(\pi) = (\Omega(\pi), B(\pi))$ where $\Omega(\pi) = \Omega_1(\pi) \cap \Omega_2(\pi)$ and $\Psi(\pi) = \{(T(e) \cap \Omega(\pi), H(e) \cap \Omega(\pi)) \mid e \in \Xi \text{ and } T(e) \cap \Omega(\pi) \neq \phi \text{ and } H(e) \cap \Omega(\pi) \neq \phi\}$. We have $\Omega_1(\pi) \cap \Omega_2(\pi) \subseteq \Gamma$ and each hyperarc in $\Psi(\pi)$ is a subhyperarc of Δ^* for all $\pi \in \Pi = \Pi_1 \cap \Pi_2$. So $Z(\pi) = (\Omega(\pi), \Psi(\pi))$ is a weak induced subhypergraph of Δ^* for all $\pi \in \Pi = \Pi_1 \cap \Pi_2$. That is, $\Delta_1 \cap_R \Delta_2 = (\Delta^*, \Omega, \Psi, \Pi)$ is a soft directed hypergraph of Δ^* since all the conditions for a soft directed hypergraph are satisfied. □

Theorem 7.4. *Let $\Delta^* = (\Gamma, \Xi)$ be a simple directed hypergraph and $\Delta_1 = (\Delta^*, \Omega_1, \Psi_1, \Pi_1)$ and $\Delta_2 = (\Delta^*, \Omega_2, \Psi_2, \Pi_2)$ be two soft directed hypergraphs of Δ^* such that $\Pi_1 \cap \Pi_2 \neq \phi$ and $\Omega_1(\pi) \cap \Omega_2(\pi) \neq \phi$ for all $\pi \in \Pi_1 \cap \Pi_2$. Then $\Delta_1 \cap_R \Delta_2$ is a soft weak induced subhypergraph of $\Delta_1 \cap_E \Delta_2$.*

Proof. By Theorems 5.3 and 7.3, we have $\Delta_1 \cap_E \Delta_2$ and $\Delta_1 \cap_R \Delta_2$ are soft directed hypergraphs of Δ^*. Assume that $\Delta_1 \cap_E \Delta_2 = \Delta_{EI} = (\Delta^*, \Omega_{EI}, \Psi_{EI}, \Pi_{EI})$ and $\Delta_1 \cap_R \Delta_2 = \Delta_{RI} = (\Delta^*, \Omega_{RI}, \Psi_{RI}, \Pi_{RI})$. By the definitions of extended intersection and restricted intersection of two soft directed hypergraphs, the parameter set Π_{EI} of Δ_{EI} is $\Pi_1 \cup \Pi_2$ and the parameter set Π_{RI} of Δ_{RI} is $\Pi_1 \cap \Pi_2$. Clearly we have $\Pi_{RI} \subseteq \Pi_{EI}$ since $\Pi_1 \cap \Pi_2 \subseteq \Pi_1 \cup \Pi_2$. If $\pi \in \Pi_{RI} = \Pi_1 \cap \Pi_2$, $Z_{RI}(\pi) = (\Omega_{RI}(\pi), \Psi_{RI}(\pi))$, where $\Omega_{RI}(\pi) = \Omega_1(\pi) \cap \Omega_2(\pi)$ and $\Psi_{RI}(\pi) = \{(T(e) \cap \Omega_{RI}(\pi), H(e) \cap \Omega_{RI}(\pi)) \mid e \in \Xi$ and $T(e) \cap \Omega_{RI}(\pi) \neq \phi$ and $H(e) \cap \Omega_{RI}(\pi) \neq \phi\}$ and $Z_{EI}(\pi) = (\Omega_{EI}(\pi), \Psi_{EI}(\pi))$, where $\Omega_{EI}(\pi) = \Omega_1(\pi) \cap \Omega_2(\pi)$ and $\Psi_{EI}(\pi) = \{(T(e) \cap \Omega_{EI}(\pi), H(e) \cap \Omega_{EI}(\pi)) \mid e \in \Xi$ and $T(e) \cap \Omega_{EI}(\pi) \neq \phi$ and $H(e) \cap \Omega_{EI}(\pi) \neq \phi\}$. Clearly $Z_{RI}(\pi)$ is a weak induced subhypergraph of $Z_{EI}(\pi), \forall \pi \in \Pi_{RI} = \Pi_1 \cap \Pi_2$, since both dh-parts are identical. That is, we have

(a) $\Pi_{RI} \subseteq \Pi_{EI}$,

(b) For all $\pi \in \Pi_{RI}$, $Z_{RI}(\pi) = (\Omega_{RI}(\pi), \Omega_{RI}(\pi))$ is a weak induced subhypergraph of $Z_{EI}(\pi) = (\Omega_{EI}(\pi), \Psi_{EI}(\pi))$.

Hence $\Delta_1 \cap_R \Delta_2$ is a soft weak induced subhypergraph of $\Delta_1 \cap_E \Delta_2$. □

Theorem 7.5. *Let $\Delta^* = (\Gamma, \Xi)$ be a simple directed hypergraph and $\Delta_1 = (\Delta^*, \Omega_1, \Psi_1, \Pi_1)$ and $\Delta_2 = (\Delta^*, \Omega_2, \Psi_2, \Pi_2)$ be two soft directed hypergraphs of Δ^* such that $\Pi_1 \cap \Pi_2 \neq \phi$ and $\Omega_1(\pi) \cap \Omega_2(\pi) \neq \phi$ for all $\pi \in \Pi_1 \cap \Pi_2$. Then $\Delta_1 \cap_R \Delta_2$ is a soft weak induced subhypergraph of $\Delta_1 \cup_E \Delta_2$.*

Proof. By Theorems 4.3 and 7.3, we have $\Delta_1 \cup_E \Delta_2$ and $\Delta_1 \cap_R \Delta_2$ are soft directed hypergraphs of Δ^*. Assume that $\Delta_1 \cup_E \Delta_2 = \Delta_{EU} = (\Delta^*, \Omega_{EU}, \Psi_{EU}, \Pi_{EU})$ and $\Delta_1 \cap_R \Delta_2 = \Delta_{RI} = (\Delta^*, \Omega_{RI}, \Psi_{RI}, \Pi_{RI})$. By the definitions of extended union and restricted intersection of two soft directed hypergraphs, the parameter set Π_{EU} of Δ_{EU} is $\Pi_1 \cup \Pi_2$ and the parameter set Π_{RI} of Δ_{RI} is $\Pi_1 \cap \Pi_2$. Clearly we have $\Pi_{RI} \subseteq \Pi_{EU}$ since $\Pi_1 \cap \Pi_2 \subseteq \Pi_1 \cup \Pi_2$. If $\pi \in \Pi_{RI} = \Pi_1 \cap \Pi_2$, $Z_{RI}(\pi) = (\Omega_{RI}(\pi), \Psi_{RI}(\pi))$, where $\Omega_{RI}(\pi) = \Omega_1(\pi) \cap \Omega_2(\pi)$ and $\Psi_{RI}(\pi) = \{(T(e) \cap \Omega_{RI}(\pi), H(e) \cap \Omega_{RI}(\pi)) \mid e \in \Xi$ and $T(e) \cap \Omega_{RI}(\pi) \neq \phi$ and $H(e) \cap \Omega_{RI}(\pi) \neq \phi\}$ and $Z_{EU}(\pi) = (\Omega_{EU}(\pi), \Psi_{EU}(\pi))$, where $\Omega_{EU}(\pi) = \Omega_1(\pi) \cup \Omega_2(\pi)$ and $\Psi_{EU}(\pi) = \{(T(e) \cap \Omega_{EU}(\pi), H(e) \cap \Omega_{EU}(\pi)) \mid e \in \Xi$ and $T(e) \cap \Omega_{EU}(\pi) \neq \phi$ and $H(e) \cap \Omega_{EU}(\pi) \neq \phi\}$. Clearly $Z_{RI}(\pi)$ is a weak induced subhypergraph of $Z_{EU}(\pi), \forall \pi \in \Pi_{RI} = \Pi_1 \cap \Pi_2$. That is, we have

(a) $\Pi_{RI} \subseteq \Pi_{EU}$,

(b) For all $\pi \in \Pi_{RI}$, $Z_{RI}(\pi) = (\Omega_{RI}(\pi), \Psi_{RI}(\pi))$ is a weak induced subhypergraph of $Z_{EU}(\pi) = (\Omega_{EU}(\pi), \Psi_{EU}(\pi))$.

Hence $\Delta_1 \cap_R \Delta_2$ is a soft weak induced subhypergraph of $\Delta_1 \cup_E \Delta_2$. □

8. Conclusion

The introduction of soft directed hypergraphs stemmed from incorporating soft set principles into directed hypergraphs. Through parameterization, soft directed hypergraphs generate a sequence of descriptions for intricate relations depicted by directed hypergraphs. Undoubtedly, the incorporation of parameterization tools renders soft directed hypergraphs a pivotal component in the realm of directed hypergraph theory.

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