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Asymptotic average shadowable points for homeomorphisms

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ABSTRACT

In this paper, given a homeomorphism f of a compact metric space X, we show that the set of all asymptotic average shadowable points of f is an open and invariant set and f has the asymptotic average shadowing property if and only if the set of all asymptotic average shadowable points of f is X if and only if any Borel probability measure μ of X has the asymptotic average shadowing property.

Keywords: shadowing property, shadowable point, asymptotic average shadowing property, aymptotic average shadowable point, Borel probability measure

2020 Mathematics Subject Classification: 37B65.

1. Introduction

The asymptotic average shadowing property was introduced by Gu [2]. Gu proved in [2] that if a homomorphism f of a compact metric space X has the asymptotic average shadowing property, then it is chain transitive. Honary and Bahabadi [3] proved that if a diffeomorphism f of a compact smooth manifold M has the asymptotic average shadowing property and M is not finite and $\dim M = 2$, then the C^1 interior of the set of all C^1 diffeomorphisms with the asymptotic average shadowing property is characterized by the set of Ω stable diffeomorphisms, and Lee [7] proved that if $\dim M \geq 2$ then the C^1 interior of the set of all C^1 diffeomorphisms with the asymptotic average shadowing property is characterized by the set of weak hyperbolic (dominated splitting) diffeomorphisms.

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Also Lee [8] proved that for C^1 generic diffeomorphism f of a compact smooth manifold M with dim$M \geq 2$ then f is weakly hyperbolic. Thus, the asymptotic mean shadowing property is an interesting topic to study dynamical systems (topological and smooth dynamical systems). Regarding the various shadowing points, Morales [9] was the first to introduce the shadowable points. After his research, many valuable results are published in [1, 5, 6, 4] using the various shadowable points. The asymptotic average shadowable point was introduced by Rego and Arbieto [10]. As the notion of [10], we will study in the paper. More specifically, given a homeomorphism f of a compact metric space X, we prove that the set of all asymptotic average shadowable points of f is an open and invariant set and f has the asymptotic average shadowing property if and only if the set of all asymptotic average shadowable points of f is X if and only if any Borel probability measure μ of X has the asymptotic average shadowing property.

2. Basic notions and proof of theorem

Let (X, d) be a compact metric space with metric d and let $f : X \to X$ be a homeomorphism. For any $\delta > 0$, An infinite sequence $\{x_i : i \in \mathbb{Z}\}$ is said to be *δ-pseudo-orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. We say that f has the *shadowing property* if for every $\epsilon > 0$ there exists $\delta > 0$ such that every δ pseudo-orbit $\{x_i : i \in \mathbb{Z}\}$ can be ϵ shadowed by some point in X, i.e. there exists $y \in X$ such that $d(f^i(y), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. Then the point $y \in X$ is called the *shadowable point* of f. Let $Sh(f)$ be the set of all shadowable points of f. Morales [9] proved that $Sh(f)$ is an invariant set and f has the shadowing property if and only if $Sh(f) = X$.

Now we are concerned with a kind of shadowing property and a kind of shadowable points of f. It is a different notion of the shadowing property.

An infinite sequence $\{x_i : i \in \mathbb{Z}\}$ is said to be *asymptotic pseudo-orbit* of f if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) = 0, \text{ and } \lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{-1} d(f(x_i), x_{i+1}) = 0.$$

A homeomorphism f of X has the *asymptotic average shadowing property* if any asymptotic average pseudo-orbit $\{x_i : i \in \mathbb{Z}\}$ can be asymptotic average shadowed by some point in X, i.e. there exists $y \in X$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y), x_i) = 0, \text{ and } \lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{-1} d(f^i(y), x_i) = 0.$$

Then the point $y \in X$ is called *asymptotic average shadowable point* of f. Denote by $AEVSh(f)$ the set of all asymptotic average shadowable points of f.

We say that f is *chain transitive* if, for any $\delta > 0$, there exists a sequence $\{x_i : i = 0, 1, \ldots, n\}$ such that $d(f(x_i), x_{i+1}) < \delta$ for all $i = 0, 1, \ldots, n$. Gu [2] proved that if f has the asymptotic average shadowing property, then it is chain transitive. Thus, the asymptotic average shadowing property is not equivalent to the shadowing property (see, [2] Section 5.1).

Lemma 2.1. *Let f be a homeomorphism of X. Then $AEVSh(f)$ is an open set.*

Proof. Assume that $AEVSh(f) \neq \emptyset$. Let $y \in AEVSh(f)$ and any asymptotic average pseudo orbit through y can be asymptotic average shadowed by some point in X. Take $\epsilon > 0$. Let $x \in B(y, \epsilon)$ and $\xi = \{x_i : i \in \mathbb{Z}\}$ be an asymptotic average pseudo orbit through x. We make an infinite sequence $\zeta = \{y_i : i \in \mathbb{Z}\}$ as follows:

$y_i = x_i$ for $i \neq 0$ and $y_i = y$ for $i = 0$. Then we see that

$$\begin{aligned}\sum_{i=0}^{n-1} d(f(y_i), y_{i+1}) &= d(f(y), y_1) + d(f(y_1), y_2) + \cdots + d(f(y_{n-1}), y_n) \\ &= d(f(y), x_1) + d(f(x_1), x_2) + \cdots + d(f(x_{n-1}), x_n) \\ &\leq \text{diam}X + \sum_{i=0}^{n-1} d(f(x_i), x_{i+1}),\end{aligned}$$

and

$$\begin{aligned}\sum_{i=-n}^{-1} d(f(y_i), y_{i+1}) &= d(f(y_{-n}), y_{-n+1}) + +d(f(y_{-n+1}), y_{-n+2}) + \cdots + d(f(y_{-1}), y) \\ &= d(f(x_{-n}), x_{-n+1}) + d(f(x_{-n+1}), x_{-n+2}) + \cdots + d(f(x_{-1}), y) \\ &\leq \sum_{i=-n}^{-1} d(f(x_i), x_{i+1}) + \text{diam}X.\end{aligned}$$

Since ξ is an asymptotic average pseudo orbit of f, we have that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(y_i), y_{i+1}) &\leq \lim_{n\to\infty} \frac{1}{n}\Big(\text{diam}X + \sum_{i=0}^{n-1} d(f(x_i), x_{i+1})\Big) \\ &= \lim_{n\to\infty} \frac{\text{diam}X}{n} + \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) = 0,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{-1} d(f(y_i), y_{i+1}) &\leq \lim_{n\to\infty} \frac{1}{n}\Big(\sum_{i=-n}^{-1} d(f(x_i), x_{i+1}) + \text{diam}X\Big) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{-1} d(f(x_i), x_{i+1}) + \lim_{n\to\infty} \frac{\text{diam}X}{n} = 0.\end{aligned}$$

Thus ζ is an asymptotic pseudo obit of f through y. Since $y \in AEVSh(f)$, there is a point $z \in X$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), y_i) = 0, \text{ and } \lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{-1} d(f^i(z), y_i) = 0.$$

Since $y_i = x_i$ for $i \neq 0$ and $y_0 = y$, we also see that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d\left(f^i(z), x_i\right) = 0, \text{ and } \lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{-1} d(f^i(z), x_i) = 0.$$

Thus the asymptotic pseudo orbit ξ can be asymptotic average shadowed by $z \in X$. This means that $x \in AEVSh(f)$ and so $B(y,\epsilon) \subset ASEVSh(f)$. It implies that $ASEVSh(f)$ is an open set.

□

A set $\Lambda \subset X$ is an *invariant* if $f(\Lambda) \subset \Lambda$.

Lemma 2.2. *Let f be a homeomorphism of X. Then $AEVSh(f)$ is an invariant set.*

Proof. For any $x \in AEVSh(f)$, let $\xi = \{x_i : i \in \mathbb{Z}\}$ be an asymptotic average pseudo orbit of f through $f(x)$. Since ξ is an asymptotic average pseudo orbit of f, we have

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{i=-n}^{n-1} d(f(x_i), x_{i+1}) = 0.$$

Since f^{-1} is uniformly continuous, take $\epsilon > 0$ such that $d(a,b) < \epsilon/4 (a, b \in X)$ implies $d(f^{-1}(a), f^{-1}(b)) < \epsilon$. Let $j = \#\{i \in \mathbb{N}^+ : d(f(x_i), x_{i+1}) \geq \epsilon/4\}$. Then we have

$$\begin{aligned}\sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) &= \sum_{i=0}^{j} d(f(x_i), x_{i+1}) + \sum_{i=j+1}^{n-1} d(f(x_i), x_{i+1}) \\ &< \sum_{i=0}^{j} d(f(x_i), x_{i+1}) + \sum_{i=j+1}^{n-1} \frac{\epsilon}{4}.\end{aligned}$$

Consider the sequence $\eta = \{f^{-1}(x_i) : i \in \mathbb{Z}\}$. Since

$$\sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) < \sum_{i=0}^{j} d(f(x_i), x_{i+1}) + \sum_{i=j+1}^{n-1} \frac{\epsilon}{4},$$

we have

$$\begin{aligned}\sum_{i=0}^{n-1} d(f^{-1}(f(x_i)), f^{-1}(x_{i+1})) &= \sum_{i=0}^{j} d(f^{-1}(f(x_i)), f^{-1}(x_{i+1})) + \sum_{i=j+1}^{n-1} d(f^{-1}(f(x_i)), f^{-1}(x_{i+1})) \\ &< \sum_{i=0}^{j} \dim X + \sum_{i=j+1}^{n-1} \epsilon.\end{aligned}$$

Since

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) = 0,$$

we see that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^{-1}(f(x_i)), f^{-1}(x_{i+1})) = 0.$$

Similarly, we can see that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{-1} d(f^{-1}(f(x_i)), f^{-1}(x_{i+1})) = 0.$$

Then ζ is an asymptotic average pseudo orbit of f through x. Since $x \in AEVSh(f)$, there is $z \in X$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), f^{-1}(x_i)) = 0, \text{ and } \lim_{n\to\infty}\frac{1}{n}\sum_{i=-n}^{-1} d(f^i(z), f^{-1}(x_i)) = 0.$$

By uniform continuity of f, take $r > 0$ such that $d(f(a), f(b)) < r$ whenever $d(a, b) < r/4(a, b \in X)$. Let $l = \#\{i \in \mathbb{N}^+ : d(f^i(z), f^{-1}(x_i)) \geq r/4\}$. Then we see that

$$\sum_{i=0}^{n-1} d(f^i(z), f^{-1}(x_i)) = \sum_{i=0}^{l} d(f^i(z), f^{-1}(x_i)) + \sum_{i=l+1}^{n-1} d(f^i(z), f^{-1}(x_i)).$$

. Then we see that

$$\begin{aligned}\sum_{i=0}^{n-1} d(f^i(z), f^{-1}(x_i)) &= \sum_{i=0}^{l} d(f^i(z), f^{-1}(x_i)) + \sum_{i=l+1}^{n-1} \frac{r}{4}\\ &< \sum_{i=0}^{l} \text{diam}X + \sum_{i=l+1}^{n-1} \frac{r}{4}.\end{aligned}$$

Thus we have that

$$\begin{aligned}\sum_{i=0}^{n-1} d\left(f(f^i(z)), x_i\right) &= \sum_{i=0}^{l} d(f(f^i(z)), x_i) + \sum_{i=l+1}^{n-1} d(f(f^i(z)), x_i)\\ &< \sum_{i=0}^{l} d(f(f^i(z)), x_i) + \sum_{i=l+1}^{n-1} r\\ &< \sum_{i=0}^{l} \text{diam}X + \sum_{i=l+1}^{n-1} r.\end{aligned}$$

Since $f(f^i(z)) = f^i(f(z))$ for all $i \in \mathbb{Z}$, and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), f^{-1}(x_i)) = 0,$$

we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} d(f^i(f(z)), x_i) = 0.$$

Similarly, we can see that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=-n}^{-1} d(f^i(f(z)), x_i) = 0.$$

It implies that ξ is an asymptotic average shadowed by $f(z)$. Hence $f(x) \in AEVSh(f)$, and so $AEVSh(f)$ is an invariant set. $\square$

Lemma 2.3. *Let f be a homeomorphism of X. If $AEVSh(f) = X$, then f has the asymptotic average shadowing property.*

Proof. Suppose that for each $k \in \mathbb{N}$ there is an asymptotic average pseudo orbit $\xi^k = \{x_i^k : i \in \mathbb{Z}\}$ which can not be asymptotic average shadowed by a point in X. Since X is compact, we assume that $x_0^k \to p \in X$ as $k \to \infty$. Since $p \in AEVSh(f) = X$, there is an asymptotic average pseudo orbit through p can be asymptotic average shadowed by a point in X. For any small $\epsilon > 0$, there is $l \in \mathbb{N}$ such that $d(x_0^l, p) < \epsilon$ and $d(f(x_0^l), f(p)) < \epsilon$. We construct a sequence $\zeta = \{y_i : i \in \mathbb{Z}\}$ as follows: $y_i = x_i^l$ if $i \neq 0$ and $y_0 = p$ if $i = 0$. Then we have that

$$\begin{aligned}\sum_{i=0}^{n-1} d(f(y_i), y_{i+1}) &= d(f(p), y_1) + d(f(y_1), y_2) + \cdots + d(f(y_{n-1}), y_n)\\ &= d(f(p), x_1^l) + d(f(x_1^l), x_2^l) + \cdots + d(f(x_{n-1}^l), x_n^l)\\ &\leq d(f(p), f(x_0^l)) + d(f(x_0^l), x_1^l) + +d(f(x_1^l), x_2^l) + \cdots + d(f(x_{n-1}^l), x_n^l)\\ &< \epsilon + \text{diam}X + \sum_{i=0}^{n-1} d(f(x_i^l), x_{i+1}^l),\end{aligned}$$

and

$$\begin{aligned}\sum_{i=-n}^{-1} d(f(y_i), y_{i+1}) &= d(f(y_{-n}), y_{-n+1}) + +d(f(y_{-n+1}), y_{-n+2}) + \cdots + d(f(y_{-1}), y(= p))\\ &= d(f(x_{-n}^l), x_{-n+1}^l) + d(f(x_{-n+1}^l), x_{-n+2}^l) + \cdots + d(f(x_{-1}^l), p)\\ &\leq d(f(x_{-n}^l), x_{-n+1}^l) + d(f(x_{-n+1}^l), x_{-n+2}^l) + \cdots + d(f(x_{-1}^l), x_0^l) + d(x_0^l, p)\\ &< \sum_{i=-n}^{-1} d(f(x_i^l), x_{i+1}^l) + \text{diam}X + \epsilon.\end{aligned}$$

Thus we see that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(y_i), y_{i+1}) < \lim_{n\to\infty} \frac{1}{n}\Big(\epsilon + \text{diam}X + \sum_{i=0}^{n-1} d(f(x_i^l), x_{i+1}^l)\Big) = 0,$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{-1} d(f(y_i), y_{i+1}) < \lim_{n\to\infty} \frac{1}{n}\Big(\sum_{i=-n}^{-1} d(f(x_i^l), x_{i+1}^l) + \text{diam}X + \epsilon\Big) = 0.$$

This means that $\zeta = \{y_i : i \in \mathbb{Z}\}$ is an asymptotic average pseudo orbit of f through p. Since $p \in AEVSh(f)$, there is $z \in X$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), y_i) = 0, \text{ and } \lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{-1} d(f^i(z), y_i) = 0.$$

Since $y_i = x_i^l$ for $i \neq 0$ and $y_0 = p$, we also see that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i^l) = 0, \text{ and} lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{-1} d(f^i(z), x_i^l) = 0.$$

This is a contradiction.

□

Theorem 2.4. *Let f be a homeomorphism of X. $AEVSh(f) = X$ if and only if f has the asymptotic average shadowing property.*

Proof. It is clear that if f has the asymptotic average shadowing property then $AEVSh(f) = X$. Thus we will prove the converse part. As Lemma 2.3, we have that if a homeomorphism f has the asymptotic average shadowing property then $AEVSh(f) = X$. This completes the proof. □

Kawaguchi [4] proved that for any ergodic probability measure μ of X, if $\mu(Sh(f)) = 1$ then f has the shadowing property, where μ of X is ergodic if either $\mu(A) = 0$ or $\mu(A) = 1$ for any Borel set $A \subset X$. For any Borel probability measure μ of X, we say that μ is *invariant* for f if $\mu(A) = \mu(f^{-1}(A)) = \mu(f(A))$ for any Borel set $A \subset X$. Let $\mathcal{M}_f(X)$ be the set of all invariant Borel probability measures of X.

About the result, we introduced the following notion. We say that $\mu \in \mathcal{M}_f(X)$ has the asymptotic average shadowing property if there is a Borel set $A \subset X$ such that $\mu(A) = 1$ and any asymptotic average pseudo orbit $\{x_i : i \in \mathbb{Z}\}$ through $A(x_0 \in A)$ can be asymptotic shadowed by a point in X. It is observed that μ has the asymptotic average shadowing property for f, then $\mu(AEVSh(f)) = 1$.

Lemma 2.5. *For any $\mu \in \mathcal{M}_f(X)$, if μ has the asymptotic average shadowing property then $AEVSh(f) = X$.*

Proof. For any $x \in X$, there is an invariant measure μ of X such that $\mu(\omega(x)) = 1$, where $\omega(x)$ is the omega limit set of x. Since μ has the asymptotic average shadowing property, $\mu(AEVSh(f)) = 1$. Then we can choose $y \in \omega(x) \cap AEVSh(f)$. By Lemma 2.1, there is $r > 0$ such that $B(y, r) \subset AEVSh(f)$. Since $y \in \omega(f)$, there is $j > 0$ such that $f^j(x) \in B(y, r) \subset AEVSh(f)$. By Lemma 2.2, $AEVSh(f)$ is invariant. We have $x \in AEVSh(f)$, and so $AEVSh(f) = X$. □

Theorem 2.6. *Let f be a homeomorphism of X. Any Borel probability measure μ of X has the asymptotic average shadowing property if and only if f has the asymptotic average shadowing property.*

Proof. Suppose that any Borel probability measure μ of X has the asymptotic average shadowing property. By Lemma 2.5, we see that $AEVSh(f) = X$. By Theorem 2.4, f has the asymptotic average shadowing property.

Conversely, , we assume that f has the asymptotic average shadowing property. By the definition of the asymptotic average shadowing property, it is clear that every Borel probability measure μ of X has the asymptotic average shadowing property. $\square$

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